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Suggested Solutions for Tutorial 3

1

A_1 By the 2x2-method we have

$$A_1^{-1} = \frac{1}{4}\begin{bmatrix} 5 & 2 \\ 3 & 2 \end{bmatrix}$$

(You can get the same result by the Gauss-Jordan method).

A_2

$$\left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 1 & 0 \\ 7 & 8 & 9 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} -4 \quad -7 \end{array} \sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & -6 & -12 & -7 & 0 & 1 \end{array}\right] \begin{array}{c} -2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & -3 & -6 & -4 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \end{array}\right]$$

Row echelon form has a zero-row. A_2 does not have an inverse.

A_3

$$\left[\begin{array}{ccc|ccc} 2 & 2 & -1 & 1 & 0 & 0 \\ 2 & 3 & -1 & 0 & 1 & 0 \\ -2 & 3 & 4 & 0 & 0 & 1 \end{array}\right] \begin{array}{c} -1 \quad 1 \end{array} \sim \left[\begin{array}{ccc|ccc} 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 5 & 3 & 1 & 0 & 1 \end{array}\right] \begin{array}{c} -5 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 2 & -1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 3 & 6 & -5 & 1 \end{array}\right] \begin{array}{c} \frac{1}{3} \quad \boxed{\frac{1}{3}} \end{array} \sim \left[\begin{array}{ccc|ccc} 2 & 2 & 0 & 3 & -\frac{5}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -\frac{5}{3} & \frac{1}{3} \end{array}\right] \begin{array}{c} -2 \end{array}$$

$$\sim \left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 5 & -\frac{11}{3} & \frac{1}{3} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -\frac{5}{3} & \frac{1}{3} \end{array}\right] \boxed{\frac{1}{2}} \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{5}{2} & -\frac{11}{6} & \frac{1}{6} \\ 0 & 1 & 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 2 & -\frac{5}{3} & \frac{1}{3} \end{array}\right]$$

So

$$A_3^{-1} = \begin{bmatrix} \frac{5}{2} & -\frac{11}{6} & \frac{1}{6} \\ -1 & 1 & 0 \\ 2 & -\frac{5}{3} & \frac{1}{3} \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 15 & -11 & 1 \\ -6 & 6 & 0 \\ 12 & -10 & 2 \end{bmatrix}$$

$\underline{A_4}$

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -3 & 0 & 1 & 0 & 0 \\ 3 & 0 & 4 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 5 & -2 & 0 & 0 & 0 & 1 \end{array}\right] \quad -3 \quad -1$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 1 & 4 & -2 & -1 & 0 & 0 & 1 \end{array}\right] \quad -1$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 & -1 & -1 & 0 & 1 \end{array}\right] \quad -2 \quad \sim \left[\begin{array}{cccc|cccc} 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 2 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 5 & -1 & -2 & 1 \end{array}\right] \quad -2 \quad 3 \quad -1$$

$$\sim \left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 4 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 21 & -2 & -8 & 3 \\ 0 & 0 & 1 & 0 & -3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 5 & -1 & -2 & 1 \end{array}\right]$$

So $A_4^{-1} = \begin{bmatrix} 4 & 0 & -1 & 0 \\ 21 & -2 & -8 & 3 \\ -3 & 0 & 1 & 0 \\ 5 & -1 & -2 & 1 \end{bmatrix}$

2 a) $AX = B - 3X \iff AX + 3X = B$

$\iff AX + 3IX = B \iff (A+3I)X = B$

$\iff X = (A+3I)^{-1}B$ if $(A+3I)^{-1}$ exists!

$A+3I = \begin{bmatrix} 4 & 3 \\ 1 & 5 \end{bmatrix} \quad (A+3I)^{-1} = \frac{1}{17}\begin{bmatrix} 5 & -3 \\ -1 & 4 \end{bmatrix}$

So $\underline{\underline{X}} = (A+3I)^{-1}B = \frac{1}{17}\begin{bmatrix} 5 & -3 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \underline{\underline{\frac{1}{17}\begin{bmatrix} 7 & 16 \\ 2 & 7 \end{bmatrix}}}$

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b) $AXB^{-1} = C \iff A^{-1}AXB^{-1} = A^{-1}C \iff XB^{-1} = A^{-1}C$

$\iff XB^{-1}B = A^{-1}CB \iff X = A^{-1}CB$

$A = \begin{bmatrix} 1 & 3 \\ 1 & 2 \end{bmatrix} \quad A^{-1} = \frac{1}{-1}\begin{bmatrix} 2 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}$

So $\underline{\underline{X}} = A^{-1}CB = \left(\begin{bmatrix} -2 & 3 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$= \underline{\underline{\begin{bmatrix} -6 & -16 \\ 3 & 8 \end{bmatrix}}}$

2c) $XC = B^T + XA - 2I \iff XC - XA = B^T - 2I$

$\iff X(C-A) = (B^T - 2I) \iff X = (B^T - 2I)\underline{(C-A)^{-1}}$

if this exists!

$C - A = \begin{bmatrix} 0 & -2 \\ -1 & -2 \end{bmatrix}$

So $(C-A)^{-1} = \frac{1}{-2}\begin{bmatrix} -2 & 2 \\ 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -1 & 0 \end{bmatrix}$

So $\underline{\underline{X}} = (B^T - 2I)(C-A)^{-1} = \left(\begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\right) \cdot \frac{1}{2}\begin{bmatrix} 2 & -2 \\ -1 & 0 \end{bmatrix}$

$= \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 5 & 1 \end{bmatrix}\begin{bmatrix} 2 & -2 \\ -1 & 0 \end{bmatrix} = \frac{1}{2}\underline{\underline{\begin{bmatrix} -1 & 0 \\ 9 & -10 \end{bmatrix}}}$

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3) We have a system equivalent to the matrix equation $Ax = b$ where $A = \begin{bmatrix} 2 & 2 & -1 \\ 2 & 3 & -1 \\ -2 & 3 & 4 \end{bmatrix}$, $x = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $b = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$

So $Ax = b \iff A^{-1}Ax = A^{-1}b \iff x = A^{-1}b$.

Using $A^{-1} = A_3^{-1}$ from 1 we get:

$\underline{\underline{x}} = A^{-1}b = \frac{1}{6}\begin{bmatrix} 15 & -11 & 1 \\ -6 & 6 & 0 \\ 12 & -10 & 2 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} -12 \\ 6 \\ -12 \end{bmatrix} = \underline{\underline{\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}}}$

(4) We try to invert the Matrix using Gauss-Jordan.

$$\left[\begin{array}{cc|cc} -1 & 3 & 1 & 0 \\ c & 5 & 0 & 1 \end{array}\right] \overset{(c)}{\sim} \left[\begin{array}{cc|cc} -1 & 3 & 1 & 0 \\ 0 & 5+3c & c & 1 \end{array}\right]$$

The bottom row is zero when $5+3c=0 \Leftrightarrow c=-\frac{5}{3}$.

~~For all other~~ So the matrix has no inverse for this c.

For all other c we may divide by $5+3c$ and continue:

$$\left[\begin{array}{cc|cc} -1 & 3 & 1 & 0 \\ 0 & 1 & \frac{c}{5+3c} & \frac{1}{5+3c} \end{array}\right] \overset{(-3)}{\sim} \left[\begin{array}{cc|cc} -1 & 0 & \frac{5}{5+3c} & \frac{-3}{5+3c} \\ 0 & 1 & \frac{c}{5+3c} & \frac{1}{5+3c} \end{array}\right] \boxed{-1}$$

$$1-\frac{3c}{5+3c}=\frac{5+\cancel{3c}-\cancel{3c}}{5+3c}$$

$$\sim \left[\begin{array}{cc|cc} 1 & 0 & \frac{-5}{5+3c} & \frac{3}{5+3c} \\ 0 & 1 & \frac{c}{5+3c} & \frac{1}{5+3c} \end{array}\right]$$

So the inverse exists for $c \neq -\frac{5}{3}$ and it is:

$$\frac{1}{5+3c}\begin{bmatrix} -5 & 3 \\ c & 1 \end{bmatrix}$$

(The same result can be obtained easier with the 2x2-formula! This method works for nxn-matrices though)

(5) ☆ $A^2+3A-2I=0$

$\iff A^2+3A=2I$

$\iff A(A+3I)=2I$

$\iff (A)\,\frac{1}{2}(A+3I)=I$

This shows that the inverse of A is $\frac{1}{2}(A+3I)$!!

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(6) a) Rotating by the same angle in the opposite direction is the inverse map.

b) No inverse exists. The inverse would have to map the x-axis onto all of $\mathbb{R}^2$.

c) S is its own inverse! Reflecting 2 times has no effect, we have $[S][S]=I$.

7) Put the vectors as <u>rows</u> in a matrix and row reduce:

a)

$$\begin{array}{c} v_1 \\ v_2 \\ v_3 \\ v_4 \end{array}\begin{bmatrix} 1 & 0 & 2 & 3 \\ 3 & 1 & -1 & 2 \\ 0 & 1 & -1 & 1 \\ 5 & -1 & 5 & 6 \end{bmatrix} \begin{matrix} -3 & -5 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -7 & -7 \\ 0 & 1 & -1 & 1 \\ 0 & -1 & -5 & -9 \end{bmatrix} \begin{matrix} -1 & 1 \end{matrix} \sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -7 & -7 \\ 0 & 0 & 6 & 8 \\ 0 & 0 & -12 & -16 \end{bmatrix} \begin{matrix} 2 & \boxed{\tfrac{1}{2}} \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -7 & -7 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

these vectors form a basis for S!

b) ...So S has dimension 3; $\dim(S) = 3$

c) This corresponds to solving the system:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 2 & -7 & 3 & 0 \\ 3 & -7 & 4 & 0 \end{array}\right] \begin{matrix} -2 & -3 \end{matrix} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & -7 & 3 & -2 \\ 0 & -7 & 4 & -3 \end{array}\right] 7$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 12 \\ 0 & 0 & 4 & 11 \end{array}\right] -\frac{4}{3} \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 3 & 12 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

No solutions, so <u>no</u> the vector w is not in S.

d) No, for example, w is not a linear combination of $v_1, \ldots, v_4$.

(or just note that $\dim S = 3 \neq 4 = \dim \mathbb{R}^4$

8/8

8) a) Col(M) = Span of columns. We write the columns as rows and reduce:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 0 & 1 \\ -2 & 4 & 4 \\ 3 & 2 & 4 \end{bmatrix} \begin{matrix} -2 \\ 2 \\ -3 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 8 & 10 \\ 0 & -4 & -5 \end{bmatrix} \begin{matrix} 2 \\ -1 \end{matrix} \sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -4 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{basis!} \; (\boxed{-1})$$

So $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} \right\}$ is a basis for Col(M).

Thus rank M = dim Col(M) = 2

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b) $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is in the null space iff it solves the System $Mx=0$

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 3 & 0 \\ 2 & 0 & 4 & 2 & 0 \\ 3 & 1 & 4 & 4 & 0 \end{array}\right] \begin{matrix} -2 \\ -3 \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 2 & -2 & 3 & 0 \\ 0 & -4 & 8 & -4 & 0 \\ 0 & -5 & 10 & -5 & 0 \end{array}\right] \begin{matrix} \boxed{-\frac{1}{4}} \\ \boxed{-\frac{1}{5}} \end{matrix}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 2 & -2 & 3 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right] \begin{matrix} -1 \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 2 & -2 & 3 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \begin{matrix} -2 \end{matrix}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

take $x_3 = s$, $x_4 = t$. Then

$x_1 = -2s - t$
$x_2 = 2s - t$ $(s, t \in \mathbb{R})$
$x_3 = s$
$x_4 = t$

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ So $\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul(A)

So the dimension of Nul(A) is 2. dim Nul(A) = 2.

7 Alternative solution

a) We put the vectors as columns in a matrix and row-reduce.

$$\begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 1 & 1 & -1 \\ 2 & -1 & -1 & 5 \\ 3 & 2 & 1 & 6 \end{bmatrix} (-2)(-3) \sim \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & -7 & -1 & -5 \\ 0 & -7 & 1 & -9 \end{bmatrix} (7)(7) \sim \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 6 & -12 \\ 0 & 0 & 8 & -16 \end{bmatrix} \left(-\tfrac{4}{3}\right)$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 5 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 6 & -12 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Pivots in columns 1,2,3, therefore a basis for ~~col~~ S is: $\rightarrow$

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

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b) By a), $\dim(S) = 3$

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c) We solve:

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & -1 & -1 & 0 \\ 3 & 2 & 1 & 0 \end{array}\right] (-2)(-3) \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & -7 & -1 & -2 \\ 0 & -7 & 1 & -3 \end{array}\right] (7) \sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 6 & 12 \\ 0 & 0 & 8 & 11 \end{array}\right] \left(-\tfrac{8}{6}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 3 & 0 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 6 & 12 \\ 0 & 0 & 0 & -5 \end{array}\right]$$

No solutions! Answer: No, $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \notin S$.

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d) No, for example $\begin{bmatrix} 1 \\ 2 \\ 0 \\ 0 \end{bmatrix} \notin S$, so S does not span $\mathbb{R}^4$.

8 Alternative Solution

We start by finding a basis for $Nul(M)$: $MX=0 \iff$

b)

$$\left[\begin{array}{cccc|c} 1 & 2 & -2 & 3 & 0 \\ 2 & 0 & 4 & 2 & 0 \\ 3 & 1 & 4 & 4 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 2 & -2 & 3 & 0 \\ 0 & -4 & 8 & -4 & 0 \\ 0 & -5 & 10 & -5 & 0 \end{array}\right] \begin{array}{c} \\ \boxed{-\frac{1}{4}} \\ \boxed{-\frac{1}{5}} \end{array} \sim \left[\begin{array}{cccc|c} 1 & 2 & -2 & 3 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \end{array}\right]$$

pivots.

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 1 & 0 \\ 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \iff \begin{cases} x_1 = -2x_3 - x_4 \\ x_2 = 2x_3 - x_4 \\ x_3 \text{ is free} \\ x_4 \text{ is free} \end{cases} \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \quad s,t \in \mathbb{R}$$

Therefore $\left\{ \begin{bmatrix} -2 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $Nul(M)$.

For a) we note that column 1 and 2 were pivot columns in the echelon form of M.

Therefore, $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $Col(M)$

__Remark:__ Note that a subspace can have more than one basis. Both $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix} \right\}$ and $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right\}$ span the same subspace ($Col(M)$). Both answers are correct.