

Solutions for assignment #4

Section 2.5

23.(a) $y(t) = y_0 e^{-\beta t}$.

(b) From part (a), $dx/dt = -\alpha xy_0 e^{-\beta t}$. Separating variables, $dx/x = -\alpha y_0 e^{-\beta t} dt$. Integrating both sides, the solution is $x(t) = x_0 e^{-\alpha y_0 (1 - e^{-\beta t})/\beta}$.

(c) As $t \rightarrow \infty$, $y(t) \rightarrow 0$ and $x(t) \rightarrow x_0 e^{-\alpha y_0/\beta}$. Over a long period of time, the proportion of carriers vanishes. Therefore the proportion of the population that escapes the epidemic is the proportion of susceptibles left at that time, $x_0 e^{-\alpha y_0/\beta}$.

Section 2.6

4. First divide both sides by $(2xy + 2)$. We now have $M(x, y) = y$ and $N(x, y) = x$. Since $M_y = N_x = 0$, the resulting equation is exact. Integrating M with respect to x , while holding y constant, results in $\psi(x, y) = xy + h(y)$. Differentiating with respect to y , $\psi_y = x + h'(y)$. Setting $\psi_y = N$, we find that $h'(y) = 0$, and hence $h(y) = 0$ is acceptable. Therefore the solution is defined implicitly as $xy = c$. Note that if $xy + 1 = 0$, the equation is trivially satisfied.

10. $M(x, y) = y/x + 6x$ and $N(x, y) = \ln x - 2$. Since $M_y = N_x = 1/x$, the given equation is exact. Integrating N with respect to y , while holding x constant, results in $\psi(x, y) = y \ln x - 2y + h(x)$. Differentiating with respect to x , $\psi_x = y/x + h'(x)$. Setting $\psi_x = M$, we find that $h'(x) = 6x$, and hence $h(x) = 3x^2$. Therefore the solution is defined implicitly as $3x^2 + y \ln x - 2y = c$.

16. $M(x, y) = y e^{2xy} + x$ and $N(x, y) = bx e^{2xy}$. Note that $M_y = e^{2xy} + 2xy e^{2xy}$, and $N_x = b e^{2xy} + 2bxy e^{2xy}$. The given equation is exact, as long as $b = 1$. Integrating N with respect to y , while holding x constant, results in $\psi(x, y) = e^{2xy}/2 + h(x)$. Now differentiating with respect to x , $\psi_x = y e^{2xy} + h'(x)$. Setting $\psi_x = M$, we find that $h'(x) = x$, and hence $h(x) = x^2/2$. We conclude that $\psi(x, y) = e^{2xy}/2 + x^2/2$. Hence the solution is given implicitly as $e^{2xy} + x^2 = c$.

21. $M_y = 1$ and $N_x = 2$. Multiply both sides by the integrating factor $\mu(x, y) = y$ to obtain $y^2 dx + (2xy - y^2 e^y) dy = 0$. Let $\bar{M} = yM$ and $\bar{N} = yN$. It is easy to see that $\bar{M}_y = \bar{N}_x$, and hence the latter ODE is exact. Integrating \bar{M} with respect to x yields $\psi(x, y) = xy^2 + h(y)$. Equating ψ_y with \bar{N} results in $h'(y) = -y^2 e^y$, and hence $h(y) = -e^y(y^2 - 2y + 2)$. Thus $\psi(x, y) = xy^2 - e^y(y^2 - 2y + 2)$, and the solution is defined implicitly by $xy^2 - e^y(y^2 - 2y + 2) = c$.