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Tutorial 6 – SUGGESTED SOLUTIONS

1) $u = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ $v = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ $w = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$

a) $(u+2v)\cdot w = \left(\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right) \cdot \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = 9-4+1 = \underline{\underline{6}}$

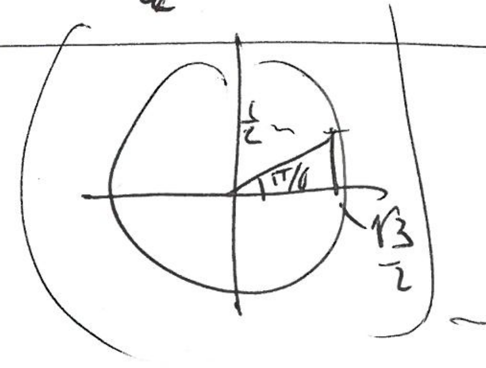
b) $\|w + 3u \cancel{+} - v\| = \left\| \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\| = \left\| \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix} \right\| = \sqrt{25+16+16} = \underline{\underline{\sqrt{57}}}$

c) $(u+v)\cdot w = \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} = 6-3+1 = 4 \neq 0$, so no, the vectors $u+v$ and w are not orthogonal to each other.

d) $\frac{1}{\|w\|} w = \frac{1}{\sqrt{3^2+(-1)^2+1^2}} w = \frac{1}{\sqrt{11}} w = \underline{\frac{1}{\sqrt{11}} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}}$

e) We have $\cos(\alpha) \overset{\text{angle between } u \text{ and } v}{=} \frac{u \cdot v}{\|u\|\|v\|} = \frac{3}{\sqrt{6}\sqrt{2}} = \frac{3}{\sqrt{3}\sqrt{2}\sqrt{2}} = \frac{3}{\sqrt{3}} \cdot \frac{1}{2} = \frac{\sqrt{3}}{2}$

So $\cos(\alpha) = \frac{\sqrt{3}}{2}$. This means that $\underline{\underline{\alpha = \frac{\pi}{6}}}$



(2) Let $v_1 = (1, 2, -1)$, $v_2 = (1, 0, 1)$, $v_3 = (3, 2, 3)$.

Then $v_1 \cdot v_2 = 0$ and $v_1 \cdot v_2 = 0$, but $v_2 \cdot v_3 = 6 \neq 0$,

So no $\{v_1, v_2, v_3\}$ is not an orthogonal set.

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(3) $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is orthogonal to both $\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ if

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = 0 \quad \text{and} \quad \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = 0$$

$$\Leftrightarrow \begin{cases} 3x + z = 0 \\ 2x + y + z = 0 \end{cases}$$

Take for example $x = t$. Then $z = -3t$ and

$y = -2x - z = -2t - (-3t) = t$ so $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$

take for example $t = 1$ to get the vector

$\begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix}$ which satisfies the condition (there are others of course).

Answer

(4) $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ lies in the plane if and only if 3/6

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 3 \\ -5 \\ 2 \end{bmatrix} = 0 \iff \underline{\underline{3x - 5y + 2z = 0}}$$ ← Equation for the plane.

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(5) $$\begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \cdot \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} = \cos\alpha(-\sin\alpha) + \sin\alpha\cos\alpha = 0$$

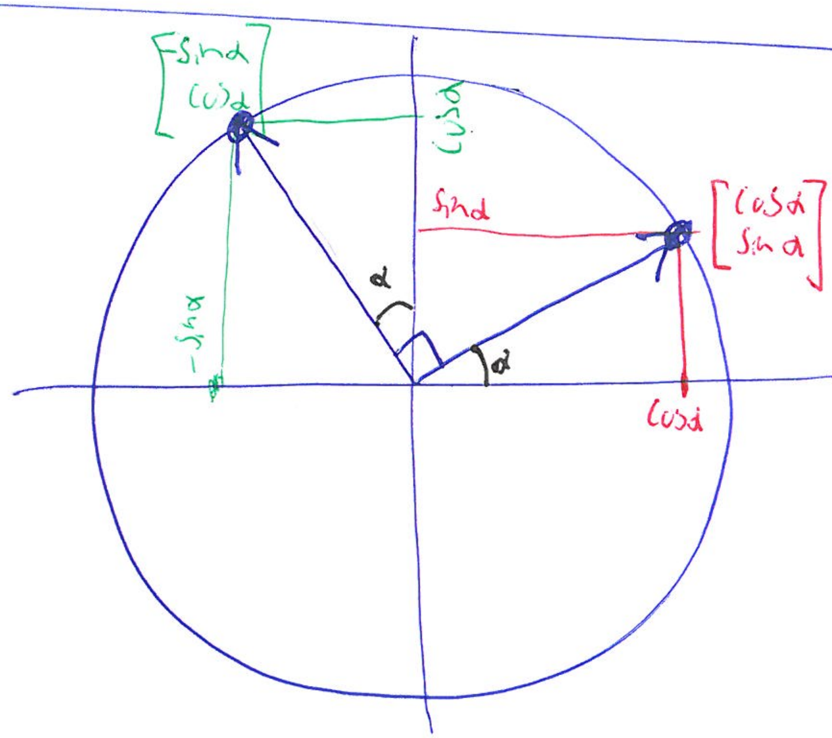
So the vectors are orthogonal.

Now $\left\| \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} \right\| = \sqrt{(\cos(\alpha))^2 + (\sin(\alpha))^2} = \sqrt{1} = 1$ by trigonometry

~~So~~ and similarly $\left\| \begin{bmatrix} -\sin\alpha \\ \cos\alpha \end{bmatrix} \right\| = \sqrt{(-\sin\alpha)^2 + (\cos\alpha)^2} = \sqrt{1} = 1$.

Thus the vectors are orthogonal to each other and have length one: it is an orthonormal set.

Picture:

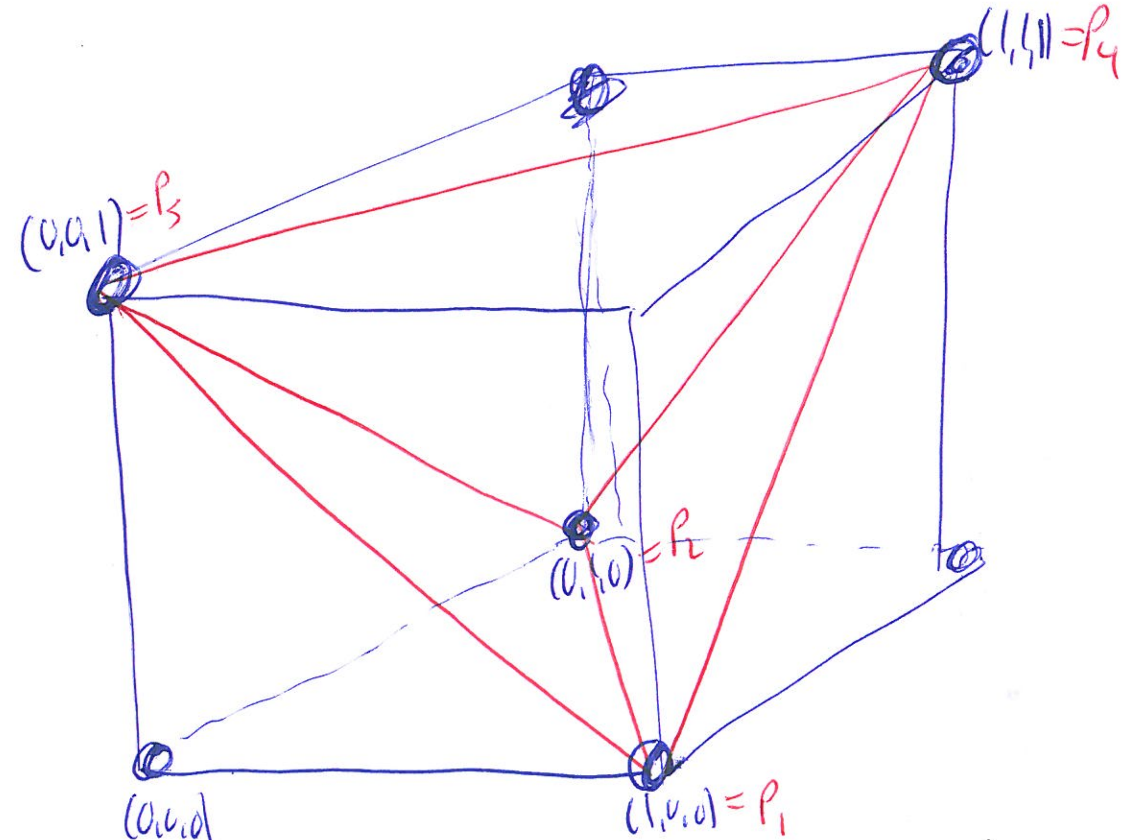


6 ☆ It's enough to show that all the sides of the tetrahedron has the same length.

Let $P_1 = (1,0,0)$, $P_2 = (0,1,0)$, $P_3 = (0,0,1)$, $P_4 = (1,1,1)$

Then
$$
\left.\begin{aligned}
P_4 - P_1 &= (0,1,1) \Rightarrow \|P_4 - P_1\| = \sqrt{2} \\
P_4 - P_2 &= (1,0,1) \Rightarrow \|P_4 - P_2\| = \sqrt{2} \\
P_4 - P_3 &= (1,1,0) \Rightarrow \|P_4 - P_3\| = \sqrt{2} \\
P_3 - P_2 &= (0,-1,1) \Rightarrow \|P_3 - P_2\| = \sqrt{2} \\
P_2 - P_1 &= (-1,1,0) \Rightarrow \|P_2 - P_1\| = \sqrt{2} \\
P_1 - P_3 &= (1,0,-1) \Rightarrow \|P_1 - P_3\| = \sqrt{2}
\end{aligned}\right\} \text{OK!}
$$

Picture:



(It's also easy to check that all angles are $\frac{\pi}{3}$)

(7) $\text{Proj}_{(3,1,2)}(1,2,-5) = \dfrac{(1,2,-5)\cdot(3,1,2)}{(3,1,2)\cdot(3,1,2)}(3,1,2) = \underline{\underline{\dfrac{-5}{14}(3,1,2)}}$

$\text{Proj}_{(1,2,1,-1)}(2,3,8,6) = \underline{\underline{\dfrac{10}{7}(1,2,1,-1)}}$

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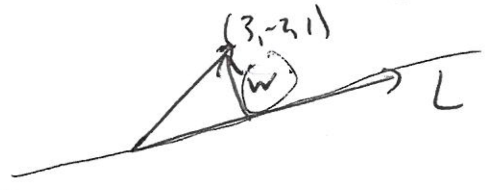
(8) We project $(3,-2,1)$ onto $(1,2,-2)$:

$\text{proj}_{(1,2,-2)}(3,-2,1) = \dfrac{-3}{9}(1,2,-2) = \underline{\underline{-\dfrac{1}{3}(1,2,-2)}}$ ← Point on L closest to $(3,-2,1)$

Now take $w = (3,-2,1) - \text{proj}_{(1,2,-2)}(3,-2,1) = (3,-2,1) - \left(-\frac{1}{3}\right)(1,2,-2)$
$= \frac{1}{3}\big((9,-6,3) + (1,2,-2)\big) = \frac{1}{3}(10,-4,1)$. Then the distance to

L is $\|w\| = \frac{1}{3}\sqrt{100+16+1} = \frac{1}{3}\sqrt{117} = \frac{1}{3}\sqrt{3\cdot 3\cdot 13} = \frac{\sqrt{3\cdot 3}}{3}\sqrt{13} = \underline{\underline{\sqrt{13}}}$



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(9) let $u = (1,4,2)$.

$\underline{\underline{\text{Proj}_w u = \frac{15}{14}w}}$. Next we have $u - \text{proj}_w u = (1,4,2) - \frac{15}{14}(1,2,3)$
$= \frac{1}{14}\big((14,56,28) - (15,30,45)\big) = \frac{1}{14}(-1,26,-17)$

Answer

$\underline{\text{thus}}$ $\underset{u}{(1,4,2)} = \boxed{\underbrace{\frac{15}{14}(1,2,3)}_{\parallel w} + \underbrace{\frac{1}{14}(-1,26,-17)}_{\perp w}}$ is the desired decomposition.

Parallel to w   Perpendicular to w

10) Write $v=(1,0,1,1,1)$, $v_1=(1,1,-1,-1,0)$, $v_2=(2,-2,0,0,1)$, $v_3=(1,1,1,1,0)$ 6/6

a) Yes, because $v_1 \cdot v_2 = 0$, $v_2 \cdot v_3 = 0$, and $v_1 \cdot v_3 = 0$

b) $\text{proj}_W v = \text{proj}_{v_1} v + \text{proj}_{v_2} v + \text{proj}_{v_3} v = \frac{-1}{4} v_1 + \frac{3}{9} v_2 + \frac{3}{4} v_3$

$= \frac{-3}{12} v_1 + \frac{4}{12} v_2 + \frac{9}{12} v_3 = \frac{1}{12}(-3v_1 + 4v_2 + 9v_3)$

$= \frac{1}{12}\Big((-3,-3,3,3,0) + (8,-8,0,0,4) + (9,9,9,9,0)\Big) = \underline{\frac{1}{12}(14,-2,12,12,4)}$

c) We have $v - \text{proj}_W v = (1,0,1,1,1) - \frac{1}{12}(14,-2,12,12,4)$

$= \frac{1}{12}\Big((12,0,12,12,12) - (14,-2,12,12,4)\Big) = \frac{1}{12}(-2,2,0,0,8) = \frac{1}{6}(-1,1,0,0,4)$

So the distance between v and W is $\|v - \text{proj}_W v\| = \left\|\frac{1}{6}(-1,1,0,0,4)\right\|$

$= \frac{1}{6}\sqrt{1+1+16} = \frac{1}{6}\sqrt{18} = \frac{1}{6}\sqrt{9\cdot 2} = \frac{1}{6}\sqrt{9}\sqrt{2} = \frac{3}{6}\sqrt{2} = \frac{\sqrt{2}}{2}$

11) the vector $v = \begin{bmatrix}1\\2\end{bmatrix}$ is on the line. We have

$\text{proj}_v(1,0) = \frac{1}{5}\begin{bmatrix}1\\2\end{bmatrix}$ and $\text{proj}_v(0,1) = \frac{2}{5}\begin{bmatrix}1\\2\end{bmatrix}$

Thus $\begin{bmatrix}\frac{1}{5}\\ \frac{2}{5}\end{bmatrix}$ and $\begin{bmatrix}\frac{2}{5}\\ \frac{4}{5}\end{bmatrix}$ are columns 1 and 2 of the

Standard matrix: Let $T = \begin{bmatrix}\frac{1}{5} & \frac{2}{5}\\ \frac{2}{5} & \frac{4}{5}\end{bmatrix} = \frac{1}{5}\begin{bmatrix}1 & 2\\ 2 & 4\end{bmatrix}$

This is the projection matrix!