# Tutorial 2

MATH 1104 B • October 5, 2016 • Jonathan Nilsson

Work alone or in small groups with the following problems during the tutorial. Your TA is available to help you both during the tutorial and during their weekly office hours. You are not expected to be able to solve all problems in one hour, but you may want to complete the exercises at home. Problems marked by ★ can be tricky and should probably be saved for last. Suggested solutions will be posted after the tutorial.

## Linear maps

1. Let $T$ be the linear map whose standard matrix is $\begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 1 & 1 & 2 \\ 5 & -3 & 1 & 0 \end{bmatrix}$.
   (a) $T$ is map from $\mathbb{R}^m$ to $\mathbb{R}^n$. What are $m$ and $n$?
   (b) Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Find $\mathbf{T}(\mathbf{v_1})$, $\mathbf{T}(\mathbf{v_2})$, and $\mathbf{T}(\mathbf{v_1} + \mathbf{v_2})$.
   (c) Try to calculate $\mathbf{T}(\mathbf{T}(\mathbf{v_1}))$. You can't. Explain why!
   (d) Find all vectors $\mathbf{v}$ such that $\mathbf{T}(\mathbf{v}) = \mathbf{0}$.
   (e) For what vectors $\mathbf{w}$ can we find $\mathbf{v}$ such that $\mathbf{T}(\mathbf{v}) = \mathbf{w}$?

2. A linear map $R : \mathbb{R}^2 \to \mathbb{R}^2$ has standard matrix $[R] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.
   (a) Find $R(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$ and $R(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$.
   (b) Describe the map $R$ geometrically.
   *Hint: Try to multiply a few vectors with $[R]$ - draw a picture!*
   (c) Find another linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathbf{T}(\mathbf{R}(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v}$. Write down the standard matrix for $T$.
   (d) What do you think the product of the matrices $[T]$ and $[R]$ is going to be? Multiply and verify your guess!

3. We know that for the linear map $F : \mathbb{R}^2 \to \mathbb{R}^3$ we have

$$F(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad F(\begin{bmatrix} 1 \\ -1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   (a) Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
   (b) Use your result to find $F(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$.
   (c) Find the standard matrix for $F$.

4. ★ Let $S$ and $T$ be any two linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let $Q$ be the their composition: $Q(\mathbf{v}) = S(T(\mathbf{v}))$. Show that $Q$ is also a linear map.

## Matrix Algebra

5. Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 & 5 \\ 3 & 2 & -2 \end{bmatrix}$.

   Which of the following eight expressions are defined? Evaluate the ones that are!

   $AB \qquad BA \qquad AA \qquad BB \qquad AB^T \qquad B^TA \qquad (A+B^T)A \qquad (A+A^T)B$

6. Consider the following matrices:

$$C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 3 \\ 5 & 0 \\ 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

   In each problem, determine what size the matrix $X$ must be. Then find all matrices $X$ that solves each given matrix equation. The matrix $I$ always means the identity matrix of the appropriate size.
   *Hint: Solve each equation algebraically before you insert any matrices.*

   (a) $CD = 2X + I$
   (b) $X + (DC - 3I) = 0$
   (c) $(E - I)^2C + 3X^T = 0$

7. Here are some common errors in simplifying matrix expressions. Explain what's wrong and correct the right hand side of the expressions.

   (a) $AB + CB = B(A + C)$
   (b) $((A + B)C)^T = (A^T + B^T)C^T$
   (c) $AB + 3B = (A + 3)B$
   (d) $(A + B)^2 = A^2 + 2AB + B^2$

8. ★ Calculate $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{123}$.
   *Hint: It may help to think of the matrix as a linear map!*