# MAST 330/2 LEC A ASSIGNMENT #3

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**2.3.2.** A tank initially contains 120 L of pure water. A mixture containing a concentration of $\gamma$ g/L of salt enters the tank at a rate of 2 L/min, and the well-stirred mixture leaves the tank at the same rate. Find an expression in terms of $\gamma$ for the amount of salt in the tank at any time $t$. Also find the limiting amount of salt in the tank as $t \to \infty$.

*Solution.* The model is

$$\frac{dQ}{dt} = r_{in}\gamma - r_{out}\frac{Q}{V}$$

where $Q$ is the amount of salt in a tank of volume $V$ at time $t$, $r_{in}$ and $r_{out}$ are flow rates in and out, respectively, and $\gamma$ is the concentration of salt flowing in. As given, $r_{in} = r_{out} = 2\,\text{L/min}$, and $V = 120\,\text{L}$. Thus

$$\frac{dQ}{dt} = 2\gamma - \frac{Q}{60}$$

Rewrite this equation in the standard form

$$\frac{dQ}{dt} + \frac{1}{60}Q = 2\gamma$$

and recognize it is like $\frac{d}{dt}y(t) + ay(t) = g(t)$ , which has integrating factor $e^{at}$. Thus,

$$\mu(t) = e^{t/60}$$

Multiplying both sides of the model equation by $\mu(t)$,

$$e^{t/60}\frac{dQ}{dt} + e^{t/60}\frac{1}{60}Q = \frac{d}{dt}\left(e^{t/60}Q\right) = 2e^{t/60}\gamma$$

integrating,

$$\begin{aligned} e^{t/60}Q &= \int 2e^{t/60}\gamma\,dt + C \\ &= 120\gamma e^{t/60} + C \end{aligned}$$

then isolating $Q$ gives

$$\begin{aligned} Q &= \frac{120\gamma e^{t/60} + C}{e^{t/60}} \\ &= 120\gamma + Ce^{-t/60} \end{aligned}$$

Initially, the tank contains 120 L of pure water, *i.e.* $Q(0) = 0$. To satisfy this initial condition we must set $C = -120\gamma$

$$Q(0) = 0 = 120\gamma + Ce^0 \implies C = -120\gamma$$

Thus

$$Q(t) = 120\gamma - 120\gamma e^{-t/60} = 120\gamma\left(1 - e^{-t/60}\right)$$

The limiting amount of salt in the tank as $t \to \infty$ is

$$\lim_{t\to\infty} Q(t) = \lim_{t\to\infty} 120\gamma\left(1 - e^{-t/60}\right) = 120\gamma\left(1 - \lim_{t\to\infty}\frac{1}{e^{t/60}}\right) = 120\gamma$$

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**2.3.20.** A ball with mass 0.15 kg is thrown upward with initial velocity 20 m/s from the roof of a building 30 m high. Neglect air resistance. (a) Find the maximum height above the ground that the ball reaches. (b) Assuming that the ball misses the building on the way down, find the time that it hits the ground. (c) Plot the graphs of velocity and position versus time.

*Solution.* Assume $v$ is positive in the up direction, *i.e.* when the ball is climbing.

$$F = ma \implies -mg = m\frac{dv}{dt} \implies \frac{dv}{dt} = -g$$

$$\frac{dv}{dt} = -g \implies v = -gt + C \quad v(0) = 20 \implies C = 0 \therefore v(t) = 20 - gt$$

$$\frac{dx}{dt} = 20 - gt \implies x = 20t - \frac{1}{2}gt^2 + C \quad x(0) = 30 \implies C = 30 \therefore x(t) = 20t - \frac{1}{2}gt^2 + 30$$

Now, (a) at max height $v = 0$, so

$$v(t) = 0 = 20 - gt \implies t = \frac{20}{9.8} = 2.04\,\text{s}$$

Thus

$$x_{max} = x(2.04) = 20(2.04) - \frac{1}{2}(9.8)(2.04)^2 + 30 = 50.4\,\text{m}$$

(b) When the ball hits the ground $x = 0$, so

$$x(t) = 0 = 20t - \frac{1}{2}gt^2 + 30$$

which is just a quadratic equation:

$$t = \frac{-20 \pm \sqrt{(20)^2 - 4\left(\frac{g}{2}\right)(30)}}{2\left(-\frac{g}{2}\right)} = -1.17(\text{impossible}) \text{ or } 5.25\,\text{s}$$

Thus at $t = 5.25$ seconds, the ball hits the ground.