TUT 2 SUGGESTED SOLUTIONS

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Tutorial 2

MATH 1104 B • October 5, 2016 • Jonathan Nilsson

Work alone or in small groups with the following problems during the tutorial. Your TA is available to help you both during the tutorial and during their weekly office hours. You are not expected to be able to solve all problems in one hour, but you may want to complete the exercises at home. Problems marked by ★ can be tricky and should probably be saved for last. Suggested solutions will be posted after the tutorial.

Linear maps

1. Let T be the linear map whose standard matrix is $\begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 1 & 1 & 2 \\ 5 & -3 & 1 & 0 \end{bmatrix}$.

   (a) T is map from $\mathbb{R}^m$ to $\mathbb{R}^n$. What are m and n?

   (b) Let $\mathbf{v_1} = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{v_2} = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. Find $\mathbf{T}(\mathbf{v_1})$, $\mathbf{T}(\mathbf{v_2})$, and $\mathbf{T}(\mathbf{v_1}+\mathbf{v_2})$.

   (c) Try to calculate $\mathbf{T}(\mathbf{T}(\mathbf{v_1}))$. You can't. Explain why!

   (d) Find all vectors $\mathbf{v}$ such that $\mathbf{T}(\mathbf{v}) = \mathbf{0}$.

   (e) For what vectors $\mathbf{w}$ can we find $\mathbf{v}$ such that $\mathbf{T}(\mathbf{v}) = \mathbf{w}$?

2. A linear map $R : \mathbb{R}^2 \to \mathbb{R}^2$ has standard matrix $[R] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

   (a) Find $R(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$ and $R(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$.

   (b) Describe the map R geometrically.
   Hint: Try to multiply a few vectors with $[R]$ - draw a picture!

   (c) Find another linear map $T : \mathbb{R}^2 \to \mathbb{R}^2$ such that $\mathbf{T}(\mathbf{R}(\mathbf{v})) = \mathbf{v}$ for all $\mathbf{v}$. Write down the standard matrix for T.

   (d) What do you think the product of the matrices $[T]$ and $[R]$ is going to be? Multiply and verify your guess!

3. We know that for the linear map $F : \mathbb{R}^2 \to \mathbb{R}^3$ we have

$$F(\begin{bmatrix} 2 \\ 1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad F(\begin{bmatrix} 1 \\ -1 \end{bmatrix}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

   (a) Write $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ as a linear combination of $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

   (b) Use your result to find $F(\begin{bmatrix} 1 \\ 0 \end{bmatrix})$.

   (c) Find the standard matrix for F.

4. ★ Let S and T be any two linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. Let Q be the their composition: $Q(\mathbf{v}) = S(T(\mathbf{v}))$. Show that Q is also a linear map.

Matrix Algebra

5. Let $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 & 5 \\ 3 & 2 & -2 \end{bmatrix}$.

   Which of the following eight expressions are defined? Evaluate the ones that are!

   $$AB \qquad BA \qquad AA \qquad BB \qquad AB^T \qquad B^TA \qquad (A+B^T)A \qquad (A+A^T)B$$

6. Consider the following matrices:

   $$C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} -1 & 3 \\ 5 & 0 \\ 0 & 1 \end{bmatrix} \qquad E = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

   In each problem, determine what size the matrix X must be. Then find all matrices X that solves each given matrix equation. The matrix I always means the identity matrix of the appropriate size.
   Hint: Solve each equation algebraically before you insert any matrices.

   (a) $CD = 2X + I$
   (b) $X + (DC - 3I) = 0$
   (c) $(E - I)^2C + 3X^T = 0$

7. Here are some common errors in simplifying matrix expressions. Explain what's wrong and correct the right hand side of the expressions.

   (a) $AB + CB = B(A + C)$
   (b) $((A + B)C)^T = (A^T + B^T)C^T$
   (c) $AB + 3B = (A + 3)B$
   (d) $(A + B)^2 = A^2 + 2AB + B^2$

8. ★ Calculate $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{123}$.
   Hint: It may help to think of the matrix as a linear map!

Jonathan Nilsson

SUGGESTED SOLUTIONS FOR TUTORIAL 2

LINEAR MAPS

1

$$[T] = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 1 & 1 & 2 \\ 5 & -3 & 1 & 0 \end{bmatrix}$$

a) T is a map from $\mathbb{R}^4$ to $\mathbb{R}^3$

b) $$T(v_1) = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 1 & 1 & 2 \\ 5 & -3 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix}$$

$$T(v_2) = \begin{bmatrix} 1 & 0 & 2 & 3 \\ -1 & 1 & 1 & 2 \\ 5 & -3 & 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 11 \end{bmatrix}$$

Since T is linear, we have $T(v_1+v_2) = T(v_1)+T(v_2) = \begin{bmatrix} -1 \\ 0 \\ -2 \end{bmatrix} + \begin{bmatrix} 7 \\ 1 \\ 11 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 9 \end{bmatrix}$
(or just compute it directly).

c) T takes vector from $\mathbb{R}^4$ and gives vectors of $\mathbb{R}^3$ as output. $T(v_1)$ is in $\mathbb{R}^3$, so we cannot apply T to it again.

d) $$T\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \iff \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ -1 & 1 & 1 & 2 & 0 \\ 5 & -3 & 1 & 0 & 0 \end{array}\right] \begin{matrix} 1 & -5 \end{matrix} \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 3 & 5 & 0 \\ 0 & -3 & -9 & -15 & 0 \end{array}\right] 3$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & 0 \\ 0 & 1 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \iff \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = s\begin{bmatrix} -2 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ -5 \\ 0 \\ 1 \end{bmatrix}$$

x_3, x_4 free.
$x_3 = s$, $x_4 = t$

$s, t \in \mathbb{R}$.
All vectors x of this form satisfies $Tx = 0$

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d) When does $TX = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ have a solution?

$$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & a \\ -1 & 1 & 1 & 2 & b \\ 5 & -3 & 1 & 0 & c \end{array}\right] \begin{array}{cc} 1 & -5 \\ & \\ & \end{array} \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & a \\ 0 & 1 & 3 & 5 & a+b \\ 0 & -3 & -9 & -15 & -5a+c \end{array}\right] \begin{array}{c} \\ 3 \\ \\ \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 3 & a \\ 0 & 1 & 3 & 5 & a+b \\ 0 & 0 & 0 & 0 & -2a+3b+c \end{array}\right]$$

So $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ can be written as $T(x)$, for some x precisely when

$-2a+3b+c=0$.

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2) $R: \mathbb{R}^2 \to \mathbb{R}^2 \qquad [R] = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

a) $R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$R\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$

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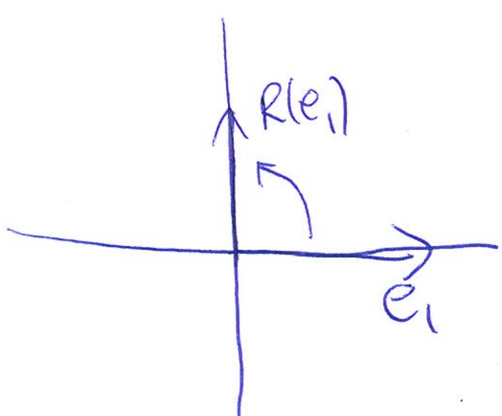
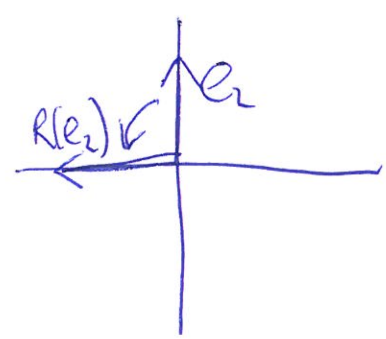
b)





R is "rotation by $\frac{\pi}{2}$ counterclockwise in the plane".

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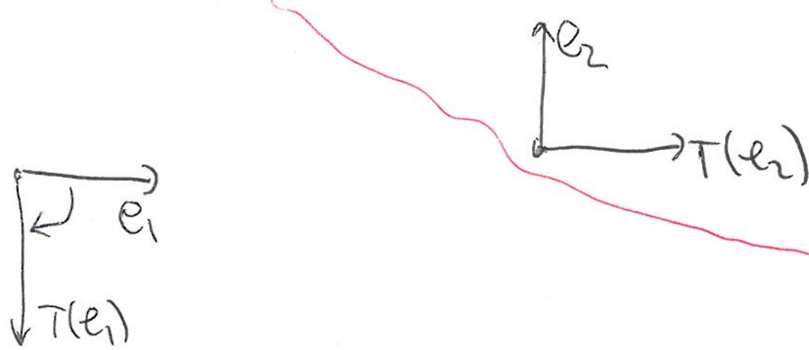
c) Let's take T to be the linear map that rotates in the other direction: $\frac{\pi}{2}$ clockwise! This should "undo" what R did so that $T(R(v)) = v$, right?

The columns of T is determined by where the vectors $\begin{bmatrix}1\\0\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$ are sent by T.

Geometrically we see that $T\begin{bmatrix}1\\0\end{bmatrix}$ should be $\begin{bmatrix}0\\-1\end{bmatrix}$ and $T\begin{bmatrix}0\\1\end{bmatrix}$ should be $\begin{bmatrix}1\\0\end{bmatrix}$.



Thus the standard matrix of T is $[T] = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}$

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d) First doing R and then T should do nothing to the vector, so we should have $[T][R] = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = I$.

We test this: $[T][R] = \begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix}\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

So our geometrical intuition was right!

③ $F:\mathbb{R}^2 \to \mathbb{R}^3$ $\quad F\left(\begin{bmatrix}2\\1\end{bmatrix}\right) = \begin{bmatrix}1\\1\\0\end{bmatrix}, \quad F\left(\begin{bmatrix}1\\-1\end{bmatrix}\right) = \begin{bmatrix}1\\2\\3\end{bmatrix}$ 4/8

a) $c_1\begin{bmatrix}2\\1\end{bmatrix} + c_2\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} \Leftrightarrow \left[\begin{array}{cc|c}2&1&1\\1&-1&0\end{array}\right]$ -2

$$\sim \left[\begin{array}{cc|c}0&3&1\\1&-1&0\end{array}\right] \boxed{\tfrac{1}{3}} \sim \left[\begin{array}{cc|c}1&-1&0\\0&1&\frac{1}{3}\end{array}\right] 1 \sim \left[\begin{array}{cc|c}1&0&\frac{1}{3}\\0&1&\frac{1}{3}\end{array}\right]$$

So $\frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$

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b) Since F is linear we have $F\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = F\left(\frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\-1\end{bmatrix}\right)$

$$= F\left(\frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix}\right) + F\left(\frac{1}{3}\begin{bmatrix}1\\-1\end{bmatrix}\right) = \frac{1}{3}F\left(\begin{bmatrix}2\\1\end{bmatrix}\right) + \frac{1}{3}F\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix}1\\1\\0\end{bmatrix} + \frac{1}{3}\begin{bmatrix}1\\2\\3\end{bmatrix} = \frac{1}{3}\begin{bmatrix}2\\3\\3\end{bmatrix} = \begin{bmatrix}\frac{2}{3}\\1\\1\end{bmatrix}$$

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c) $[F]$ has $F\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $F\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$ as columns. To find $F\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$ we write $\begin{bmatrix}0\\1\end{bmatrix}$ as a linear combination of $\begin{bmatrix}2\\1\end{bmatrix}$ and $\begin{bmatrix}1\\-1\end{bmatrix}$ and use the linearity of F to find $F\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$:

$$\left[\begin{array}{cc|c}2&1&0\\1&-1&1\end{array}\right] -2 \sim \left[\begin{array}{cc|c}0&3&-2\\1&-1&1\end{array}\right] \boxed{\tfrac{1}{3}} \sim \left[\begin{array}{cc|c}1&-1&1\\0&1&-\frac{2}{3}\end{array}\right] 1$$

$$\sim \left[\begin{array}{cc|c}1&0&\frac{1}{3}\\0&1&\frac{2}{3}\end{array}\right].$$

Thus $\frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix} - \frac{2}{3}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}0\\1\end{bmatrix}$ Cont. →

$\hookrightarrow$ So since F is linear we have

$$F\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = F\left(\frac{1}{3}\begin{bmatrix}2\\1\end{bmatrix} + \left(-\frac{2}{3}\right)\begin{bmatrix}1\\-1\end{bmatrix}\right) = \frac{1}{3}F\left(\begin{bmatrix}2\\1\end{bmatrix}\right) + \left(-\frac{2}{3}\right)F\left(\begin{bmatrix}1\\-1\end{bmatrix}\right)$$

$$= \frac{1}{3}\begin{bmatrix}1\\1\\0\end{bmatrix} + \left(-\frac{2}{3}\right)\begin{bmatrix}1\\2\\3\end{bmatrix} = \frac{1}{3}\left(\begin{bmatrix}1\\1\\0\end{bmatrix} - 2\begin{bmatrix}1\\2\\3\end{bmatrix}\right) = \frac{1}{3}\begin{bmatrix}-1\\-3\\-6\end{bmatrix} = \begin{bmatrix}-\frac{1}{3}\\-1\\-2\end{bmatrix}.$$

We have found $F\left(\begin{bmatrix}1\\0\end{bmatrix}\right)$ and $F\left(\begin{bmatrix}0\\1\end{bmatrix}\right)$. We put them as columns of a matrix. This gives the standard matrix of F!

$$[F] = \begin{bmatrix}\frac{2}{3} & -\frac{1}{3}\\ 1 & -1\\ 1 & -2\end{bmatrix}$$

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(4) ☆ We have

$$Q(u+v) = S(T(u+v)) \overset{\text{because } T \text{ is linear}}{=} S(T(u)+T(v))$$

$$\underset{\text{because } S \text{ is linear.}}{=} S(T(u)) + S(T(v)) = Q(u) + Q(v) \quad \text{for all } u, v \in \mathbb{R}^n.$$

Similarly, $\underline{Q(\lambda v)} = S(T(\lambda v)) \overset{\text{because } T \text{ is linear}}{=} S(\lambda T(v)) \underset{\text{because } S \text{ is linear}}{=} \lambda S(T(v)) = \underline{\lambda Q(v)}$

for all $v \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$.

We have shown that
- $Q(u+v) = Q(u) + Q(v)$

and
- $Q(\lambda v) = \lambda Q(v)$.

Thus Q is linear!

5) $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 & 5 \\ 3 & 2 & -2 \end{bmatrix}$

MATRIX ALGEBRA

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A is $[2\times 2]$, B is $[2\times 3]$, so it makes sense to compute:

$$AB = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & -1 & 5 \\ 3 & 2 & -2 \end{bmatrix} = \begin{bmatrix} -6 & -5 & 9 \\ 3 & -1 & 13 \end{bmatrix}$$

BA is <u>undefined</u>.

$$A^2 = AA = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} -5 & -4 \\ 6 & -5 \end{bmatrix}$$

$B^2 = BB$ is <u>undefined</u>, <u>so is AB^T</u> since B^T is $[3\times 2]$.

$$B^TA = \begin{bmatrix} 0 & 3 \\ -1 & 2 \\ 5 & -2 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 9 & 3 \\ 5 & 4 \\ -1 & -12 \end{bmatrix}$$

$(A+B^T)A$ is <u>undefined</u>, we cannot add $A+B^T$.

Finally:

$$(A+A^T)B = \left(\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ -2 & 1 \end{bmatrix}\right)B = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}}_{}^{A+A^T}\begin{bmatrix} 0 & -1 & 5 \\ 3 & 2 & -2 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 8 \\ 6 & 3 & 1 \end{bmatrix}$$

Note that we could also have done: $(A+A^T)B = AB + A^TB$

these we found before!

6) $C = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ $D = \begin{bmatrix} -1 & 3 \\ 5 & 0 \\ 0 & 1 \end{bmatrix}$ $E = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ 7/8

(Size of X will follow once we solve the equations)

a) $CD = 2X + I \iff 2X = CD - I \iff X = \frac{1}{2}(CD - I)$

So $\underline{X} = \frac{1}{2}\left(\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 3 \\ 5 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{1}{2}\left(\begin{bmatrix} 9 & 4 \\ 5 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$= \frac{1}{2}\begin{bmatrix} 8 & 4 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ \frac{5}{2} & 0 \end{bmatrix}$

b) $X + DC - 3I = 0 \iff$

$X = 3I - DC$

So $\underline{X} = 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} -1 & 3 \\ 5 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} - \begin{bmatrix} -1 & 1 & 2 \\ 5 & 10 & 5 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 4 & -1 & -2 \\ -5 & -7 & -5 \\ 0 & -1 & 2 \end{bmatrix}$

c) $(E-I)^2 C + 3X^T = 0 \iff 3X^T = -(E-I)^2 C$

$\iff X^T = -\frac{1}{3}(E-I)^2 C$. We find X^T first:

$X^T = -\frac{1}{3}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)^2 \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -\frac{1}{3}\left(\overbrace{\begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix}}^{E-I}\overbrace{\begin{bmatrix} 0 & 2 \\ 3 & 3 \end{bmatrix}}^{E-I}\right)\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$= -\frac{1}{3}\begin{bmatrix} 6 & 6 \\ 9 & 15 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -\begin{bmatrix} 2 & 2 \\ 3 & 5 \end{bmatrix}\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \end{bmatrix} = -\begin{bmatrix} 2 & 6 & 4 \\ 3 & 11 & 8 \end{bmatrix}$

So $X^T = \begin{bmatrix} -2 & -6 & -4 \\ -3 & -11 & -8 \end{bmatrix}$ Transposing again we get $X = \begin{bmatrix} -2 & -3 \\ -6 & -11 \\ -4 & -8 \end{bmatrix}$

8/8

7 a) $AB+CB = (A+C)B$ ← B should be on the right since B was on the right on the other side of the equation.

b) $((A+B)C)^T = C^T(A+B)^T = C^T(A^T+B^T)$ ← Taking transpose reverses the order of matrix multiplication. $(AB)^T = B^TA^T$.

c) $AB+3B = AB+3IB = (A+3I)B$

"A+3" doesn't make sence since A is a matrix and 3 is a scalar. However, if we write B as $I \cdot B$ we may still factor the expression!

d) $(A+B)^2 = (A+B)(A+B) = A^2+AB+BA+B^2$

these might be different, so we can't write them as 2AB!

8 ☆ Find $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{123}$

Since this map rotates vectors by $\frac{\pi}{2}$ (see problem 2), the product $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{123}$ rotates vectors by $\frac{\pi}{2}$ 123 times. But rotating 4 times does nothing! Thus we note that since $123 = 4\cdot 30 + 3$ that

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{123} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{4\cdot 30+3} = \Bigg(\underbrace{\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{4}}_{\text{does nothing! Identity map!}}\Bigg)^{30} \cdot \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{3} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{3} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

30 full rotations, +3 rot.

(Alternatively, compute some powers and note that $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^4 = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and proceed the same way).