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Tutorial 1 – Suggested Solutions

1

The first system:

$$\begin{cases} 4x + y = -5 \\ 3x + 4y = 6 \\ -x + 2y = 8 \end{cases} \longleftrightarrow \left[\begin{array}{cc|c} 4 & 1 & -5 \\ 3 & 4 & 6 \\ -1 & 2 & 8 \end{array}\right] \sim \left[\begin{array}{cc|c} -1 & 2 & 8 \\ 3 & 4 & 6 \\ 4 & 1 & -5 \end{array}\right] \; 3 \; 4$$

$$\sim \left[\begin{array}{cc|c} -1 & 2 & 8 \\ 0 & 10 & 30 \\ 0 & 9 & 27 \end{array}\right] \begin{array}{c} \\ \boxed{\tfrac{1}{10}} \\ \boxed{\tfrac{1}{9}} \end{array} \sim \left[\begin{array}{cc|c} -1 & 2 & 8 \\ 0 & 1 & 3 \\ 0 & 1 & 3 \end{array}\right] -1 \sim \left[\begin{array}{cc|c} -1 & 2 & 8 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

Pivots

We have reached a row echelon form. There are two pivots and two variables. Hence the system has a unique solution.

We continue:

$$\left[\begin{array}{cc|c} -1 & 2 & 8 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] -2 \sim \left[\begin{array}{cc|c} -1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right] \boxed{-1} \sim \left[\begin{array}{ccc} 1 & 0 & -2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{array}\right]$$

This is the reduced row echelon form. (RREF)

The RREF corresponds to the system $\begin{cases} x = -2 \\ y = 3 \end{cases}$ which is our unique solution.

Next system:

$$\begin{cases} x - y + z = 2 \\ x + y - 2z = 7 \\ 3x - 7y + 9z = -1 \end{cases} \longleftrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 1 & 1 & -2 & 7 \\ 3 & -7 & 9 & -1 \end{array}\right] -1 \; -3 \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 2 & -3 & 5 \\ 0 & -4 & 6 & -7 \end{array}\right] 2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 2 & -3 & 5 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

REF

Pivots

Since there is a pivot in the rightmost column the system will have no solutions (the last eq. says $0 = 3$).

We find the RREF anyway:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 2 & -3 & 5 \\ 0 & 0 & 0 & 3 \end{array}\right] \begin{array}{c} \\ \boxed{\tfrac{1}{2}} \\ \boxed{\tfrac{1}{3}} \end{array} \sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & -\frac{3}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 1 \end{array}\right] -\frac{5}{2} \; -2$$

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] 1 \sim \left[\begin{array}{ccc|c} 1 & 0 & -\frac{1}{2} & 0 \\ 0 & 1 & -\frac{3}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

RREF

No solutions.

<u>Last System</u>

$$\begin{cases} x - y + 3z = -5 \\ -2x + 3y - 2z = 7 \end{cases} \longleftrightarrow \left[\begin{array}{ccc|c} 1 & -1 & 3 & -5 \\ -2 & 3 & -2 & 7 \end{array}\right] 2$$

pivots

$$\sim \left[\begin{array}{ccc|c} 1 & -1 & 3 & -5 \\ 0 & 1 & 4 & -3 \end{array}\right]$$

This is ~~a~~ REF. There is one more variable than pivots, so the solution space will have one free variable (that is, infinitely many solutions).

We continue:

$$\left[\begin{array}{ccc|c} 1 & -1 & 3 & -5 \\ 0 & 1 & 4 & -3 \end{array}\right] 1 \sim \left[\begin{array}{ccc|c} 1 & 0 & 7 & -8 \\ 0 & 1 & 4 & -3 \end{array}\right]$$

This is the RREF. Since the third column does not have a pivot, z will be our free variable

The RREF corresponds to the system

$$\begin{cases} x \quad -7z = -8 \\ \quad y + 4z = -3 \end{cases}$$

z is free, so solving for x and y we get:

$$\begin{cases} x = -8 + 7z \\ y = -3 - 4z \\ z \text{ is free} \end{cases}$$

Our answer.

Note that this can be written $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -8 \\ -3 \\ 0 \end{bmatrix} + z \begin{bmatrix} 7 \\ -4 \\ 1 \end{bmatrix}$

$z \in \mathbb{R}$.

② This time I don't draw the vertical line for "=". It is optional.

$$\begin{bmatrix} 1 & 3 & 2 & 3 & -1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 2 & 6 & 0 & 2 & -2 & 1 & 1 \\ 1 & 3 & -1 & 0 & -1 & 0 & -1 \end{bmatrix} \begin{matrix} -2 \quad -1 \\ \\ \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 3 & 2 & 3 & -1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & -4 & -4 & 0 & -1 & -7 \\ 0 & 0 & -3 & -3 & 0 & -1 & -5 \end{bmatrix} \begin{matrix} \\ 4 \quad 3 \\ \\ \\ \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 2 & 3 & -1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{matrix} \\ \\ -1 \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 3 & 2 & 3 & -1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivots

REF

The last column has no pivot, so there are solutions! There will be 6 variables and 3 pivots means $6-3=3$ free variables!

We continue:

$$\begin{bmatrix} 1 & 3 & 2 & 3 & -1 & 1 & 4 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \\ -2 \\ 1 \quad \boxed{-1} \\ \\ \end{matrix} \sim \begin{bmatrix} 1 & 3 & 0 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

RREF

As an equation system this is:

$$\begin{cases} x_1 + 3x_2 + x_4 - x_5 = 1 \\ x_3 + x_4 = 2 \\ x_6 = -1 \end{cases}$$

The non-pivot-column variables we take as free: x_2, x_4, and x_5 are free. Solving the system in terms of these yield:

$$\boxed{\begin{cases} x_1 = 1 - 3x_2 - x_4 + x_5 \\ x_2 \text{ is free} \\ x_3 = 2 - x_4 \\ x_4 \text{ is free} \\ x_5 \text{ is free} \\ x_6 = -1 \end{cases}}$$

Our answer

In vector form this becomes:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \\ 0 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

where $x_2, x_4, x_5 \in \mathbb{R}$.

3 We row reduce as usual:

$$\begin{cases} x - by = -1 \\ bx + 2by = 2 \end{cases} \Longleftrightarrow \begin{cases} x - by = -1 \\ (b^2+2b)y = b+2 \end{cases} \quad (*)$$

pivot

pivot???

The number of solutions depend on wheather or not $b^2+2b=0$.
Since $b^2+2b = b(b+2) = 0 \iff b=0$ or $b=-2$ we investigate those cases seperately:

When $\underline{b=0}$ the system $(*)$ is $\begin{cases} x = -1 \\ 0y = 2 \end{cases}$

which has $\boxed{\text{no solutions}}$

When $\underline{b=-2}$ the system $(*)$ is $\begin{cases} x + 2y = -1 \\ 0y = 0 \end{cases}$

Which has infinitely many solutions:

$$\boxed{\begin{cases} x = -1-2y \\ y \text{ is free} \end{cases}} \quad \text{or} \quad \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix} + y\begin{bmatrix} -2 \\ 1 \end{bmatrix} \text{ in vector form.} \quad (y \in \mathbb{R})$$

For all other b (that is $b \neq 0$ and $b \neq -2$) we continue from $(*)$:

$$\begin{cases} x - by = -1 \\ (b^2+2b)y = b+2 \end{cases} \boxed{\tfrac{1}{b^2+2b}} \Longleftrightarrow \begin{cases} x - by = -1 \\ y = \frac{1}{b} \end{cases} \quad \frac{b+2}{b^2+2b} \quad b$$

$$\Longleftrightarrow \boxed{\begin{cases} x = 0 \\ y = \frac{1}{b} \end{cases}}$$ Unique solution.

In summary: when $b=0$: no solutions, when $b=-2$: infinitely many sol., when $b \neq 0$ and $b \neq -2$: a unique solution.

4

First:

$$\begin{bmatrix} 1 & 2 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 2 & -2 & 1 & 0 \end{bmatrix} \; 1 \; -2 \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 3 & 2 & 2 \\ 0 & -6 & -1 & -2 \end{bmatrix} \; 2 \sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 3 & 2 & 2 \\ 0 & 0 & 3 & 2 \end{bmatrix} \begin{matrix} \\ \boxed{\tfrac{1}{3}} \\ \boxed{\tfrac{1}{3}} \end{matrix}$$

REF

$$\sim \begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{2}{3} \end{bmatrix} \; -1 \; -\tfrac{2}{3} \sim \begin{bmatrix} 1 & 2 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & \frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{3} \end{bmatrix} \; -2 \sim \begin{bmatrix} 1 & 0 & 0 & -\frac{1}{9} \\ 0 & 1 & 0 & \frac{2}{9} \\ 0 & 0 & 1 & \frac{2}{3} \end{bmatrix}$$

RREF

Second:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \\ 1 & 2 & 3 \end{bmatrix} \; -4 \; -7 \; -1 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \\ 0 & 0 & 0 \end{bmatrix} \; -2 \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; \boxed{-\tfrac{1}{3}}$$

REF

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \; -2 \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

RREF

Third:

REF

$$\begin{bmatrix} 3 & 1 & 2 & 0 \\ 2 & 1 & 2 & 3 \end{bmatrix} \; -\tfrac{2}{3} \sim \begin{bmatrix} 3 & 1 & 2 & 0 \\ 0 & \frac{1}{3} & \frac{2}{3} & 3 \end{bmatrix} \; \boxed{3} \sim \begin{bmatrix} 3 & 1 & 2 & 0 \\ 0 & 1 & 2 & 9 \end{bmatrix} \; -1$$

$$\sim \begin{bmatrix} 3 & 0 & 0 & -9 \\ 0 & 1 & 2 & 9 \end{bmatrix}$$

RREF

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(5) Your answer obviously depends on the digits you start with. Most likely you will find a <u>unique solution</u> - otherwise your phone number is very special!

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(6) Linear dependence is a property that a <u>collection</u> of vectors might have. The student probably thought that it was a property that a single ~~vector~~ vector had.

~~[illegible]~~

Analogy: are the lines l_1 and l_2 parallel?

"l_1 is but l_2 isn't"

↑

<u>Makes no sense!</u>

7 a) Suppose that $\lambda_1 V_1 + \lambda_2 V_2 + \lambda_3 V_3 = V_4$.

This is equivalent to $\lambda_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

$$\Leftrightarrow \begin{bmatrix} \lambda_1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} \lambda_2 \\ 0 \\ -\lambda_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\lambda_3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \Leftrightarrow \begin{bmatrix} \lambda_1 + \lambda_2 \\ 2\lambda_3 \\ -\lambda_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\Leftrightarrow \begin{cases} \lambda_1 + \lambda_2 = 1 \\ 2\lambda_3 = 2 \\ -\lambda_2 = 3 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = 4 \\ 2\lambda_3 = 2 \\ -\lambda_2 = 3 \end{cases} \Leftrightarrow \begin{cases} \lambda_1 = 4 \\ \lambda_2 = -3 \\ \lambda_3 = 1 \end{cases}$$

So we have $4V_1 + (-3)V_2 + 1V_3 = V_4$ and the answer is <u>YES.</u>

b) ~~From a) we~~ This follows from a) for example:

$$4V_1 - 3V_2 + V_3 - V_4 + 0V_5 = 0$$

A nontrivial linear combination is zero $\Leftrightarrow$ $\{V_1, \ldots, V_5\}$ is lin. dependent.

c) $\{V_1, V_2, V_3\}$ spans $\mathbb{R}^3$ if and only if every $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$ can be written as a linear combination of $\{V_1, V_2, V_3\}$.

As in a) we get the system $\begin{cases} \lambda_1 + \lambda_2 = a \\ 2\lambda_3 = b \\ -\lambda_2 = c \end{cases}$

$$\Rightarrow \begin{cases} \lambda_1 = a - c \\ \lambda_2 = -c \\ \lambda_3 = \frac{b}{2} \end{cases}$$

This system has a solution $\lambda_1, \lambda_2, \lambda_3$ for each a, b, c, so <u>YES</u>, $\{V_1, V_2, V_3\}$ spans $\mathbb{R}^3$.

d) we solve $\lambda_2 v_2 + \lambda_3 v_3 + \lambda_5 v_5 = 0$

$$\Leftrightarrow \lambda_2 \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + \lambda_3 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} + \lambda_5 \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Leftrightarrow \begin{cases} \lambda_2 + 3\lambda_5 = 0 \\ 2\lambda_3 + 2\lambda_5 = 0 \\ -\lambda_2 + \lambda_5 = 0 \end{cases}$$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right] \text{(1)} \sim \left[\begin{array}{ccc|c} 1 & 0 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right]$$

3 var, 3 pivots $\Rightarrow$ unique solution. In fact we get $(\lambda_2, \lambda_3, \lambda_5) = (0,0,0)$

Conclusion: The only way that $\lambda_2 v_2 + \lambda_3 v_3 + \lambda_5 v_5 = 0$ is that $(\lambda_1, \lambda_2, \lambda_3) = (0,0,0)$. Thus $\{v_2, v_3, v_5\}$ is linearly independent.

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f) As in d) we have $\lambda_3 v_3 + \lambda_4 v_4 + \lambda_5 v_5 = 0$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 2 & 2 & 2 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \boxed{\tfrac{1}{2}} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 3 & 1 & 0 \end{array}\right] \text{(-3)} \sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right]$$

unique sol

Conclusion As in d) $\{v_3, v_4, v_5\}$ are lin. independent.

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e) As in d), f): $\lambda_2 v_2 + \lambda_4 v_4 + \lambda_5 v_5 = 0$

$$\Leftrightarrow \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 2 & 2 & 0 \\ -1 & 3 & 1 & 0 \end{array}\right] \text{(1)} \boxed{\tfrac{1}{2}} \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right] \text{(-4)} \sim \left[\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

This has infinitely many solutions. So there are infinitely many ways in which $\lambda_2 v_2 + \lambda_4 v_4 + \lambda_5 v_5 = 0$.

Thus $\{v_2, v_4, v_5\}$ are linearly dependent.

8) The three vectors are lin. indep if the only lin. comb. of them that is zero is the trivial combination.

As in (7) we have $\lambda_1\begin{bmatrix}1\\2\\1\end{bmatrix}+\lambda_2\begin{bmatrix}1\\1\\3\end{bmatrix}+\lambda_3\begin{bmatrix}0\\c\\-3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$\Leftrightarrow \left[\begin{array}{ccc|c}1&1&0&0\\2&1&c&0\\1&3&-3&0\end{array}\right]\begin{array}{c}(-2)\ (-1)\\ \\ \\ \end{array} \sim \left[\begin{array}{ccc|c}1&1&0&0\\0&-1&c&0\\0&2&-3&0\end{array}\right]\begin{array}{c}\\(2)\\ \\ \end{array} \sim \left[\begin{array}{ccc|c}1&1&0&0\\0&-1&c&0\\0&0&2c-3&0\end{array}\right]$$

$$\underset{\text{pivot?}}{2c-3}$$

If $2c-3\neq 0$ the system has a unique solution ~~...~~ $(0,0,0)$ and the vectors are independent.

For $2c-3=0 \Leftrightarrow c=\frac{3}{2}$ the system has infinitely many solutions and the vectors are linearly dependent.

~~Answer: ...~~

Answer: ~~...~~

For $c\neq\frac{3}{2}$ the vectors are linearly independent.