ADDITION OF TWO POWER SERIES

$$\sum_{n=2}^{\infty} n(n-1)\, C_n X^{n-2}$$

$$\sum_{n=0}^{\infty} C_n X^{n+1}.$$

$$\sum_{n=2}^{\infty} n(n-1)\, C_n X^{n-2} + \sum_{n=0}^{\infty} C_n X^{n+1}.$$

$$= 2(2-1)\, C_2 X^{2-2} + \sum_{n=3}^{\infty} n(n-1)\, C_n X^{n-2}$$

$$+ \sum_{n=0}^{\infty} C_n X^{n+1}.$$

$$= 2 C_2 X^{0} + \sum_{n=3}^{\infty} n(n-1)\, C_n X^{n-2}$$

$$+ \sum_{n=0}^{\infty} C_n X^{n+1}.$$

$$= 2c_2 + \underbrace{\overset{n=3}{3(3-1)}}_{6} c_3 x^1 + \overset{n=0}{c_0 x^1}$$

$$+ \overset{n=4}{4(4-1) c_4 x^2} + \overset{n=1}{c_1 x^2}$$

$$+ \underbrace{\overset{n=5}{5(5-1)}}_{(k+2)(k+1)} \underset{c_{k+2}}{c_5} x^{\overset{k}{3}} + \overset{n=2}{\underset{c_{k-1}}{c_2}} x^{3 \leftarrow k}.$$

$$= 2c_2 + \sum_{k=1}^{\infty} (k+2)(k+1) c_{k+2} x^k.$$

$$+ \sum_{k=1}^{\infty} c_{k-1} x^k.$$

$$\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1} =$$

$$\sum_{n=3}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=0}^{\infty} C_n x^{n+1}$$

$$\sum_{n=3}^{\infty} n(n-1) C_n x^{n-2}$$

LET [redacted] $k = n-2 \rightarrow n = k+2$

$$\boxed{\sum_{k=1}^{\infty} (k+2)(k+1) C_{k+2} x^{k}}$$

$$\sum_{n=0}^{\infty} C_n x^{n+1} = C_0 x + C_1 x^2 + C_2 x^3$$

LET [redacted] $k = n+1 \rightarrow n = k-1$

$$\boxed{\sum_{k=1}^{\infty} C_{k-1} x^{k}} = C_0 x + C_1 x^2 + C_2 x^3$$

$$\sum_{n=3}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+1}$$

$$= \sum_{k=1}^{\infty} (k+2)(k+1)\, c_{k+2}\, x^{k}$$

$$+ \sum_{k=1}^{\infty} c_{k-1}\, x^{k}.$$

$$= \sum_{k=1}^{\infty} \left\{ (k+2)(k+1)\, c_{k+2} + c_{k-1} \right\} x^{k}.$$

$$= 2C_2 + \sum_{k=1}^{\infty} \left\{ (k+2)(k+1)C_{k+2} + C_{k-1} \right\} x^k$$

POWER SERIES SOLUTION.

SOLVE:

$$y'' + xy = 0 .$$ (AIRY'S EQUATION)

ASSUME A SOLUTION:

$$y = \sum_{n=0}^{\infty} C_n x^n .$$

$$y' = \frac{d}{dx} \sum_{n=0}^{\infty} C_n x^n .$$

$$= \sum_{n=0}^{\infty} C_n n x^{n-1} .$$

$$= \sum_{n=1}^{\infty} C_n n x^{n-1} .$$

$$y'' = \frac{d}{dx} y' = \frac{d}{dx} \sum_{n=1}^{\infty} c_n n x^{n-1}.$$

$$= \sum_{n=1}^{\infty} c_n n(n-1) x^{n-2}.$$

$$= \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2}.$$

$$xy = x \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_n x^n \cdot x$$

$$= \sum_{n=0}^{\infty} c_n x^{n+1}.$$

THEN

$$y'' + xy = \sum_{n=2}^{\infty} c_n n(n-1) x^{n-2} + \sum_{n=0}^{\infty} c_n x^{n+}$$

$$= 0$$

$$2C_2 + \sum_{k=1}^{\infty} \left\{ (k+1)(k+2)\, C_{k+2} + C_{k-1} \right\} x^k = 0$$

SINCE THE ABOVE EXPRESSION IS ZERO ALL THE COEFFICIENT MUST BE ZERO

$$b_0 + b_1 x + b_2 x^2 + b_3 x^3 + b_4 x^4 + \ldots + b_n x^n + \ldots = 0$$

$b_0 = 0$

$b_1 = 0$

$b_2 = 0$

$b_3 = 0$

$\vdots \quad \vdots$

$b_n = 0$

etc.

THEN

$$2C_2 = 0 \longrightarrow C_2 = 0$$

$$(k+1)(k+2)\, C_{k+2} + C_{k-1} = 0 \, .$$

OR

$$C_{k+2} = -\frac{C_{k-1}}{(k+1)(k+2)} \qquad k = 1, 2, 3, 4, 5, \ldots$$

(A) IS THE RECURRENCE RELATION

$k = 1$ $\qquad C_3 = -\dfrac{C_0}{2 \cdot 3}.$

$k = 2$ $\qquad C_4 = -\dfrac{C_1}{3 \cdot 4}.$

$k = 3$ $\qquad C_5 = -\dfrac{\overset{=0}{C_2}}{4 \cdot 5} = 0$

$k = 4$ $\qquad C_6 = -\dfrac{C_3}{5 \cdot 6} = -\left\{-\dfrac{1}{5 \cdot 6 \cdot 2 \cdot 3} C_0\right\}$

$$= \frac{1}{2 \cdot 3 \cdot 5 \cdot 6} C_0$$

$k = 5$ $\qquad C_7 = -\dfrac{C_4}{6 \cdot 7} = \dfrac{1}{3 \cdot 4 \cdot 6 \cdot 7} C_1$

$k = 6$ $\qquad C_8 = -\dfrac{C_5}{7 \cdot 8} = 0$

$k=7$ $$c_9 = -\frac{c_6}{8\cdot 9} = -\frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}\, c_0$$

$k=8$ $$c_{10} = -\frac{c_7}{9\cdot 10} = -\frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}\, c_1$$

$k=9$ $$c_{11} = -\frac{c_8 = 0}{10\cdot 11} = 0$$

$$y = c_0 + c_1 x + \overset{0}{\cancel{c_2}} x^2 + c_3 x^3 + c_4 x^4 + \overset{0}{\cancel{c_5}} x^5 + c_6 x^6$$
$$+ c_7 x^7 + \underset{0}{\cancel{c_8}} x^8 + c_9 x^9 + c_{10} x^{10} + \underset{0}{\cancel{c_{11}}} x^{11} -$$

$$y = c_0 + c_1 x - \frac{c_0}{2\cdot 3} x^3 - \frac{c_1}{3\cdot 4} x^4$$

$$+ \frac{1}{2\cdot 3\cdot 5\cdot 6} c_0 x^6 + \frac{1}{3\cdot 4\cdot 6\cdot 7} c_1 x^7 .$$

$$\phi - \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} c_0 x^9 - \frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} c_1 x^{10}$$

LET

$$y = y_1 c_0 + y_2 c_1 ,$$

WHERE:

$$Y_1 = 1 - \frac{1}{2\cdot 3}x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6}x^6 - \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9}x^9 + \dots$$

$$= 1 + \sum_{k=1}^{\alpha} \frac{(-1)^k}{2\cdot 3 \cdots (3k-1)(3k)} x^{3k}.$$

$$Y_2 = x - \frac{1}{3\cdot 4}x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7}x^7$$

$$- \frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10}x^{10} + \dots = \cancel{x + \sum_{k=1}^{\alpha} \frac{(-1)^k}{3\cdot 4 \cdots (3k}}$$

$$= x + \sum_{k=1}^{\alpha} \frac{(-1)^k}{3\cdot 4 \cdots (3k)(3k+1)} x^{3k+1}.$$

THEREFORE THE SOLUTION IS

$$Y(x) = C_0 \left\{ 1 + \sum_{k=1}^{\alpha} \frac{(-1)^k}{2\cdot 3 \cdots (3k-1)(3k)} x^{3k} \right\}$$

$$+ C_1 \left\{ x + \sum_{k=1}^{\alpha} \frac{(-1)^k}{3\cdot 4 \cdots (3k)(k+1)} x^{3k+1} \right\}$$

$$Y = \left\{ 1 - \frac{1}{2\cdot 3} x^3 + \frac{1}{2\cdot 3\cdot 5\cdot 6} x^6 - \frac{1}{2\cdot 3\cdot 5\cdot 6\cdot 8\cdot 9} x^9 + \cdots \right\} C_0$$

$$+ C_1 \left\{ x - \frac{c_1}{3\cdot 4} x^4 + \frac{1}{3\cdot 4\cdot 6\cdot 7} x^7 - \frac{1}{3\cdot 4\cdot 6\cdot 7\cdot 9\cdot 10} x^{10} + \cdots \right\}$$

GIVEN

$$y'' + x y' + y = 0$$

FIND THE GENERAL SOLUTION AS A MACLAURIN SERIES

ASSUME THE SOLUTION TO BE IN THE FORM:

$$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots + a_n x^n + \ldots$$

$$= \sum_{n=0}^{\infty} a_n x^n .$$

$$y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} n(n-1) x^{n-2} .$$

$$\sum_{n=2}^{\infty} a_n n (n-1) x^{n-2} + x \sum_{n=1}^{\infty} n a_n x^{n-1}$$

$$+ \sum_{n=0}^{\infty} a_n x^n = 0 .$$

1st.

$$\sum_{n=2}^{\alpha} n(n-1)\,a_n\,x^{n-2}$$

$$K = n \cancel{+} 2$$

LET $n = k+2 \longrightarrow n = 2 \qquad \cancel{2} = k + \cancel{2}$

$k = 0$

$$\sum_{k=0}^{\alpha} (k+2)(k+2-1)\,a_{k+2}\,x^{k+2-2}$$

$$= \sum_{k=0}^{\alpha} (k+2)(k+1)\,a_{k+2}\,x^{k}.$$

2nd.

$$x \sum_{n=1}^{\alpha} n\,a_n\,x^{n-1}$$

$$= \sum_{n=1}^{\alpha} n\,a_n\,x^{n}$$

LET $n = k.$

$$\sum_{k=1}^{\alpha} k\,a_k\,x^{k} = \sum_{k=0}^{\alpha} k\,a_k\,x^{k}.$$

3rd LET $n = k$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{k=0}^{\infty} a_k x^k .$$

THEREFORE,

$$\sum_{k=0}^{\infty} (k+2)(k+1) a_{k+2} x^k$$

$$+ \sum_{k=0}^{\infty} k a_k x^k + \sum_{k=0}^{\infty} a_k x^k = 0$$

$$\sum_{k=0}^{\infty} \left\{ (k+2)(k+1) a_{k+2} + (k+1) a_k \right\} x^k = 0$$

OR

$$(k+2)(k+1)\,a_{k+2} + (k+1)\,a_k = 0$$

$$a_{k+2} = -\frac{\cancel{k+1}}{(k+2)\cancel{(k+1)}}\,a_k$$

$$= -\frac{1}{k+2}\,a_k \qquad k = 0, 1, 2, 3, 4, \ldots$$

RECURRENCE RELATION

$k=0$ $\quad a_2 = -\dfrac{1}{2}\,a_0$

$k=1$ $\quad a_3 = -\dfrac{1}{3}\,a_1.$

$k=2$ $\quad a_4 = -\dfrac{1}{4}\,a_2 = -\dfrac{1}{4}\left(-\dfrac{1}{2}\,a_0\right) = \dfrac{1}{2\cdot 4}\,a_0$

$k=3$ $\quad a_5 = -\dfrac{1}{5}\,a_3 = -\dfrac{1}{5}\left(-\dfrac{1}{3}\,a_1\right)$

$$= \frac{1}{3\cdot 5}\,a_1.$$

$k=4 \qquad a_6 = -\dfrac{1}{6}\, a_4 = -\dfrac{1}{6}\left(\dfrac{1}{2.4}\right) a_0$

$$= -\frac{1}{2.4.6}\, a_0 .$$

$k=5 \qquad a_7 = -\dfrac{1}{7}\, a_5 = -\dfrac{1}{7}\left(\dfrac{1}{3.5}\, a_1\right)$

$$= -\frac{1}{3.5.7}\, a_1$$

WE CAN DIVIDE THE COEFFICIENTS INTO TWO GROUPS.

1st. $k = 2, 4, 6, 8, \ldots$

$a_2 = -\dfrac{a_0}{2}, \quad a_4 = -\dfrac{a_2}{4} = (-1)^2 \dfrac{a_0}{2.3}$

$a_6 = -\dfrac{a_4}{6} = (-1)^3 \dfrac{a_0}{2.4.6}, \ \ldots\ldots$

$\cancel{k=1,3,5,\ldots}$

2nd. $k = 1, 3, 5, 7, \ldots$

$a_3 = -\dfrac{a_1}{3}, \quad a_5 = -\dfrac{a_3}{5} = (-1)^2 \dfrac{a_1}{3.5},$

$a_7 = (-1)^3 \dfrac{a_1}{3.5.7}, \ \ldots\ldots$

$$a_{2k} = (-1)^k \frac{a_0}{2\cdot 4\cdot 6 \cdots 2k} = (-1)^k \frac{a_0}{2^k \cdot k!}$$

$$a_{2k+1} = (-1)^k \frac{a_1}{1\cdot 3\cdot 5 \cdots (2k+1)} = (-1)^k \frac{2^k k! \, a_1}{(2k+1)!}$$

$$k = 1, 2, 3, \ldots$$

THEREFORE THE SOLUTION IS:

$$y = a_0 \left\{ 1 - \frac{x^2}{2} + \cdots + \frac{(-1)^k x^{2k}}{2^k \cdot k!} + \cdots \right\}$$

$$+ \, a_1 \left\{ x - \frac{x^3}{1\cdot 3} + \cdots + \frac{(-1)^k 2^k k! \, x^{2k+1}}{(2k+1)!} + \cdots \right\}$$

~~RATIO TEST 1st SERIES~~

$$\cancel{\frac{(-1)^{k+1} x^{2(k+1)}}{2^{k+1} (k+1)!}} \Bigg/ \frac{2^k \cdot \cancel{k!}}{(-1)^k \cdot \cancel{k!}} = \cancel{\frac{(-1)^k (-1) x^{2k} \cdot x^2}{2^k 2 \, k! (k+1)} \; \frac{}{(-1)^k \, k!}}$$

RATIO TEST

1st SERIES.

$$\frac{(-1)^{k+1}\, x^{2(k+1)}}{2^{k+1}\,(k+1)!}\;\frac{2^k\, k!}{(-1)^k\, x^{2k}} =$$

$$\frac{\cancel{(-1)^k}(-1)\,\cancel{x^{2k}}\,x^2}{\cancel{2^k}\,2\,\cancel{k!}\,(k+1)}\;\frac{\cancel{2^k}\,\cancel{k!}}{\cancel{(-1)^k}\,\cancel{x^{2k}}} = -\frac{x^2}{2(k+1)}$$

$$\lim_{k\to\infty} -\frac{x^2}{2(k+1)} \longrightarrow 0 < 1 \longrightarrow$$ SERIES CONVERGES.

2nd SERIES

$$\frac{(-1)^{k+1}\, 2^{k+1}\,(k+1)!\; x^{2(k+1)+1}}{(2(k+1)+1)!}\;\frac{(2k+1)!}{(-1)^k\, 2^k\, k!\; x^{2k+1}}$$

$$\frac{\cancel{(-1)^k}(-1)\,\cancel{2^k}\,2^1\,\cancel{k!}\,(k+1)\,\cancel{x^{2k+1}}\,x^2}{(2(k+1)+1)!}\cdot\frac{(2k+1)!}{\cancel{(-1)^k}\,\cancel{2^k}\,\cancel{k!}\,\cancel{x^{2k+1}}}$$

$$= -2\,\frac{(k+1)\,(2k+1)!\,x^2\,\cancel{(2k+1)!}}{((2k+1)+2)!}$$

$$= -2\,\frac{(k+1)\,\cancel{(2k+1)!}\;x^2}{\cancel{(2k+1)!}\,(2k+2)(2k+3)}$$

$$= -2\,\frac{(k+1)\quad x^2}{4k^2+6k+4k+6} = -2\,\frac{k+1}{4k^2+10k+6}$$

$$\lim_{k\to\infty}\left\{-2\,\frac{k+1\quad x^2}{4k^2+10k+6}\right\} \longrightarrow 0 \longrightarrow \text{CONVERG}$$

SOLVE LEGENDRE'S EQUATION

$$(1-x^2)y'' - 2xy' + p(p+1)y = 0 .$$

BY THE METHOD OF POWER SERIES

p is a constant.

Assume a solution of the form

$$y = \sum_{n=0}^{\infty} a_n x^n$$

$$y' = \sum_{n=1}^{\infty} \alpha_n n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} .$$

LIKE BEFORE

$$\sum_{n=2}^{\infty} a_n n(n-1) x^{n-2}$$

LET $n = k + 2 \longrightarrow$ LOWER LIMIT $k + 2 = 2$

$k = 0$

EXPONENT OF x

$$x^{n-2} = x^{k+2-2} = x^{k}.$$

THEN

$$y'' = \sum_{n=2}^{\infty} a_n n(n-1) x^{n-2} = \sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^{k}$$

$$-2xy' = \sum_{n=0}^{\infty} -2 n a_n x^{n} = \sum_{k=0}^{\infty} -2k a_k x^{k}$$

$$-x^{2} y'' = -\sum_{k=0}^{\infty} a_{k+2} (k+2)(k+1) x^{k+2}. = A$$

$$Ⓐ = -\left\{ a_2 (2)(1)\, x^2 + a_3 (3)(2)\, x^3 + \ldots\ldots \right\}$$

IF WE LET $\ell = k + 2$

$$Ⓐ = -\sum_{\ell=0}^{\infty} a_\ell (\ell)(\ell - 1)\, x^\ell$$

$$= -\left\{ \underbrace{\cancel{a_0 (0)(-1)\, x^0}}_{0} + \underbrace{\cancel{a_1 (1)(0)\, x^1}}_{0} \right.$$

$$\left. + a_2 (2)(1)\, x^2 + a_3 (3)(2)\, x^3 + \ldots \right\}$$

$$\sum_{k=0}^{\alpha} a_{k+2} (k+2)(k+1) x^{k}.$$

$$- \sum_{\ell=0}^{\alpha} a_{\ell} (\ell)(\ell-1) x^{\ell}$$

$$- \sum_{k=0}^{\alpha} 2 k a_k x^{k}$$

$$+ \sum_{k=0}^{\alpha} p(p+1) a_k x^{k} = 0$$

CHANGE THE D.V. INTO m

$$\sum_{m=0}^{\alpha} a_{m+2} (m+2)(m+1) x^{m}$$

$$- \sum_{m=0}^{\alpha} a_m (m)(m-1) x^{m}$$

$$- \sum_{m=0}^{\alpha} 2 m a_m x^{m}$$

$$+ \sum^{\alpha} p(p+1) a_m x^{m} = 0.$$

OR.

$$\sum_{m=0}^{\alpha} \Big\{ a_{m+2}(m+2)(m+1) - a_m\, m(m-1) - 2m a_m + p(p+1) a_m \Big\} x^m = 0.$$

$$a_{m+2}(m+2)(m+1) - a_m \Big\{ m(m-1) + 2m - p(p+1) \Big\} = 0.$$

$$a_{m+2} = \frac{m^2 - \cancel{m} + \cancel{2}\overset{1}{}m - p^2 - p}{(m+2)(m+1)} a_m.$$

$$= \frac{(m^2 + m) - p^2 - p}{(m+2)(m+1)} = \frac{m(m+1) - p(p+1)}{(m+2)(m+1)}$$

$$a_{m+2} = \frac{(m-p)(m+p+1)}{(m+2)(m+1)} a_m.$$

THEREFORE.

$$y = a_0 \left\{ 1 - \frac{p(p+1)}{2!} x^2 + \frac{p(p-2)(p+1)(p+3)}{4!} x^4 - + \cdots \right.$$

$$+ a_1 \left\{ x - \frac{(p-1)(p+2)}{3!} x^3 + \frac{(p-1)(p-3)(p+2)(p+4)}{5!} x^5 - \right.$$

$$\left. + \cdots \cdots \right\}$$

SHOW THAT THE FOLLOWING POWER SERIES

$$y = \sum_{n=1}^{\infty} \frac{1}{n\,2^n} x^n$$

IS A SOLUTION OF THE ODE.

$$(2-x)y'' - y' = 0.$$

$$y' = \sum_{n=1}^{\infty} \frac{1}{\cancel{n}\,2^n} \cancel{n}\, x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} \frac{1}{2^n} (n-1)\, x^{n-2}.$$

$$2y'' = \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} (n-1)\, x^{n-2}.$$

$$\frac{1}{(n+1)\,2^{n+1}}\; x^{n+1}\; \frac{n\,2^{n}}{x^{n}}$$

$$= \frac{1}{(n+1)\,\cancel{2^{n}}\,2}\; \cancel{x^{n}}\,x\; \frac{n\,\cancel{2^{n}}}{\cancel{x^{n}}}$$

$$= \frac{1}{2}\;\frac{n}{n+1}\;x$$

$$\lim_{n \to \infty} \frac{1}{2}\;\frac{n}{n+1}\,x \;=\; \frac{1}{2}\,x$$

$$2y'' = \sum_{n=2}^{\infty} \frac{1}{2^{n-1}} (n-1) x^{n-2} =$$

LET $n = k+1$

$$\sum_{k=1}^{\infty} \frac{1}{2^{k}} k x^{k-1} .$$

$$- \sum_{k=2}^{\infty} \frac{1}{2^{k}} (k-1) x^{k-1}$$

$$- \sum_{k=1}^{\infty} \frac{1}{2^{k}} x^{k-1} = 0 .$$

$$\sum_{k=1}^{\infty} \left\{ \frac{1}{2^{k}} k - \frac{1}{2^{k}} \right\} x^{k-1}$$

$$- \sum_{k=2}^{\infty} \frac{1}{2^{k}} (k-1) x^{k-1} = 0$$

$$\left(\cancel{\frac{1}{2^{(1)}}} (1) - \cancel{\frac{1}{2^{1}}} \right) x^{0} + \sum_{k=2}^{\infty} \left\{ \frac{1}{2^{k}} k - \frac{1}{2^{k}} - \frac{1}{2^{k}} (k-1) \right\} x^{k-1}$$

$$\cancel{\frac{1}{2^{k}}} k - \cancel{\frac{1}{2^{k}}} - \cancel{\frac{1}{2^{k}} k} + \cancel{\frac{1}{2^{k}}} = 0 .$$