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Tutorial 4 – Suggested Solutions

(1)

$$\begin{vmatrix}2&5\\8&7\end{vmatrix} = 14-40 = \underline{\underline{-26}}$$

Expand!

$$\begin{vmatrix}1&1&0\\0&5&3\\3&0&-2\end{vmatrix} = \begin{vmatrix}1&0&0\\0&5&3\\3&-3&-2\end{vmatrix} = 1\cdot\begin{vmatrix}5&3\\-3&-2\end{vmatrix}(+0+0) = -10-(-9) = \underline{\underline{-1}}$$

Expand!

$$\begin{vmatrix}1&2&-1\\1&1&-3\\-2&1&4\end{vmatrix} = \begin{vmatrix}1&2&-1\\0&-1&-2\\0&5&2\end{vmatrix} = 1\begin{vmatrix}-1&-2\\5&2\end{vmatrix} = -2+10 = \underline{\underline{8}}$$

Expand! Expand!

$$\begin{vmatrix}1&0&1&0\\0&1&2&-3\\3&0&4&0\\1&1&5&-2\end{vmatrix} = \begin{vmatrix}1&0&0&0\\0&1&2&-3\\3&0&1&0\\1&1&4&-2\end{vmatrix} = \begin{vmatrix}1&2&-3\\0&1&0\\1&4&-2\end{vmatrix} = \begin{vmatrix}1&-3\\1&-2\end{vmatrix} = -2+3 = \underline{\underline{1}}$$

(2) a) A matrix A has an inverse only if $\det(A) \neq 0$.

Since $\begin{vmatrix} 4 & 3m \\ m & 2m \end{vmatrix} = m \begin{vmatrix} 4 & 3m \\ 1 & 2 \end{vmatrix} = m(8-3m)$,

the matrix $\begin{bmatrix} 4 & 3m \\ m & 2m \end{bmatrix}$ has an inverse for all m except

$m=0$ and $m = \frac{8}{3}$.

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b) The system can be written $\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 137 \\ 129 \\ 385 \end{bmatrix}$.

This system has a unique solution when $\det(A) \neq 0$. We compute

$$\det(A) = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 1 \\ 0 & 0 & -1 \\ 3 & -1 & 1 \end{vmatrix} = -(-1)\begin{vmatrix} 3 & 1 \\ 3 & -1 \end{vmatrix} = 3\begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = \underline{\underline{-6}} \neq 0$$

(2)

So yes, the system does have a unique solution.

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c) We form the matrix having the vectors as columns, (rows would also work). The vectors are linearly dependent when the determinant of this matrix is zero. We have

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{vmatrix} \overset{(-1)}{=} \begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0,$$

expand!

so yes, the three vectors are linearly dependent.

3

We solve

$$0 = \begin{vmatrix} x & 1 & x & 1 \\ 2 & x & -3 & 1 \\ 1 & x & 1 & x \\ x & 2x & 0 & 0 \end{vmatrix} = x \begin{vmatrix} x+1 & x+1 & x+1 & x+1 \\ 2 & x & -3 & 1 \\ 1 & x & 1 & x \\ 1 & 2 & 0 & 0 \end{vmatrix}$$

factor out x factor out $(x+1)$

$$= x(x+1) \begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & x & -3 & 1 \\ 1 & x & 1 & x \\ 1 & 2 & 0 & 0 \end{vmatrix} = x(x+1) \begin{vmatrix} 1 & -1 & 1 & 1 \\ 2 & x-4 & -3 & 1 \\ 1 & x-2 & 1 & x \\ 1 & 0 & 0 & 0 \end{vmatrix} = -x(x+1) \begin{vmatrix} -1 & 1 & 1 \\ x-4 & -3 & 1 \\ x-2 & 1 & x \end{vmatrix}$$

Expand!

$$= -x(x+1) \begin{vmatrix} 0 & 1 & 0 \\ x-7 & -3 & 4 \\ x-1 & 1 & x-1 \end{vmatrix} = x(x+1) \begin{vmatrix} x-7 & 4 \\ x-1 & x-1 \end{vmatrix} = x(x+1)(x-1) \begin{vmatrix} x-7 & 4 \\ 1 & 1 \end{vmatrix}$$

Expand! factor out $x-1$

$$= x(x+1)(x-1)\big(1(x-7) - 4 \cdot 1\big) = \underline{x(x+1)(x-1)(x-11)}$$

So the four solutions to the equations are $\underline{0, -1, 1, \text{ and } 11}$.

4

The system can be written $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}}_{b}$ with

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 1 & 1 & -3 \\ -2 & 1 & 4 \end{bmatrix}.$$ By problem 1, $\det(A) = 8$.

By Cramer's rule we have

$$x = \frac{1}{8}\begin{array}{c} b \\ \begin{vmatrix} \boxed{\begin{matrix} 0 \\ 1 \\ -1 \end{matrix}} & \begin{matrix} 2 \\ 1 \\ 1 \end{matrix} & \begin{matrix} -1 \\ -3 \\ 4 \end{matrix} \end{vmatrix} \end{array} \; (1) = \frac{1}{8}\begin{array}{c} \text{expand!} \\ \begin{vmatrix} 0 & 2 & -1 \\ 1 & 1 & -3 \\ 0 & 2 & 1 \end{vmatrix} \end{array} = -\frac{1}{8}\begin{vmatrix} 2 & -1 \\ 2 & 1 \end{vmatrix} = -\frac{1}{8}(2+2) = \underline{\underline{-\frac{1}{2}}}$$

and $$y = \frac{1}{8}\begin{array}{c} b \\ \begin{vmatrix} \begin{matrix} 1 \\ 1 \\ -2 \end{matrix} & \boxed{\begin{matrix} 0 \\ 1 \\ -1 \end{matrix}} & \begin{matrix} -1 \\ -3 \\ 4 \end{matrix} \end{vmatrix} \end{array} \; (1) = \frac{1}{8}\begin{array}{c} \text{expand!} \\ \begin{vmatrix} 1 & 0 & -1 \\ 1 & 1 & -3 \\ -1 & 0 & 1 \end{vmatrix} \end{array} = \frac{1}{8}\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = \frac{1}{8}(1-1) = \underline{\underline{0}}$$

and $$z = \frac{1}{8}\begin{array}{c} b \\ \begin{vmatrix} \begin{matrix} 1 \\ 1 \\ -2 \end{matrix} & \begin{matrix} 2 \\ 1 \\ 1 \end{matrix} & \boxed{\begin{matrix} 0 \\ 1 \\ -1 \end{matrix}} \end{vmatrix} \end{array} \; (1) = \frac{1}{8}\begin{array}{c} \text{expand!} \\ \begin{vmatrix} 1 & 2 & 0 \\ 1 & 1 & 1 \\ -1 & 2 & 0 \end{vmatrix} \end{array} = -\frac{1}{8}\begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} = -\frac{1}{8}(2+2) = \underline{\underline{-\frac{1}{2}}}$$

So the solution to the system is given by

$$(x, y, z) = \left(-\frac{1}{2}, 0, -\frac{1}{2}\right)$$

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(5) First, let's find the determinant: Let's call the matrix A.

$$\det(A) = \begin{vmatrix} 1 & -3 & 4 \\ 0 & 2 & 3 \\ 5 & 7 & 4 \end{vmatrix} \overset{-5}{=} \begin{vmatrix} 1 & -3 & 4 \\ 0 & 2 & 3 \\ 0 & 22 & -16 \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 22 & -16 \end{vmatrix} = 2\begin{vmatrix} 1 & 3 \\ 11 & -16 \end{vmatrix} = 2(-16-33)$$

$$= 2(-49) = \underline{\underline{-98}}$$

Now by the adjugate matrix theorem we have

$$\underline{A^{-1}} = \frac{1}{\det(A)}\operatorname{adj}(A) = -\frac{1}{98}\begin{bmatrix} \begin{vmatrix} 2 & 3 \\ 7 & 4 \end{vmatrix} & -\begin{vmatrix} 0 & 3 \\ 5 & 4 \end{vmatrix} & \begin{vmatrix} 0 & 2 \\ 5 & 7 \end{vmatrix} \\ -\begin{vmatrix} -3 & 4 \\ 7 & 4 \end{vmatrix} & \begin{vmatrix} 1 & 4 \\ 5 & 4 \end{vmatrix} & -\begin{vmatrix} 1 & -3 \\ 5 & 7 \end{vmatrix} \\ \begin{vmatrix} -3 & 4 \\ 2 & 3 \end{vmatrix} & -\begin{vmatrix} 1 & 4 \\ 0 & 3 \end{vmatrix} & \begin{vmatrix} 1 & -3 \\ 0 & 2 \end{vmatrix} \end{bmatrix}^T$$

$$= -\frac{1}{98}\begin{bmatrix} -13 & 15 & -10 \\ 40 & -16 & -22 \\ -17 & -3 & 2 \end{bmatrix}^T = \underline{\underline{\frac{1}{98}\begin{bmatrix} 13 & -40 & 17 \\ -15 & 16 & 3 \\ 10 & 22 & -2 \end{bmatrix}}}$$

Note that this is <u>really hard</u> with Gauss-Jordan in this case!

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(6) We have $\det(XY) = \det(X)\det(Y)$, and also

$1 = \det(I) = \det(XX^{-1}) = \det(X)\det(X^{-1})$, so

$\det(X^{-1}) = \dfrac{1}{\det(X)}$. Therefore:

$$\det(AB^{-1}C) = \det(A)\det(B^{-1})\det(C) = 2 \cdot \tfrac{1}{3} \cdot 5 = \frac{10}{3}$$

and

$$\det(BA^3C^T) = \det(B)\det(A)^3\underbrace{\det(C^T)}_{=\det(C)} = 3 \cdot 2^3 \cdot 5 = 3 \cdot 8 \cdot 5 = 120$$

7 First, note that if A is an $n\times n$-matrix we have $\det(-A) = (-1)^n \det(A)$, because we can factor out (-1) from each row in the determinant.

Therefore, if A is $n\times n$-shaped with n odd and skew-symmetric we have:

$$\det(A) = \det(A^T) = \det(-A) = (-1)^n \det(A) = -\det(A)$$

So $2\det(A) = 0$ and $\det(A) = 0$. Therefore A is not invertible!

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8 a) We have

$$\underbrace{\begin{bmatrix} 8 & 5 \\ -10 & -7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{v} = \begin{bmatrix} 3 \\ -3 \end{bmatrix} = \underbrace{3}_{\lambda} \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{v}$$

So $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is an eigenvector with eigenvalue 3.

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b) We solve $Av = \lambda v$ for $\lambda = -2$:

$$Av = -2v \iff (A+2I)v = 0 \iff \underbrace{\begin{bmatrix} 10 & 5 \\ -10 & -5 \end{bmatrix}}_{A+2I} \underbrace{\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{0}$$

$$\iff \begin{matrix} v_1 = -\frac{t}{2} \\ v_2 = t \end{matrix} \quad t \in \mathbb{R} \iff \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = t \begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix} \quad t \in \mathbb{R}, \text{ so}$$

every vector of form $t\begin{bmatrix} -\frac{1}{2} \\ 1 \end{bmatrix}$ (where $t \neq 0$) is an eigenvector of A with eigenvalue $\underline{-2}$. ($t=0$ is excluded, because the zero-vector is never counted as an eigenvector).

(9) We first find the eigenvalues: we solve $\det(\lambda I - B) = 0$

$$\iff \det\left(\begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} 15 & -30 \\ 4 & -7 \end{bmatrix}\right) = \begin{vmatrix} \lambda - 15 & 30 \\ -4 & \lambda + 7 \end{vmatrix} = (\lambda - 15)(\lambda + 7) + 120$$

$$= \lambda^2 - 15\lambda + 7\lambda - 105 + 120 = \lambda^2 - 8\lambda + 15 = (\lambda - 3)(\lambda - 5)$$

So the eigenvalues are $\boxed{\lambda_1 = 3 \text{ and } \lambda_2 = 5.}$

Next, let's find all eigenvectors corresponding to the eigenvalue $\underline{\lambda_1 = 3}$. We solve $BV = 3V \iff (3I - B)V = 0$

$$\iff \begin{array}{c} \left[\begin{array}{cc|c} -12 & 30 & 0 \\ -4 & 10 & 0 \end{array}\right] \end{array} \begin{array}{c} \\ -3 \quad \frac{1}{2} \end{array} \sim \left[\begin{array}{cc|c} 0 & 0 & 0 \\ -2 & 5 & 0 \end{array}\right] \longleftrightarrow \begin{cases} V_1 = \frac{5}{2}t \\ V_2 = t \end{cases}$$

$$\iff \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = t \begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix} \quad t \in \mathbb{R}.$$

So the set of eigenvectors of B with eigenvalue 3 are all vectors of form $t\begin{bmatrix} \frac{5}{2} \\ 1 \end{bmatrix}$ for $t \in \mathbb{R}$, $t \neq 0$.

Finally, let's find all eigenvectors corresponding to the eigenvalue $\underline{\lambda_2 = 5}$. We solve $BV = 5V \iff (5I - B)V = 0$

$$\iff \left[\begin{array}{cc|c} -10 & 30 & 0 \\ -4 & 12 & 0 \end{array}\right] \begin{array}{c} \frac{1}{10} \\ \\ \end{array} \sim \left[\begin{array}{cc|c} -1 & 3 & 0 \\ -4 & 12 & 0 \end{array}\right] \begin{array}{c} -4 \\ \\ \end{array} \sim \left[\begin{array}{cc|c} -1 & 3 & 0 \\ 0 & 0 & 0 \end{array}\right] \longleftrightarrow \begin{cases} V_1 = 3t \\ V_2 = t \end{cases}$$

$$\iff \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = t \begin{bmatrix} 3 \\ 1 \end{bmatrix} \quad t \in \mathbb{R}.$$

So the set of eigenvectors of B with eigenvalue 5 are all vectors of form $t\begin{bmatrix} 3 \\ 1 \end{bmatrix}$ $t \in \mathbb{R}$, $t \neq 0$.

10 (see the previous problem for details)

$$\det(\lambda I - C) = \begin{vmatrix} \lambda-2 & -1 & -1 \\ 0 & \lambda+5 & 2 \\ 0 & -8 & \lambda-3 \end{vmatrix} = (\lambda-2)\begin{vmatrix} \lambda+5 & 2 \\ -8 & \lambda-3 \end{vmatrix} = (\lambda-2)\big((\lambda+5)(\lambda-3)+16\big)$$

$$=(\lambda-2)(\lambda^2+5\lambda-3\lambda-15+16) = (\lambda-2)(\lambda^2+2\lambda+1) = (\lambda-2)(\lambda+1)(\lambda+1) \text{ so}$$

$\lambda_1 = 2$ and $\lambda_2 = -1$ are the eigenvalues.

Eigenvectors for $\lambda_1 = 2$: $Cv = 2v \iff (2I - C)v = 0$

$$\iff \left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 0 & 7 & 2 & 0 \\ 0 & -8 & -1 & 0 \end{array}\right] \; 7 \;\; -8 \sim \left[\begin{array}{ccc|c} 0 & -1 & -1 & 0 \\ 0 & 0 & -5 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right] \iff \begin{cases} v_1 = t \\ v_2 = 0 \\ v_3 = 0 \end{cases} \iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} t \\ 0 \\ 0 \end{bmatrix} \quad t \in \mathbb{R}$$

So all vectors of form $t\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ $t \in \mathbb{R}$, $t \neq 0$ are eigenvectors with eigenvalue 2.

Eigenvectors for $\underline{\lambda_2 = -1}$: $Cv = -v \iff (-I - C)v = 0$

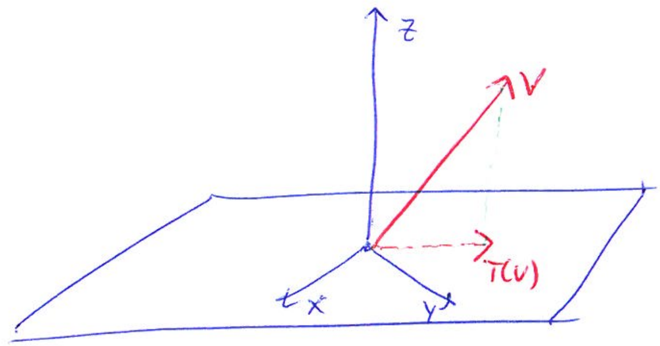
$$\iff \left[\begin{array}{ccc|c} -3 & -1 & -1 & 0 \\ 0 & 4 & 2 & 0 \\ 0 & -8 & -4 & 0 \end{array}\right] \; 2 \;\; \boxed{\tfrac{1}{2}} \sim \left[\begin{array}{ccc|c} -3 & -1 & -1 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \iff \begin{cases} v_1 = \frac{1}{3}\left(-t + \frac{t}{2}\right) \\ v_2 = -\frac{t}{2} \\ v_3 = t \end{cases}$$

$$\iff \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = t\begin{bmatrix} -\frac{1}{6} \\ -\frac{1}{2} \\ 1 \end{bmatrix} = \frac{t}{6}\begin{bmatrix} -1 \\ -3 \\ 6 \end{bmatrix} \quad \underline{t \in \mathbb{R}}$$

So all vectors of form $\frac{t}{6}\begin{bmatrix} -1 \\ -3 \\ 6 \end{bmatrix}$ for $t \in \mathbb{R}$, $t \neq 0$ are eigenvectors of C with eigenvalue -1.

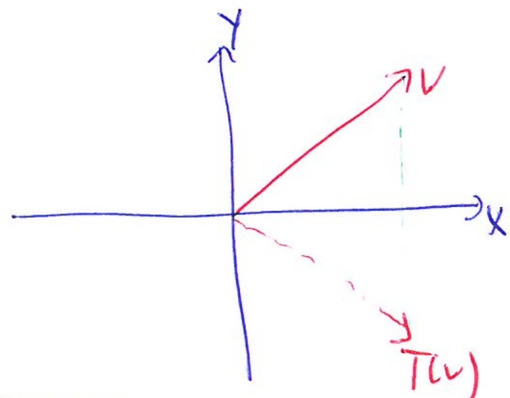
(11) ☆ a) Answers:

eigenvalues	Corr. eigenvectors
1	all nonzero vectors in the xy-plane
0	all vectors of form $t\begin{bmatrix}0\\0\\1\end{bmatrix}$ $t \neq 0$. (the z-axis).



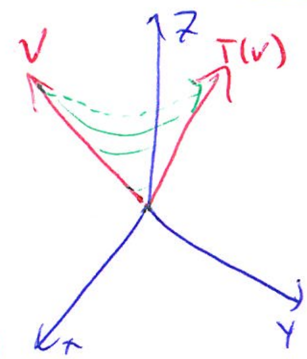
b)

eigenvalues	Corr. eigenvectors
1	$t\begin{bmatrix}1\\0\end{bmatrix}$ $t \neq 0$ (vectors on the x-axis)
-1	$t\begin{bmatrix}0\\1\end{bmatrix}$ $t \neq 0$ (vectors on the y-axis)



c)

eigenvalues	Corr. eigenvectors
1	$t\begin{bmatrix}0\\0\\1\end{bmatrix}$, $t \neq 0$. (Vectors on the z-axis).



(Note: there are actually complex eigenvalues here) More on that later!

d)

eigenvalues	Corr. eigenvectors
1	all (nonzero) vectors of $\mathbb{R}^3$