# Solutions for assignment #6

## Section 2.9

39. Substitute $v = y'$ and $v' = y''$. The equation is $2t^2v' + v^3 = 2tv$. This is a *Bernoulli* equation (See Section 2.4, Problem 27), so the substitution $z = v^{-2}$ yields $z' = -2v^{-3}v'$, and the equation turns into $2t^2v'v^3 + 1 = 2t/v^2$, i.e. into $-2t^2z'/2 + 1 = 2tz$, which in turn simplifies to $t^2z' + 2tz = (t^2z)' = 1$. Integration yields $t^2z = t + c$, which means that $z = (1/t) + (c/t^2)$. Now $y' = v = \pm\sqrt{1/z} = \pm t/\sqrt{t + c_1}$ and another integration gives

$$y = \pm\frac{2}{3}(t - 2c_1)\sqrt{t + c_1} + c_2.$$

The substitution also loses the solution $v = 0$, i.e. $y = c$.

47. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. We obtain the equation $v'v + v^2 = 2e^{-y}$, where the differentiation is with respect to $y$. This is a *Bernoulli* equation (See Section 2.4, Problem 27) and substituting $z = v^2$ we get that $z' = 2vv'$, which means that the equation reads $z' + 2z = 4e^{-y}$. The integrating factor is $\mu(y) = e^{2y}$, which turns the equation into $e^{2y}z' + 2e^{2y}z = (e^{2y}z)' = 4e^y$. Integration gives us $v^2 = z = 4e^{-y} + ce^{-2y}$. This implies that $y' = v = \pm e^{-y}\sqrt{c + 4e^y}$. Separation of variables now shows that $\pm e^y dy/\sqrt{c + 4e^y} = dt$ and then $\pm\frac{1}{2}(c + 4e^y)^{1/2} = t + d$. Algebraic manipulations then yield the implicitly defined solution $e^y = (t + c_2)^2 + c_1$.

49. Set $y' = v(y)$. Then $y'' = v'(y)(dy/dt) = v'(y)v(y)$. We obtain the equation $v'v - 3y^2 = 0$, where the differentiation is with respect to $y$. Separation of variables gives $vdv = 3y^2dy$, and after integration this turns into $v^2/2 = y^3 + c$. The initial conditions imply that $c = 0$ here, so $(y')^2 = v^2 = 2y^3$. This implies that $y' = \sqrt{2}y^{3/2}$ (the sign is determined by the initial conditions again), and this separable equation now turns into $y^{-3/2}dy = \sqrt{2}dt$. Integration yields $-2y^{-1/2} = \sqrt{2}t + d$, and the initial conditions at this point give that $d = -\sqrt{2}$. Algebraic manipulations find that $y = 2(1 - t)^{-2}$.

## Section 3.2

1.

$$W(e^{2t}, e^{-3t/2}) = \begin{vmatrix} e^{2t} & e^{-3t/2} \\ 2e^{2t} & -\frac{3}{2}e^{-3t/2} \end{vmatrix} = -\frac{7}{2}e^{t/2}.$$

5.

$$W(e^t \sin t\,, e^t \cos t) = \begin{vmatrix} e^t \sin t & e^t \cos t \\ e^t(\sin t + \cos t) & e^t(\cos t - \sin t) \end{vmatrix} = -e^{2t}\,.$$

13. $y_1'' = 2$. We see that $t^2(2) - 2(t^2) = 0$. $y_2'' = 2\,t^{-3}$, with $t^2(y_2'') - 2(y_2) = 0$. Let $y_3 = c_1 t^2 + c_2 t^{-1}$, then $y_3'' = 2c_1 + 2c_2 t^{-3}$. It is evident that $y_3$ is also a solution.

25. Clearly, $y_1 = e^t$ is a solution. $y_2' = (1+t)e^t$, $y_2'' = (2+t)e^t$. Substitution into the ODE results in $(2+t)e^t - 2(1+t)e^t + t\,e^t = 0$. Furthermore, $W(e^t, te^t) = e^{2t}$. Hence the solutions form a fundamental set of solutions.