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MAST 330/2/AA 2010
Assignment 5
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Boyce & DiPrima 2.6.25, 2.6.26, 2.6.27, 2.6.28, 2.6.30

2.6.25

$$\left(3x^2y + 2xy + y^3\right) dx + \left(x^2 + y^2\right) dy = 0$$

$$\begin{array}{lll} M(x,y) = 3x^2y + 2xy + y^3 & \implies & M_y = 3x^2 + 2x + 3y^2 \\ N(x,y) = x^2 + y^2 & \implies & N_x = 2x \end{array} \implies M_y \neq N_x \therefore \text{not exact}$$

Search for an integrating factor $\mu(x,y) \ni (\mu M)_y = (\mu N)_x$. If μ is a function of x alone, then

$$\frac{d\mu}{dx} = \mu\left(\frac{M_y - N_x}{N}\right)$$

For M, N as given,

$$\begin{aligned} \left(\frac{M_y - N_x}{N}\right) &= \frac{\left(3x^2 + 2x + 3y^2\right) - 2x}{x^2 + y^2} \\ &= 3\frac{x^2 + y^2}{x^2 + y^2} \\ &= 3 \end{aligned}$$

Therefore, ignoring irrelevant constant of integration,

$$\frac{d\mu}{dx} = 3\mu \implies \ln\mu = 3x \implies \mu = e^{3x}$$

Multiply original equation through by μ and solve this new, exact equation.

$$e^{3x}\left(3x^2y + 2xy + y^3\right) dx + e^{3x}\left(x^2 + y^2\right) dy = 0$$

$$\begin{array}{lll} M(x,y) = e^{3x}\left(3x^2y + 2xy + y^3\right) & \implies & M_y = e^{3x}\left(3x^2 + 2x + 3y^2\right) \\ N(x,y) = e^{3x}\left(x^2 + y^2\right) & \implies & N_x = e^{3x}2x + 3e^{3x}\left(x^2 + y^2\right) \end{array} \implies M_y = N_x \therefore \text{exact}$$

$$\therefore \exists\psi\left(x, y = \phi(x)\right) = c \ni \psi_x = M, \psi_y = N, \frac{d}{dx}\psi = \frac{\partial\psi}{\partial x} + \frac{\partial\psi}{\partial y}\frac{dy}{dx} = M + Ny' = 0$$

$$\begin{aligned} \psi(x,y) &= \int N\, dy \\ &= \int e^{3x}\left(x^2 + y^2\right) dy \\ &= e^{3x}\left(x^2y + \frac{y^3}{3}\right) + h(x) \end{aligned}$$

$$
\begin{aligned}
M = \psi_x = e^{3x}\left(3x^2y + 2xy + y^3\right) &= \frac{\partial}{\partial x}\left[e^{3x}\left(x^2y + \frac{y^3}{3}\right) + h(x)\right] \\
&= 3e^{3x}x^2y + e^{3x}2xy + e^{3x}y^3 + h'(x) \\
&= e^{3x}\left(3x^2y + 2xy + y^3\right) + h'(x)
\end{aligned}
$$

$$
\therefore h'(x) = 0 \implies h(x) = K
$$

$$
\therefore \psi(x,y) = e^{3x}\left(x^2y + \frac{y^3}{3}\right) + h(x) = e^{3x}\left(x^2y + \frac{y^3}{3}\right) + K
$$

$$
\frac{d}{dx}\psi = 0 \implies e^{3x}\left(x^2y + \frac{y^3}{3}\right) + K = c \implies e^{3x}\left(3x^2y + y^3\right) = 3(c - K)
$$

$$
\therefore e^{3x}\left(3x^2y + y^3\right) = C
$$

■

2.6.26

$$
y' = e^{2x} + y - 1
$$

$$
y' = e^{2x} + y - 1 \equiv \left(e^{2x} + y - 1\right)dx - dy = 0
$$

$$
\begin{aligned}
M(x,y) &= e^{2x} + y - 1 &\implies M_y &= 1 \\
N(x,y) &= -1 &\implies N_x &= 0
\end{aligned}
\implies M_y \neq N_x \therefore \text{not exact}
$$

Search for an integrating factor $\mu(x,y) \ni (\mu M)_y = (\mu N)_x$. If μ is a function of x alone, then

$$
\frac{d\mu}{dx} = \mu\left(\frac{M_y - N_x}{N}\right)
$$

For M, N as given,

$$
\begin{aligned}
\left(\frac{M_y - N_x}{N}\right) &= \frac{1-0}{-1} \\
&= -1
\end{aligned}
$$

Therefore, ignoring irrelevant constant of integration,

$$
\frac{d\mu}{dx} = -\mu \implies \ln\mu = -x \implies \mu = e^{-x}
$$

Multiply original equation through by μ and solve this new, exact equation.

$$
e^{-x}\left(e^{2x} + y - 1\right)dx - e^{-x}dy = 0
$$

$$
\begin{aligned}
M(x,y) &= e^{-x}\left(e^{2x} + y - 1\right) = e^{x} + e^{-x}y - e^{-x} &\implies M_y &= e^{-x} \\
N(x,y) &= -e^{-x} &\implies N_x &= e^{-x}
\end{aligned}
\implies M_y = N_x \therefore \text{exact}
$$

$$\therefore \exists \psi\left(x, y=\phi(x)\right)=c \ni \psi_x=M, \psi_y=N, \frac{d}{dx}\psi=\frac{\partial \psi}{\partial x}+\frac{\partial \psi}{\partial y}\frac{dy}{dx}=M+Ny'=0$$

$$\begin{aligned}\psi(x,y) &= \int N\,dy \\ &= \int -e^{-x}dy \\ &= -e^{-x}y+h(x)\end{aligned}$$

$$\begin{aligned}M=\psi_x=e^{x}+e^{-x}y-e^{-x} &= \frac{\partial}{\partial x}\left[-e^{-x}y+h(x)\right] \\ &= e^{-x}y+h'(x)\end{aligned}$$

$$\therefore h'(x)=e^{x}-e^{-x} \implies h(x)=e^{x}+e^{-x}+K$$

$$\therefore \psi(x,y)=-e^{-x}y+h(x)=-e^{-x}y+e^{x}+e^{-x}+K$$

$$\frac{d}{dx}\psi=0 \implies -e^{-x}y+e^{x}+e^{-x}+K=c \implies -e^{-x}y+e^{x}+e^{-x}=c-K$$

$$-e^{-x}y+e^{x}+e^{-x}=e^{-x}\left(-y+e^{2x}+1\right) = c-K \implies (c-K)e^{x}=-y+e^{2x}+1$$

$$\begin{aligned}\therefore y &= 1+e^{2x}-(c-K)e^{x} \\ &= 1+e^{2x}+Ce^{x}\end{aligned}$$

■

2.6.27

$$dx+\left(\frac{x}{y}-\sin y\right)dy=0$$

$$\begin{aligned}&M(x,y)=1 &&\implies M_y=0 \\ &N(x,y)=\frac{x}{y}-\sin y &&\implies N_x=\frac{1}{y} \implies M_y\neq N_x \therefore \text{not exact}\end{aligned}$$

Search for an integrating factor $\mu(x,y) \ni (\mu M)_y=(\mu N)_x$. If μ is a function of y alone, then

$$\frac{d\mu}{dy}=\mu\left(\frac{N_x-M_y}{M}\right)$$

For M, N as given,

$$\left(\frac{N_x - M_y}{M}\right) = \frac{(1/y) - 0}{1} = \frac{1}{y}$$

Therefore, ignoring irrelevant constant of integration,

$$\frac{d\mu}{dy} = \mu\frac{1}{y} \implies \ln\mu = \ln y \implies \mu = y$$

Multiply original equation through by μ,

$$ydx + (x - y\sin y)\,dy = 0$$

which can be rewritten as

$$\begin{aligned} ydx + xdy &= y\sin y\,dy \\ d(xy) &= y\sin y\,dy \end{aligned}$$

and solved by inspection on the LHS and integration by parts on the RHS

$$\begin{aligned} xy + C_1 &= \int y\sin y\,dy \\ &= \sin y - y\cos y + C_2 \quad u = y, dv = \sin y \end{aligned}$$

$$\therefore xy - \sin y + y\cos y = C$$

■

2.6.28

$$ydx + \left(2xy - e^{-2y}\right)dy = 0$$

$$\begin{aligned} M(x,y) &= y &\implies& M_y = 1 \\ N(x,y) &= \left(2xy - e^{-2y}\right) &\implies& N_x = 2y \end{aligned} \implies M_y \neq N_x \therefore \text{not exact}$$

Search for an integrating factor $\mu(x,y) \ni (\mu M)_y = (\mu N)_x$. If μ is a function of y alone, then

$$\frac{d\mu}{dy} = \mu\left(\frac{N_x - M_y}{M}\right)$$

For M, N as given,

$$\left(\frac{N_x - M_y}{M}\right) = \frac{2y - 1}{y} = 2 - \frac{1}{y}$$

Therefore, ignoring irrelevant constant of integration,

$$\frac{d\mu}{dy} = \mu\left(2 - \frac{1}{y}\right)$$

$$\begin{aligned}
\therefore \int \frac{d\mu}{\mu} &= \int \left(2 - \frac{1}{y}\right) dy \\
\ln \mu &= 2y - \ln y \\
\mu &= e^{2y - \ln y} \\
&= \frac{e^{2y}}{e^{\ln y}} \\
&= \frac{e^{2y}}{y}
\end{aligned}$$

Multiply original equation through by μ and solve this new, exact equation

$$\frac{e^{2y}}{y} y dx + \frac{e^{2y}}{y}\left(2xy - e^{-2y}\right) dy = e^{2y} dx + \left(e^{2y} 2x - \frac{1}{y}\right) dx = 0$$

$$\begin{aligned}
M(x, y) &= e^{2y} && \Longrightarrow \quad M_y = 2e^{2y} \\
N(x, y) &= 2xe^{2y} - \frac{1}{y} && \Longrightarrow \quad N_x = 2e^{2y} \quad \Longrightarrow M_y = N_x \therefore \text{exact}
\end{aligned}$$

$$\therefore \exists \psi(x, y = \phi(x)) = c \ni \psi_x = M, \psi_y = N, \frac{d}{dx}\psi = \frac{\partial \psi}{\partial x} + \frac{\partial \psi}{\partial y}\frac{dy}{dx} = M + Ny' = 0$$

$$\begin{aligned}
\psi(x, y) &= \int M\, dx \\
&= \int e^{2y} dx \\
&= xe^{2y} + h(y)
\end{aligned}$$

$$\begin{aligned}
N = \psi_y = 2xe^{2y} - \frac{1}{y} &= \frac{\partial}{\partial y}\left[xe^{2y} + h(y)\right] \\
&= 2xe^{2y} + h'(y)
\end{aligned}$$

$$\therefore h'(y) = -\frac{1}{y} \Longrightarrow h(y) = -\ln|y| + K$$

$$\therefore \psi(x, y) = xe^{2y} + h(y) = xe^{2y} - \ln|y| + K$$

$$\frac{d}{dx}\psi = 0 \Longrightarrow xe^{2y} - \ln|y| + K = c \Longrightarrow xe^{2y} - \ln|y| = c - K$$

$$\therefore xe^{2y} - \ln|y| = C$$

■

2.6.30

$$\left(4\frac{x^3}{y^2}+\frac{3}{y}\right)dx+\left(3\frac{x}{y^2}+4y\right)dy=0$$

$$\begin{array}{lll} M(x,y)=\left(4\dfrac{x^3}{y^2}+\dfrac{3}{y}\right) & \Longrightarrow & M_y=-\dfrac{8x^3}{y^3}-\dfrac{3}{y^2} \\ N(x,y)=\left(3\dfrac{x}{y^2}+4y\right) & \Longrightarrow & N_x=\dfrac{3}{y^2} \end{array} \Longrightarrow M_y \neq N_x \therefore \text{not exact}$$

Search for an integrating factor $\mu(x,y) \ni (\mu M)_y = (\mu N)_x$. If μ is a function of y alone, then

$$\frac{d\mu}{dy}=\mu\left(\frac{N_x-M_y}{M}\right)$$

For M, N as given,

$$\begin{aligned} \left(\frac{N_x-M_y}{M}\right) &= \frac{\dfrac{3}{y^2}+\dfrac{8x^3}{y^3}+\dfrac{3}{y^2}}{4\dfrac{x^3}{y^2}+\dfrac{3}{y}} \\ &= \frac{\dfrac{8x^3+6y}{y^3}}{\dfrac{4x^3+3y}{y^2}} \\ &= \frac{2}{y} \end{aligned}$$

Therefore, ignoring irrelevant constant of integration,

$$\frac{d\mu}{dy}=\mu\frac{2}{y} \Longrightarrow \ln\mu = 2\ln y = \ln y^2 \Longrightarrow \mu = y^2$$

Multiply original equation through by μ and solve this new, exact equation

$$y^2\left(4\frac{x^3}{y^2}+\frac{3}{y}\right)dx+y^2\left(3\frac{x}{y^2}+4y\right)dy=\left(4x^3+3y\right)dx+\left(3x+4y^3\right)dy=0$$

$$\begin{array}{lll} M(x,y)=4x^3+3y & \Longrightarrow & M_y=3 \\ N(x,y)=3x+4y^3 & \Longrightarrow & N_x=3 \end{array} \Longrightarrow M_y = N_x \therefore \text{exact}$$

$$\therefore \exists\psi(x,y=\phi(x))=c \ni \psi_x=M, \psi_y=N, \frac{d}{dx}\psi=\frac{\partial\psi}{\partial x}+\frac{\partial\psi}{\partial y}\frac{dy}{dx}=M+Ny'=0$$

$$\begin{aligned} \psi(x,y) &= \int M\,dx \\ &= \int\left(4x^3+3y\right)dx \\ &= x^4+3xy+h(y) \end{aligned}$$

$$
\begin{aligned}
N = \psi_y = 3x + 4y^3 &= \frac{\partial}{\partial y}\left[x^4 + 3xy + h(y)\right] \\
&= 3x + h'(y)
\end{aligned}
$$

$$\therefore h'(y) = 4y^3 \implies h(y) = y^4 + K$$

$$\therefore \psi(x, y) = x^4 + 3xy + h(y) = x^4 + 3xy + y^4 + K$$

$$\frac{d}{dx}\psi = 0 \implies x^4 + 3xy + y^4 + K = c \implies x^4 + 3xy + y^4 = c - K$$

$$\therefore x^4 + 3xy + y^4 = C$$

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