Tutorial 5 - Suggested Solutions

1

$$i(1+i)(3+i) = i(3+i+3i+i^2) = i(2+4i) = 2i+4i^2 = \underline{\underline{-4+2i}}$$

$$\overline{(2+5i)(i-3)} = \overline{(2i-6+5i^2-15i)} = \overline{-11-13i} = \underline{\underline{-11+13i}}$$

$$(1+i)^4 = \left((1+i)^2\right)^2 = (1+2i+i^2)^2 = (2i)^2 = 4i^2 = \underline{\underline{-4}}$$

$$\frac{2+5i}{i-4} = \frac{(2+5i)(-i-4)}{(i-4)(-i-4)} = \frac{-2i-8-5i^2-20i}{-i^2-4i+4i+16} = \frac{-3-22i}{17} = -\frac{3}{17} - \frac{22}{17}i$$

$$|2i+5| = \sqrt{2^2+5^2} = \sqrt{4+25} = \sqrt{29}$$

---

2 a) $z^2 = 3 \iff z^2 - 3 = 0 \iff z^2 - (\sqrt{3})^2 \iff (z-\sqrt{3})(z+\sqrt{3})$

So $z = \pm\sqrt{3}$ are the solutions.

---

b) $z^2 = -3$ ~~[illegible]~~ No real solutions! Let $z = x+iy$.

Then $(x+iy)^2 = -3 \iff x^2 + 2xiy + i^2y^2 = -3 \iff (x^2-y^2) + (2xy)i = -3$

So $\begin{cases} x^2 - y^2 = -3 \\ 2xy = 0 \implies x=0 \text{ or } y=0. \end{cases}$ $y=0$ gives $x^2 = -3 \implies$ no solutions.

So $\underline{x=0}$ this gives $-y^2 = -3 \iff y = \pm\sqrt{3}$, so

$\underline{z} = x+iy = \underline{\underline{\pm\sqrt{3}i}}$ Are the solutions.

(Note that $z^2 = -3 \implies z = \pm\sqrt{-3} = \underline{\pm i\sqrt{3}}$ also works, but does not generalize) ....

c) $z^2-2z+2iz+3-6i=0 \iff z^2+(-2+2i)z+3-6i=0$

$\iff \left(z+(-1+i)\right)^2-(-1+i)^2+3-6i=0$

$\iff (z-1+i)^2-(-2i)+3-6i=0$ Let $\underline{w=z-1+i}$

$\iff w^2=-3+4i$ let $w=x+iy$.

$\iff (x+iy)^2=-3+4i \iff \begin{cases} x^2-y^2=-3 \\ 2xy=4 \end{cases}$

Equate real and imaginary parts.

We also have

$|(x+iy)^2|=|x+iy|^2=x^2+y^2$

$=|-3+4i|=\sqrt{(-3)^2+(4)^2}=\sqrt{25}=5$

So $\begin{cases} x^2-y^2=-3 \\ x^2+y^2=5 \\ 2xy=4 \end{cases}$ add $\Rightarrow 2x^2=2 \Rightarrow x=\pm 1$

From $2xy=4$ we get $x=1 \Rightarrow y=2$ and $x=-1 \Rightarrow y=-2$

So $w_1=x_1+iy_1=1+2i$ and $w_2=-1-2i$

This gives $z_1=w_1+1-i=\underline{2+i}$ and $\underline{z_2=-3i}$

<u>Answer:</u> The solutions are $\begin{cases} z_1=2+i \\ z_2=-3i \end{cases}$

d) $2z^2 - 5iz + 15 = -iz^2 + 10z - 5i$

$\Leftrightarrow (2+i)z^2 + (-10-5i)z + 15 + 5i$ divide by $(2+i)$.

$\Leftrightarrow z^2 - \frac{(10+5i)}{2+i} z + \frac{5(3+i)}{2+i} = 0$

$\Leftrightarrow z^2 - 5\frac{(2+i)}{2+i} z + \frac{5(3+i)(2-i)}{(2+i)(2-i)} = 0$

$\Leftrightarrow z^2 - 5z + \frac{5(7-i)}{5} = 0 \Leftrightarrow z^2 - 5z + 7 - i = 0$

$\Leftrightarrow \left(z - \frac{5}{2}\right)^2 - \frac{25}{4} + 7 - i = 0$ Let $w = z - \frac{5}{2}$

$\Leftrightarrow w^2 = -\frac{3}{4} + i \Rightarrow$ Let $w = x + iy$ then

$|w|^2 = \left|-\frac{3}{4} + i\right| \Rightarrow x^2 + y^2 = \sqrt{\frac{9}{16} + \frac{16}{16}} = \sqrt{\frac{25}{16}} = \frac{5}{4}$

$(x+iy)^2 = -\frac{3}{4} + i \Leftrightarrow \begin{cases} x^2 - y^2 = -\frac{3}{4} \\ x^2 + y^2 = \frac{5}{4} \\ 2xy = 1 \quad (*) \end{cases}$ add $2x^2 = \frac{1}{2} \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\frac{1}{2}$

$(*)$ gives $x = \frac{1}{2} \Rightarrow y = 1$

$x = -\frac{1}{2} \Rightarrow y = -1$

$w_1 = \frac{1}{2} + i$

$w_2 = -\frac{1}{2} - i$

$\Rightarrow \begin{cases} z_1 = 3 + i \\ z_2 = 2 - i \end{cases}$

Diagonalize

3. $A = \begin{bmatrix} 2 & 5 \\ 5 & 2 \end{bmatrix}$ $\quad \det(\lambda I - A) = \begin{vmatrix} \lambda-2 & -5 \\ -5 & \lambda-2 \end{vmatrix} = (\lambda-2)^2 - 25 = (\lambda+3)(\lambda-7)$

Eigenvalues: -3 and 7.

$\underline{\lambda = -3}$: $\begin{bmatrix} -5 & -5 & | & 0 \\ -5 & -5 & | & 0 \end{bmatrix} \begin{matrix} (-1) \ \boxed{-\tfrac{1}{5}} \\ \end{matrix} \sim \begin{bmatrix} 1 & 1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $\quad t \in \mathbb{R},\ t \neq 0.$

$\underline{\lambda = 7}$ $\begin{bmatrix} 5 & -5 & | & 0 \\ -5 & 5 & | & 0 \end{bmatrix} \begin{matrix} (1) \ \boxed{\tfrac{1}{5}} \\ \end{matrix} \sim \begin{bmatrix} 1 & -1 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ $\quad t \in \mathbb{R},\ t \neq 0.$

Eigenvalues	Corr Eigenvectors
-3	$t(-1,1)$
7	$t(1,1)$

$\Big\}\ t \neq 0,\ t \in \mathbb{R}.$

Thus $\boxed{PDP^{-1} = A \text{ with } P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} -3 & 0 \\ 0 & 7 \end{bmatrix}}$

$B = \begin{bmatrix} 6 & 9 \\ -1 & 0 \end{bmatrix}$ $\quad \det(\lambda I - B) = \begin{vmatrix} \lambda-6 & -9 \\ 1 & \lambda \end{vmatrix} = \lambda^2 - 6\lambda + 9 = (\lambda-3)^2$

So the only eigenvalue is $\underline{\lambda = 3}$.

$\underline{\lambda = 3}$ $\begin{bmatrix} -3 & -9 & | & 0 \\ 1 & 3 & | & 0 \end{bmatrix} \begin{matrix} \\ (3) \end{matrix} \sim \begin{bmatrix} 1 & 3 & | & 0 \\ 0 & 0 & | & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -3 \\ 1 \end{bmatrix}$ $\quad t \in \mathbb{R},\ t \neq 0.$

Thus the eigenspace is 1-dimensional, and there does not exist two linearly independent eigenvectors to B.

$\boxed{\text{Thus } B \text{ is } \underline{\text{not}} \text{ diagonalizable}}$.

$$C = \begin{bmatrix} 2 & 1 & -1 & 0 \\ 0 & -3 & 0 & 4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \det(\lambda I - C) = \begin{vmatrix} \lambda-2 & -1 & 1 & 0 \\ 0 & \lambda+3 & 0 & -4 \\ 0 & 0 & \lambda-1 & 0 \\ 0 & 0 & 0 & \lambda-1 \end{vmatrix} = (\lambda-1)^2(\lambda+3)(\lambda-2)$$

Eigenvalues: $\lambda = 1, \lambda = -3, \lambda = 2$

$\underline{\lambda = 1}$

$$\left[\begin{array}{cccc|c} -1 & -1 & 1 & 0 & 0 \\ 0 & 4 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \boxed{\tfrac{1}{4}} \; (1) \sim \left[\begin{array}{cccc|c} -1 & 0 & 1 & -1 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

take $V_3 = s$, $V_4 = t$

$$\begin{cases} V_1 = s - t \\ V_2 = t \\ V_3 = s \\ V_4 = t \end{cases} \Longleftrightarrow \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$s, t \in \mathbb{R}$, $(s,t) \neq (0,0)$.

Forms eigenbasis for the eigenspace $\mathrm{Nul}(I - C)$

$\underline{\lambda = -3}$

$$\left[\begin{array}{cccc|c} -5 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \\ 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{array}\right] \; \left(-\tfrac{1}{4}\right) \; (-1) \; \boxed{-\tfrac{1}{4}} \; \boxed{-\tfrac{1}{4}} \sim \left[\begin{array}{cccc|c} -5 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} V_1 = t \\ V_2 = -5t \\ V_3 = 0 \\ V_4 = 0 \end{array}$$

$$\Rightarrow \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = t \begin{bmatrix} 1 \\ -5 \\ 0 \\ 0 \end{bmatrix} \quad \begin{array}{l} t \in \mathbb{R} \\ t \neq 0. \end{array}$$

$\underline{\lambda = 2}$

$$\left.\left[\begin{array}{cccc|c} 0 & -1 & 1 & 0 & 0 \\ 0 & 5 & 0 & -4 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]\right\} \Rightarrow \begin{array}{l} V_1 = t \\ V_2 = 0 \\ V_3 = 0 \\ V_4 = 0 \end{array} \Rightarrow \begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad t \in \mathbb{R},\ t \neq 0.$$

Next

<u>Conclusion</u>: $C = PDP^{-1}$

with $P = \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 1 & -5 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 2 \end{bmatrix}$

Diagonalize $M = \begin{bmatrix} 5 & 1 & -7 \\ 8 & 1 & -10 \\ 2 & 1 & -4 \end{bmatrix}$ $\quad \det(\lambda I - A) = \begin{vmatrix} \lambda-5 & -1 & 7 \\ -8 & \lambda-1 & 10 \\ -2 & -1 & \lambda+4 \end{vmatrix}$

$I \leftarrow (-2)$

$$= \begin{vmatrix} \lambda-3 & -1 & 7 \\ -2\lambda-6 & \lambda-1 & 10 \\ 0 & -1 & \lambda+4 \end{vmatrix} = -(-1)\begin{vmatrix} \lambda-3 & 7 \\ -2\lambda-6 & 10 \end{vmatrix} \; 2 \; + (\lambda+4)\begin{vmatrix} \lambda-3 & -1 \\ -2\lambda-6 & \lambda-1 \end{vmatrix} \; 2$$

expand

$$= \begin{vmatrix} \lambda-3 & 7 \\ -12 & 24 \end{vmatrix} + (\lambda+4)\begin{vmatrix} \lambda-3 & -1 \\ -12 & \lambda-3 \end{vmatrix} = 12\big(2(\lambda-3)+7\big) + (\lambda+4)\big((\lambda-3)^2 - 12\big)$$

$$= (\lambda-3)\big(24 + (\lambda+4)(\lambda-3)\big) + 12\underbrace{\big(7-(\lambda+4)\big)}_{= -12(\lambda-3)}$$

$$= (\lambda-3)\Big[(\lambda+4)(\lambda-3)+12\Big] = (\lambda-3)\big[\lambda^2+\lambda\big] = \lambda(\lambda+1)(\lambda-3)$$

So the eigenvalues are $0, -1$, and 3.

(You can use other methods to find the eigenvalues, for example, Sarrus Rule would give the same)

$\underline{\lambda = 0}$

$$\left[\begin{array}{ccc|c} -5 & -1 & 7 & 0 \\ -8 & -1 & 10 & 0 \\ -2 & -1 & 4 & 0 \end{array}\right] \begin{matrix} -1 \\ \\ \\ \end{matrix} \sim \left[\begin{array}{ccc|c} -5 & -1 & 7 & 0 \\ -3 & 0 & 3 & 0 \\ 3 & 0 & -3 & 0 \end{array}\right] \begin{matrix} \\ 1 \quad -\frac{1}{3} \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} -5 & -1 & 7 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{l} V_1 = t \\ V_2 = -5t + 7t = 2t \\ V_3 = t \end{array} \Leftrightarrow \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \begin{array}{l} t \in \mathbb{R} \\ t \neq 0. \end{array}$$

$\underline{\lambda = -1}$

$$\left[\begin{array}{ccc|c} -6 & -1 & 7 & 0 \\ -8 & -2 & 10 & 0 \\ -2 & -1 & 3 & 0 \end{array}\right] \begin{matrix} \\ \\ -4 \quad -3 \end{matrix} \sim \left[\begin{array}{ccc|c} 0 & 2 & -2 & 0 \\ 0 & 2 & -2 & 0 \\ -2 & -1 & 3 & 0 \end{array}\right] \begin{matrix} \\ \frac{1}{2} \quad -1 \\ \\ \end{matrix}$$

$$\sim \left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ -2 & 0 & 2 & 0 \end{array}\right] \quad \begin{array}{l} V_1 = t \\ V_2 = t \\ V_3 = t \end{array} \Rightarrow \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad \begin{array}{l} t \in \mathbb{R} \\ t \neq 0. \end{array}$$

$\underline{\lambda = 3}$

$$\left[\begin{array}{ccc|c} -2 & -1 & 7 & 0 \\ -8 & 2 & 10 & 0 \\ -2 & -1 & 7 & 0 \end{array}\right] \begin{matrix} 2 \quad -1 \\ \\ \\ \end{matrix} \sim \left[\begin{array}{ccc|c} -2 & -1 & 7 & 0 \\ -12 & 0 & 24 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} V_1 = 2t \\ V_2 = -4t + 7t = 3t \\ V_3 = t \end{array}$$

$$\boxed{\Leftrightarrow \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = t \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} \quad \begin{array}{l} t \in \mathbb{R} \\ t \neq 0. \end{array}}$$

<u>Conclusion</u> We have $M = PDP^{-1}$

with $P = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

4

$\det(\lambda I - B) = \begin{vmatrix} \lambda & 4 \\ -9 & \lambda \end{vmatrix} = \underline{\underline{\lambda^2 + 36}} = 0 \Leftrightarrow \lambda = \pm 6i$

~~and~~ $\lambda = \pm 6i$ are the eigenvalues

$\det(\lambda I - C) = \begin{vmatrix} \lambda-1 & 2 \\ -5 & \lambda-3 \end{vmatrix} = (\lambda-1)(\lambda-3) + 10$

$= \lambda^2 - 4\lambda + 13 = (\lambda-2)^2 + 9 \Rightarrow \lambda - 2 = \pm\sqrt{-9} = \pm 3i$

So $\begin{cases} \lambda_1 = 2+3i \\ \lambda_2 = 2-3i \end{cases}$ are the eigenvalues.

(Alternatively you could use our method from the lectures, which would take more time.

5

$\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} i \\ 1 \end{bmatrix}}_{v} = \begin{bmatrix} -1 \\ i \end{bmatrix} = \underbrace{i}_{\lambda}\underbrace{\begin{bmatrix} i \\ 1 \end{bmatrix}}_{v}$ ~~so~~ and

$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} i \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ i \end{bmatrix} = (-i)\begin{bmatrix} i \\ -1 \end{bmatrix}$

So: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = PDP^{-1}$

with $P = \begin{bmatrix} i & i \\ 1 & -1 \end{bmatrix}$

and $D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$

So:

Eigenvalues	Eigenvectors
i	$t\begin{bmatrix} i \\ 1 \end{bmatrix}$
$-i$	$t\begin{bmatrix} i \\ -1 \end{bmatrix}$

6 ☆ We have $Av = 3v$. Therefore

$$(A^2 + 5A - 2I)v = A \cdot A \cdot v + 5A \cdot v - 2Iv$$

$$= 3Av + 15v - 2v = 9v + 15v - 2v = \underline{22v}$$

So the corresponding eigenvalue is 22.

---

7 Let S be a solved Sudoku-matrix.

Then $$\begin{bmatrix} & & \\ & S & \\ & & \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} 45 \\ 45 \\ 45 \\ \vdots \\ 45 \end{bmatrix} = 45 \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$$ ~~Since the~~ Since each row of S contains all the integer $1, 2, \ldots, 9$ and $1 + 2 + \cdots + 9 = 45$.
