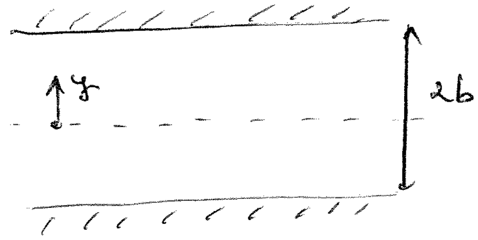
Solution Assignment #3
Summer 2016

①

#5.88 / Taylor series / Solve 5.101 / #6.107 / #6.114 / Create flow.

5.88

Pressure driven Flow, 2 stationary plates



Assumptions

* 2D flow (1)
* steady (2)
* $u = u(y)$

$$u = u_{max}\left(1 - \left(\frac{y}{b}\right)^2\right)$$

a) Evaluate the rates of linear and angular deformation (ignore)!

To solve ⇒ Evaluate $\nabla \cdot \vec{V}$ for the linear deformation in all directions
⇓ Evaluate angular deformation in all direction:

$x-y$ plane : $\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}$

$y-z$ plane : $\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}$

$z-x$ plane : $\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}$

b) Find Expression for the vorticity vector $\vec{\zeta}$

$$\vec{\zeta} = \nabla \times \vec{V} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \vec{i}\left(\overset{(1)}{\cancel{\frac{\partial w}{\partial y}}} - \overset{(1)}{\cancel{\frac{\partial v}{\partial z}}}\right) - \vec{j}\left(\overset{(1)}{\cancel{\frac{\partial w}{\partial x}}} - \overset{(1)}{\cancel{\frac{\partial u}{\partial z}}}\right) + \vec{k}\left(\frac{\partial v}{\partial x}\right.$$

$$\Rightarrow \vec{\zeta} = \vec{k}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial u}{\partial y} = u_{max}\left(-\frac{2y}{b^2}\right)$$

$$\vec{\zeta} = \left(-\frac{\partial u}{\partial y}\right)\vec{k}$$

$$\Rightarrow \boxed{\vec{\xi} = u_{max} \times \frac{2y}{b^2}\,\vec{k}}$$

We knew ~~ξ_y~~ $-b < y < b$

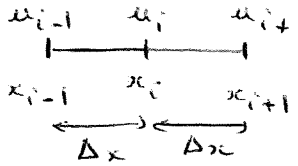
Maximum is located at $y = b$

Taylor Series

Use Taylor series to show the order of accuracy to approximate ~~dx~~ $\frac{du}{dx}$

a) $\dfrac{u_{i+1} - u_{i-1}}{2\Delta x} \simeq \dfrac{du}{dx}$

b) $\dfrac{u_i - u_{i-1}}{\Delta x} \simeq \dfrac{du}{dx}$



c) $\dfrac{u_{i+3} - u_{i-3}}{6\Delta x} \simeq \dfrac{du}{dx}$

d) $\dfrac{-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}}{12\Delta x} \simeq \dfrac{du}{dx}$

Taylor series:

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{3}(a)}{3!}(x-a)^3 + \dots + \frac{f^{n}(a)}{n!}(x-a)^n + \dots$$

$\Rightarrow$ For $\Delta x = x_{i+1} - x_i$ ~~Δx~~ $= x_i - x_{i-1}$

For Exp: Here $x = x_i + \Delta x = x_{i+1}$ and $a = x_i$

$$\Rightarrow f(x_i + \Delta x) = f(x_i) + f'(x_i)(\cancel{x_i} + \Delta x - \cancel{x_i}) + \frac{f''(x_i)}{2!}(\cancel{x_i} + \Delta x - \cancel{x_i})^2 + \dots$$

a)

(1) $$u_{i+1} = u_i + \frac{\partial u}{\partial x}\Big|_i \Delta x + \frac{\partial^2 u}{\partial x^2}\Big|_i \frac{(\Delta x)^2}{2!} + \frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(\Delta x)^3}{3!} + \dots$$

(2) $$u_{i-1} = u_i - \frac{\partial u}{\partial x}\Big|_i \Delta x + \frac{\partial^2 u}{\partial x^2}\Big|_i \frac{(\Delta x)^2}{2!} - \frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(\Delta x)^3}{3!} + \dots$$

(1) - (2) $$\Rightarrow u_{i+1} - u_{i-1} = 0 + 2\frac{\partial u}{\partial x}\Big|_i (\Delta x) + 0 + 2\frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(\Delta x)^3}{3!} + 0 + \dots$$

$$\Rightarrow \frac{u_{i+1} - u_{i-1}}{2\Delta x} = \frac{\partial u}{\partial x}\Big|_i + \underbrace{\frac{\partial^3 u}{\partial x^3}\frac{(\Delta x)^2}{3!}}_{} + \dots$$

$$\Rightarrow \frac{\partial u}{\partial x}\Big|_i \simeq \frac{u_{i+1} - u_{i-1}}{2\Delta x} + O(\Delta x^2)$$

The error is of second order $(\Delta x)^2$

b) Using Eq. (2)

$$\Rightarrow u_{i-1} - u_i = -\frac{\partial u}{\partial x}\Big|_i \Delta x + \frac{\partial^2 u}{\partial x^2}\Big|_i \frac{(\Delta x)^2}{2!} - \frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(\Delta x^3)}{3!} + \dots$$

$$\Rightarrow \frac{-u_{i-1} + u_i}{\Delta x} \simeq \frac{\partial u}{\partial x}\Big|_i \underbrace{- \frac{\partial^2 u}{\partial x^2}\Big|_i \frac{\Delta x}{2!} + \frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(\Delta x)^2}{3!}}_{\text{truncation error}} + \dots$$

$\Rightarrow$ the error is of order (Δx)

c)

(3) $$u_{i+3} = u_i + \frac{\partial u}{\partial x}\Big|_i (3\Delta x) + \frac{\partial^2 u}{\partial x^2}\Big|_i \frac{(3\Delta x)^2}{2!} + \frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(3\Delta x)^3}{3!} + \dots$$

(4) $$u_{i-3} = u_i - \frac{\partial u}{\partial x}\Big|_i (3\Delta x) + \frac{\partial^2 u}{\partial x^2}\Big|_i \frac{(3\Delta x)^2}{2!} - \frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(3\Delta x)^3}{3!} + \dots$$

(3) - (4) $$\Rightarrow u_{i+3} - u_{i-3} = 0 + 2\frac{\partial u}{\partial x}\Big|_i (3\Delta x) + 0 + 2\frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(3\Delta x)^3}{3!} + \dots$$

$$\Rightarrow \frac{u_{i+3} - u_{i-3}}{6\Delta x} \simeq \frac{\partial u}{\partial x}\Big|_i + \underbrace{\frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(3\Delta x)^2}{3!}}_{\text{truncation error.}} + \dots$$

$\Rightarrow$ The error is of order $(\Delta x)^2$

d) (5) $$u_{i+2} = u_i + \frac{\partial u}{\partial x}\Big|_i \frac{(2\Delta x)}{1!} + \frac{\partial^2 u}{\partial x^2}\Big|_i \frac{(2\Delta x)^2}{2!} + \frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(2\Delta x)^3}{3!} + \frac{\partial^4 u}{\partial x^4}\Big|_i \frac{(2\Delta x)^4}{4!} + \dots$$ (4)

(6) $$u_{i-2} = u_i - \frac{\partial u}{\partial x}\Big|_i (2\Delta x) + \frac{\partial^2 u}{\partial x^2}\Big|_i \frac{(2\Delta x)^2}{2!} - \frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(2\Delta x)^3}{3!} + \frac{\partial^4 u}{\partial x^4}\Big|_i \frac{(2\Delta x)^4}{4!} + \dots$$

(7) $$8u_{i+1} = 8u_i + 8\frac{\partial u}{\partial x}\Big|_i (\Delta x) + 8\frac{\partial^2 u}{\partial x^2}\Big|_i \frac{(\Delta x)^2}{2!} + 8\frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(\Delta x^3)}{3!} + 8\frac{\partial^4 u}{\partial x^4}\Big|_i \frac{(\Delta x)^4}{4!} + \dots$$

(8) $$8u_{i-1} = 8u_i - 8\frac{\partial u}{\partial x}\Big|_i (\Delta x) + 8\frac{\partial^2 u}{\partial x^2}\Big|_i \frac{(\Delta x)^2}{2!} - 8\frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(\Delta x)^3}{3!} + 8\frac{\partial^4 u}{\partial x^4}\Big|_i \frac{(\Delta x)^4}{4!} + \dots$$

(6) - (5) + (7) - (8)

$$\Rightarrow u_{i-2} - u_{i+2} + 8u_{i+1} - 8u_{i-1} = 0 - 2\frac{\partial u}{\partial x}\Big|_i 2\Delta x + \cancel{\frac{\partial u}{\partial x}} + 2\times 8\frac{\partial u}{\partial x}\Big|_i \Delta x + 0$$
$$- 2\frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(2\Delta x)^3}{3!} + 2\times 8\frac{\partial^3 u}{\partial x^3}\Big|_i \frac{(\Delta x)^3}{3!} + 0 + -2\frac{\partial^5 u}{\partial x^5}\Big|_i \frac{(2\Delta x)^5}{5!} + 2\times 8\frac{\partial^5 u}{\partial x^5}\Big|_i (\Delta$$

$$\Rightarrow -u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2} = \left(-4\frac{\partial u}{\partial x}\Big|_i + 16\frac{\partial u}{\partial x}\Big|_i\right)\Delta x + \cancel{\frac{\Delta x^3}{3!}\left(-16\frac{\partial^3 u}{\partial x^3}\Big|_i + 16\frac{\partial^3 u}{\partial x^3}\right)}$$
$$+ \frac{(\Delta x)^5}{5!}\left(-32\times 2 + 16\right)\frac{\partial^5 u}{\partial x^5}\Big|_i + \dots$$

$$\Rightarrow \frac{-u_{i+2} + 8u_{i+1} - 8u_{i-1} + u_{i-2}}{12\Delta x} = \frac{\partial u}{\partial x}\Big|_i \underbrace{- \frac{4}{5!}\frac{\partial^5 u}{\partial x^5}\Big|_i (\Delta x)^4 + \dots}_{\text{Truncation error}}$$

$\Rightarrow$ The error is of the order of $(\Delta x)^4$

* The one that gives a better result and is the most accurate is e) Since the error is proportional to $(\Delta x)^4$

* However, higher order approximation ~~is~~ usually requires higher computational time. Also ~~the~~ higher order approximation is sometimes not possible to use when the point doesn't exist as it is the case for the solution near the wall.

5.101 using Scheme a) and b)

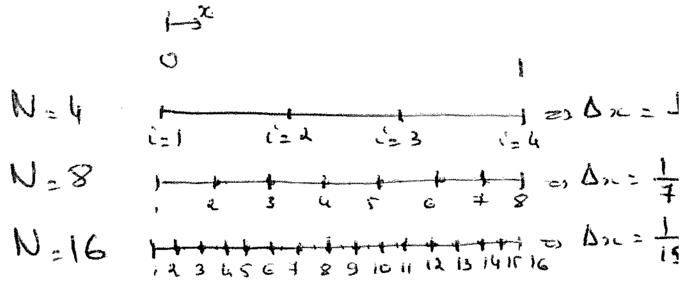
$$\frac{du}{dx} + u = 2\cos(2x) \qquad 0 \leq x \leq 1 \qquad u(0) = 0$$

$$u_{exact} = \frac{2}{5}\cos(2x) + \frac{4}{5}\sin(2x) - \frac{2}{5}e^{-x}$$

Scheme a): $\dfrac{u_{i+1} - u_{i-1}}{2\Delta x} \simeq \left.\dfrac{\partial u}{\partial x}\right|_i$

Scheme b): $\dfrac{u_i - u_{i-1}}{\Delta x} \simeq \left.\dfrac{\partial u}{\partial x}\right|_i$

- Run Computational Solver for N = 4, 8 and 16, compare results and discuss
- Show the solution is improving with N
- Plot error vs. N ⇒ results in table
- Check the order of accuracy



a) Using scheme (a)

$$\frac{u_{i+1} - u_{i-1}}{2\Delta x} + u_i = 2\cos(2x_i)$$

$$u_{i+1} - u_{i-1} + 2\Delta x\, u_i = 4\Delta x \cos(2x_i) \qquad u_1 = 0$$

$i = 2 \qquad u_3 - u_1 + 2\Delta x\, u_2 = 4\Delta x \cos(2x_2) \qquad$ (I)

→ Here we need another condition in order to solve for u. If u_4 is given we can find u_2 and u_3 (2 equation, 2 unknowns)

→ Or we could estimate u_2 using Scheme (b) and use it to find $u_3, u_4 \ldots$

Using scheme (b) $\dfrac{u_i - u_{i-1}}{\Delta x} + u_i = 2\cos(2x_i) \Rightarrow u_i - u_{i-1} + \Delta x\, u_i = 2\Delta x \cos(2x_i)$

$\Rightarrow u_i(1 + \Delta x) + u_{i-1} = 2\Delta x \cos(2x_i)$

$\Rightarrow u_2 (1 + \Delta x) + u_1 = 2\Delta x \cos(2x_2) \qquad$ where $u_1 = 0$

→ use it back in (I)

For $N = 4$ ⇒ Matrix form 4×4 Matrix $x_2 = \Delta x$
$\Delta x = \frac{1}{3}$ $x_3 = 2\Delta x$

(6)

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1+\Delta x & 0 & 0 \\ -1 & 2\Delta x & 1 & 0 \\ 0 & -1 & 2\Delta x & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\Delta x \cos((x_2)\times 2) \\ 4\Delta x \cos((x_2)\times 2) \\ 4\Delta x \cos(2 x_3) \end{bmatrix}$$ ⇒ RHS Matrix

$N = 8$ ⇒ 8×8 Matrix and $\Delta x = \frac{1}{7}$

$N = 16$ ⇒ 16×16 Matrix and $\Delta x = \frac{1}{15}$

* To solve for u ⇒ ~~SH~~ Find the inverse Matrix and Multiply it by the RHS Matrix

* Figure 1. shows the plot of the solution for $N = 4, 8$ and 16 compared to the exact solution. ⇒ We can see that for higher N (increasing the number of point in the domain, decreasing Δx), the solution is closer to the exact solution.

* Figure 2. shows the error as a function of N ⇒ The error for $u(1)$ decreases ~~with~~ for higher N ($N = 16$).

The error was calculated as $\mathcal{E} = |u_N(1) - u_{exact}(1)|$

N	$u_N(1)$	$u_{exact}(1)$	$\mathcal{E}$	Δx
4	0,18767	0,4138	0,2311	0,333
8	0,37407	0,4138	0,0397	0,1428
16	0,4058	0,4138	0,00797	0,06667

$\Delta x_8 = \frac{\Delta x_4}{2,332}$

$\mathcal{E}_8 = \frac{\mathcal{E}_4}{5,814} \simeq \frac{\mathcal{E}_4}{(2,33)^2}$

⇒ The error is of order $(\Delta x)^2$ as previously found with Taylor series in Scheme (a)

→ Do the same ~~thing~~ Analysis for scheme (b)

b) Matrix Format $N = 4$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1+\Delta x & 0 & 0 \\ 0 & 1 & 1+\Delta x & 0 \\ 0 & 0 & 1 & 1+\Delta x \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 2\Delta x \cos(2 x_2) \\ 2\Delta x \cos(2 x_3) \\ 2\Delta x \cos(2 x_4) \end{bmatrix}$$

6.107

$\Psi = Ax^3 + B(xy^2 + x^2 - y^2)$

a) Find Relation between A and B if flow is irrotational

b) Find velocity potential ϕ ?

a) $\Psi \Rightarrow$ The flow is incompressible, 2D

$\nabla \cdot \vec{V} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$

we knew $u = \frac{\partial \Psi}{\partial y} = 2Bxy - 2By$

$v = -\frac{\partial \Psi}{\partial x} = -3Ax^2 - By^2 - 2Bx$

} check incompressibility ✓

If the flow is irrotational $\Rightarrow \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

$\Rightarrow -3 \times 2Ax - \cancel{2B} - 2Bx + \cancel{2B} = 0 \Rightarrow -6Ax - 2Bx = 0$

$\Rightarrow \boxed{A = -\frac{1}{3}B}$

$\Rightarrow u = -\frac{\partial \phi}{\partial x} \Rightarrow \phi = -\int u\,dx + f(y) = -\int (2Bxy - 2By)\,dx + f(y) = -\frac{2Bx^2y}{2} + 2Bxy +$

$v = -\frac{\partial \phi}{\partial y} \Rightarrow \phi = -\int v\,dy + f(x) = +\int 3Ax^2 + By^2 + 2Bx + f(x) = 3Ax^2y + \frac{By^3}{3} + 2Bx$

$\Rightarrow \phi = -Bx^2y + 2Bxy + f(y)$

$\phi = 3Ax^2y + \frac{By^3}{3} + 2Bxy + f(x)$

if $A = -\frac{1}{3}B$

$\phi = -Bx^2y + 2Bxy + f(y)$

$\phi = -Bx^2y + \frac{By^3}{3} + 2Bxy + f(x)$

compare

$f(x) = 0$

$f(y) = \frac{By^3}{3}$

$\Rightarrow \boxed{\phi = -Bx^2y + 2Bxy + \frac{By^3}{3}}$

6.114

$\phi = Ax + Bx^2 - By^2 \quad A = 1\ m/s \ , \ B = .1\ m^{-1}s^{-1}$

Assumptions:
- * irrotational
- * 2D
- * Steady

a) obtain the velocity field

b) Stream function

c) Calculate P difference between origin and $(x,y) = (1,2)$

a) $\vec{V} = u\vec{i} + v\vec{j}$

$u = -\dfrac{\partial \phi}{\partial x} \quad , \quad v = -\dfrac{\partial \phi}{\partial y}$.

$\Rightarrow u = -A - 2Bx$.

$v = +2By$

$$\vec{V} = (-A - 2Bx)\vec{i} + 2By\vec{j}$$

$$\boxed{\vec{V} = -(1+2x)\vec{i} + 2y\vec{j}}$$

b) $u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial x}$

$\Rightarrow \psi = \int u\, dy + f(x) = -\int A + 2Bx\, dy + f(x) = -Ay - 2Bxy + f(x)$

$\psi = \int -v\, dx + f(y) = \int -2By\, dx + f(y) = -2Bxy + f(y)$

compare
$f(x) = 0$
$f(y) = -Ay$

$\Rightarrow \boxed{\psi = -Ay - 2Bxy} \Rightarrow \psi = -y - 2xy$.

c) Since ϕ exists $\Rightarrow$ irrotational

Check if the flow is incompressible $\Rightarrow \dfrac{\partial u}{\partial x} + \dfrac{\partial v}{\partial y} = -2B + 2B = 0$ ✓ incompressible

We can then apply Bernoulli on a streamline $1 \longrightarrow 2$
$(0,0) \quad (1,2)$

$P_1 + \frac{1}{2}\rho V_1^2 + \cancel{\rho g z_1} = P_2 + \frac{1}{2}\rho V_2^2 + \cancel{\rho g z_2}$ we could assume $z_1 = z_2 \Rightarrow$ not a large dif

$P_2 - P_1 = \frac{1}{2}\rho V_1^2 - \frac{1}{2}\rho V_2^2 = \frac{1}{2}\rho\left(V_1^2 - V_2^2\right)$

V_1 ? and V_2 ?

$\vec{V_1}$ at $(x,y) = (0,0)$

$\vec{V_1} = -1\vec{i} \Rightarrow \|\vec{V_1}\| = 1\ m/s$

$\vec{V_2}$ at $(x,y) = (1,2)$

$\Rightarrow \vec{V_2} = -(1+2\times1)\vec{i} + 2\times2\vec{j} = -3\vec{i} + 4\vec{j} = \vec{V_2} \Rightarrow \|V_2\| = \sqrt{(-3)^2+4^2} = 5\ m/s$

$\Rightarrow P_2 - P_1 = \frac{1}{2}\rho\left(1^2 - 5^2\right) = -\frac{1}{2}\rho\times 24$

Assume fluid is water at $T = 20°C \Rightarrow \rho = 999\ kg/m^3$

$\Rightarrow P_2 - P_1 = -\frac{1}{2}\times 999\times 24 = -11{,}988\ Pa$

$\Rightarrow \boxed{P_1 - P_2 = 11{,}988\ kPa}$

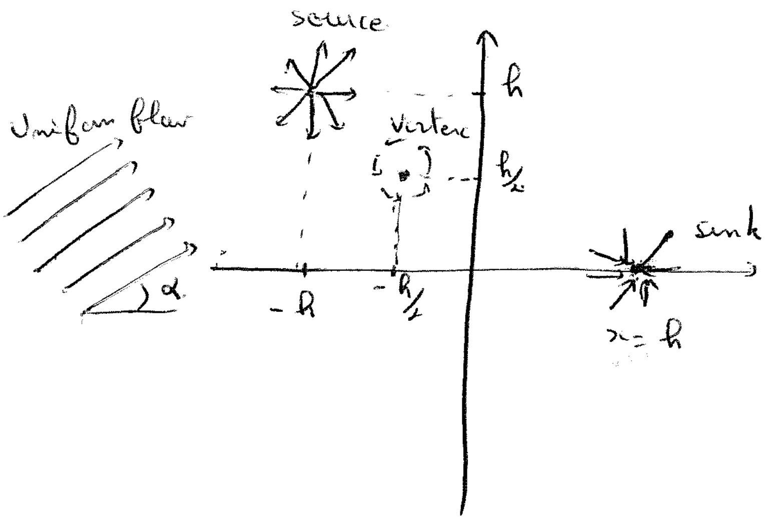
6 Create a flow of your choice

10

You can choose any combination

For this example

- A Uniform flow with an angle $\alpha = 60^\circ$ and a magnitude U
- A source of strength $q = 2\pi$ with the origin located at $(x,y) = (-h, h)$
- A sink of strength $q_2 = \frac{q}{2}$ with the origin located at $(x,y) = (h, 0)$
- A vortex counter clockwise with $\Gamma = 3\pi$, with the origin located at $(x,y) = \left(-\frac{h}{2}, \frac{h}{2}\right)$



$x = r\cos\theta$

$y = r\sin\theta$

$\Rightarrow r = \sqrt{x^2+y^2}$

The flow is irrotational and incompressible $\Rightarrow$ Satisfy Laplace Eq. $\nabla^2\Psi = 0$

$$\Rightarrow \Psi_{flow} = \Psi_{uniform} + \Psi_{source} + \Psi_{vortex} + \Psi_{sink}$$

$$\boxed{\Psi_{uniform} = Uy\cos\alpha}$$

$$\Psi_{source}\Big|_{at\ 0,0} = \frac{q}{2\pi}\theta = \frac{q}{2\pi}\operatorname{tg}^{-1}\left(\frac{y}{x}\right) \Rightarrow \text{if origin is at } (-h, h)$$

$$\Rightarrow \boxed{\Psi_{source} = \frac{q}{2\pi}\operatorname{tg}^{-1}\left(\frac{y-h}{x+h}\right)}$$

$$\Psi_{sink} = \frac{q_2}{2\pi}\theta_2 = \boxed{\frac{q}{4\pi}\operatorname{tg}^{-1}\left(\frac{y}{x-h}\right) = \Psi_{sink}}$$

$$\Psi_{vortex}\Big|_{at\ 0,0} = \frac{\Gamma}{2\pi}\ln(r) = \frac{\Gamma}{2\pi}\ln\left((x^2+y^2)^{0.5}\right) = \frac{\Gamma}{2\pi}\times 0.5\ln(x^2+y^2) = \frac{\Gamma}{4\pi}\ln(x^2+y^2)$$

if origin is at $\left(-\frac{h}{2}, \frac{h}{2}\right) \Rightarrow \boxed{\Psi_{vortex} = \frac{\Gamma}{4\pi}\ln\left(\left(x+\frac{h}{2}\right)^2 + \left(y-\frac{h}{2}\right)^2\right)}$

$$\Rightarrow \boxed{\Psi_{flow} = U_\infty y \cos\alpha + \frac{q}{2\pi} \operatorname{tg}^{-1}\left(\frac{y-h}{x+h}\right) - \frac{q}{4\pi} \operatorname{tg}^{-1}\left(\frac{y}{x-h}\right) + \frac{\Gamma}{4\pi} \ln\left(\left(x+\frac{h}{2}\right)^2 + \left(y-\frac{h}{2}\right)^2\right)}$$

You can use the contour plot in Matlab to plot the streamline.

~~The result of this flow.~~

The streamlines for this flow are plotted in Figure 3, in the next page.

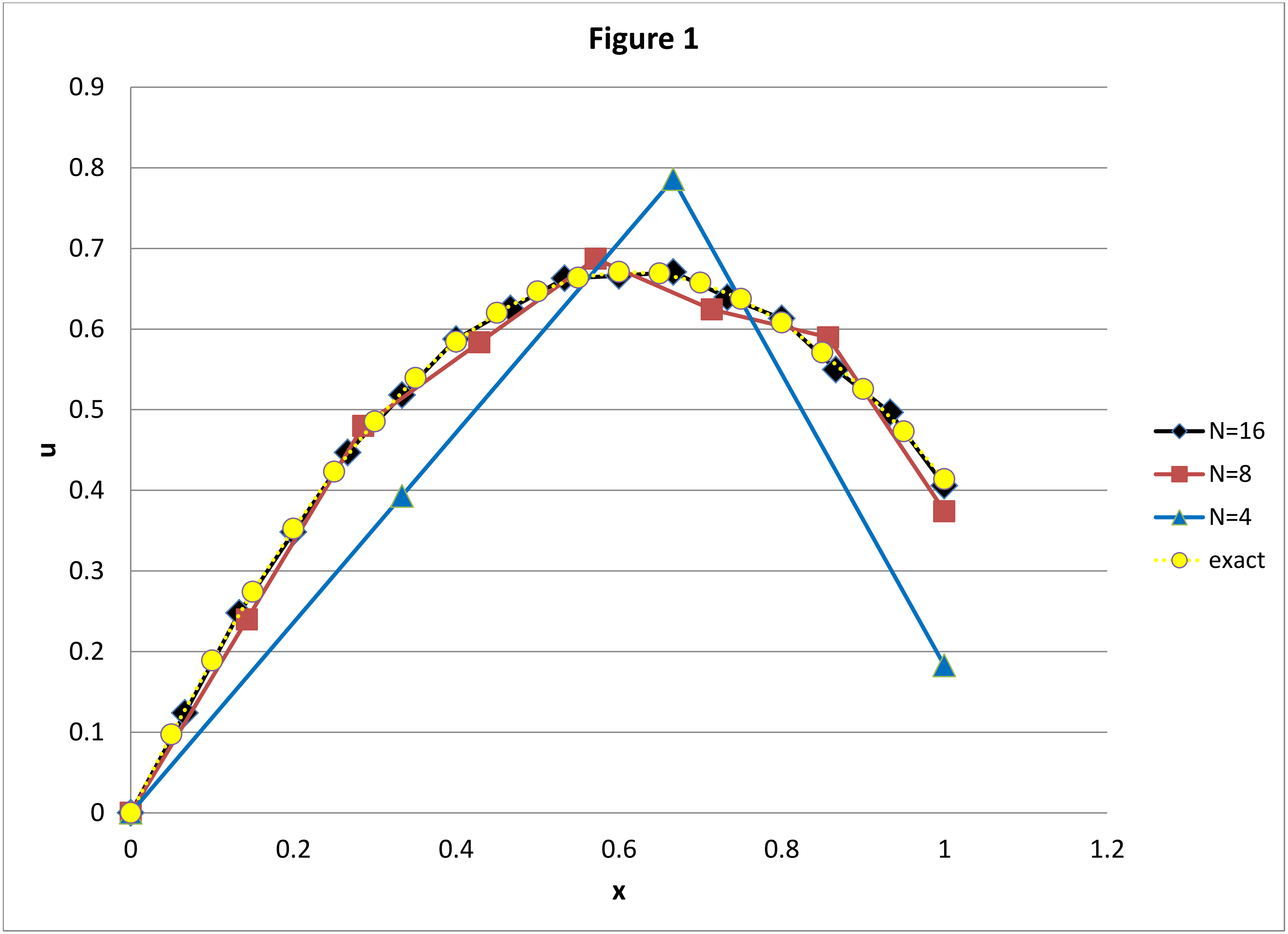


Figure 1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
u
0
0.2
0.4
0.6
0.8
1
1.2
x
N=16
N=8
N=4
exact

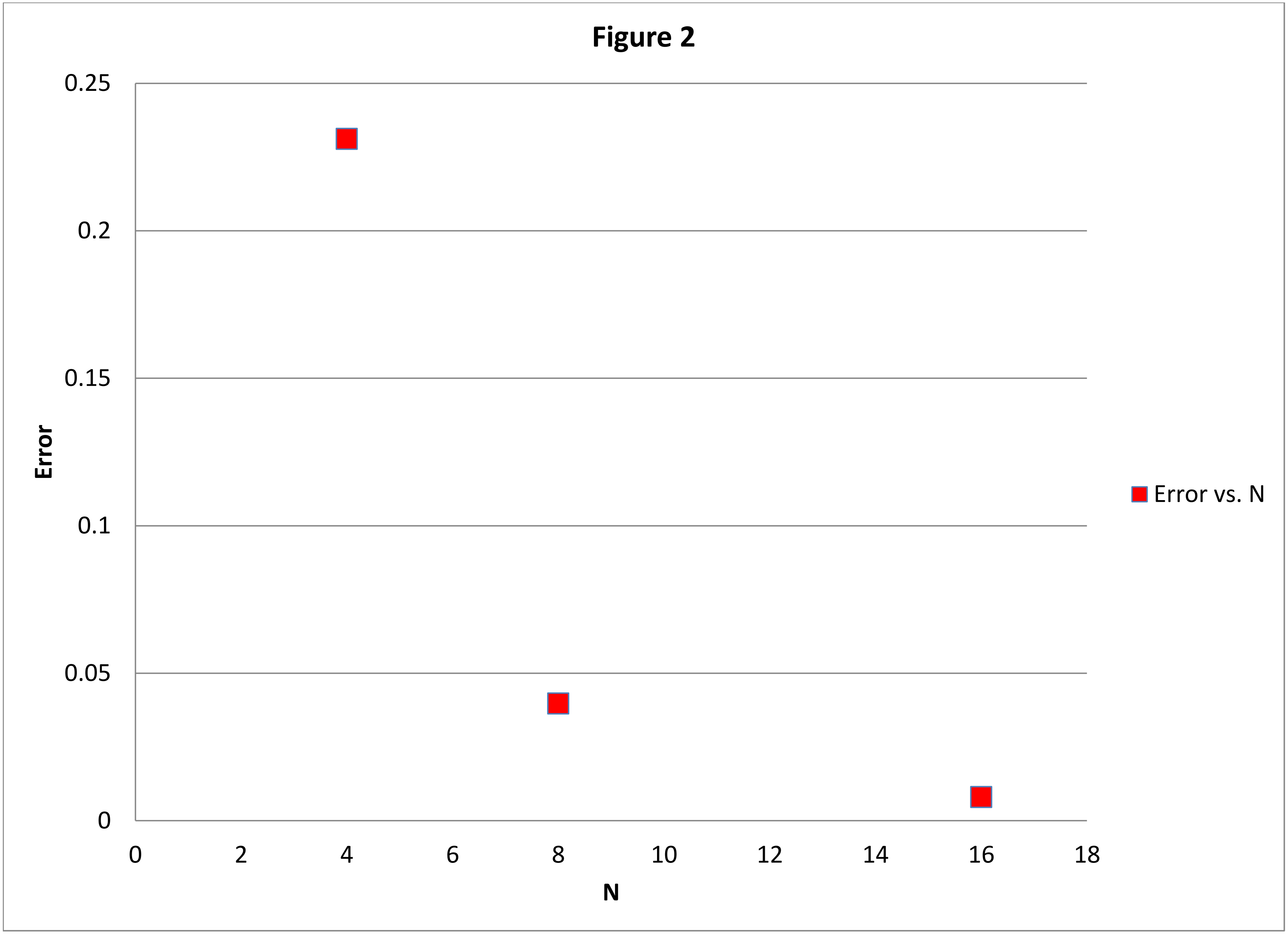


Figure 2
0.25
0.2
0.15
0.1
0.05
0
Error
0
2
4
6
8
10
12
14
16
18
N
Error vs. N

Figure 3