Solution to assignment #2

Summer 2016

5.11 / 5.73 / 5.90 / 5.94 / simplification of N-S

5.11

Incompressible laminar boundary layer parabolic variation

$u=0 \rightarrow y=0$

$u=U \rightarrow y=\delta$

$\frac{u}{U} = 2\left(\frac{y}{\delta}\right) - \left(\frac{y}{\delta}\right)^2$ where $\delta = c x^{1/2}$ where c = constant

a) Show that the simplest form of the y component is

$$\frac{v}{U} = \frac{\delta}{x}\left[\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right]$$

b) plot $\frac{v}{U}$ versus $\frac{y}{\delta}$ to find the location of Maximum $\frac{v}{U}$

Evaluate the ratio when $\delta = 5\,mm$ and $x = 0.5\,m$

a) Assumptions: * incompressible (1)

* Laminar Boundary layer (2)
* 2D flow (3)

continuity equation: $\cancel{\frac{\partial \rho}{\partial t}}^{(1)} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \cancel{\frac{\partial(\rho w)}{\partial z}}^{(3)} = 0$

$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ (I)

we know $u = \frac{2Uy}{c x^{1/2}} - \frac{y^2 U}{c^2 x}$

$\Rightarrow \frac{\partial u}{\partial x} = -\frac{1}{2} \times \frac{2Uy}{c x^{3/2}} + \frac{y^2 U}{c^2 \times x^2}$

from (I) $\Rightarrow \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} = \frac{2Uy}{2c x^{3/2}} - \frac{y^2 U}{c^2 x^2}$

$v = \int \frac{2Uy}{2c x^{3/2}} - \frac{y^2 U}{c^2 x^2}\, dy + f(x)$

$$v = \frac{1}{2}\frac{Uy^2}{c x^{3/2}} - \frac{1}{3}\frac{Uy^3}{c^2 x^2} + f x$$

(2)

$$\Rightarrow v = \frac{y^2}{c^2 x} \times \frac{1}{2}\frac{Uc}{x^{1/2}} - \frac{y^3}{c^3 x^{3/2}} \times \frac{1}{3}\frac{Uc}{x^{1/2}} + f(x)$$

$$\Rightarrow v = \frac{1}{2}\frac{y^2}{\delta^2} \times \frac{U \underbrace{c x^{1/2}}_{\delta}}{x} - \frac{1}{3}\frac{y^3}{\delta^3} \frac{Uc x^{1/2}}{x} + f(x)$$

$$\Rightarrow v = \frac{U\delta}{x}\left(\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right) + \underbrace{f(x)}_{=0}$$ we know that for $y=0$ $v=0$ due to the non-slip condition on the wall. $\Rightarrow f(x)=0$

$$\Rightarrow \frac{v}{U} = \frac{\delta}{x}\left(\frac{1}{2}\left(\frac{y}{\delta}\right)^2 - \frac{1}{3}\left(\frac{y}{\delta}\right)^3\right)$$

b) plot see Fig. 1

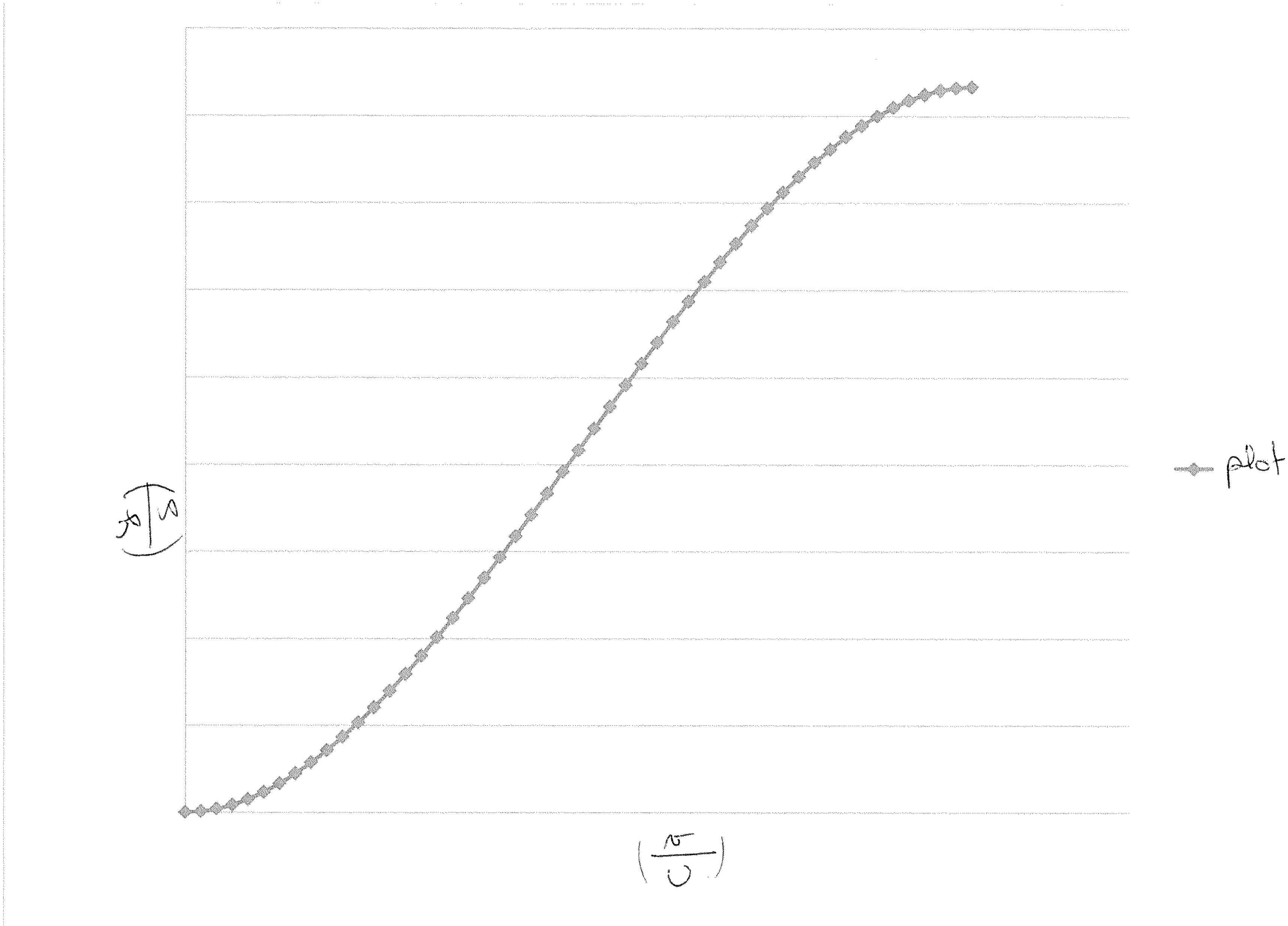
$\delta = 5$ mm $= 0{,}005$ m

$x = 5$ m $\quad \delta = c x^{1/2} \Rightarrow c = \frac{\delta}{x^{1/2}} = \frac{0{,}005}{5^{1/2}}$

We notice that the maximum value of $\frac{v}{U} = 0.00016667$ for $\frac{y}{\delta} = 1$

Or: find y for $\frac{\partial\left(\frac{v}{U}\right)}{\partial y} = 0 \Rightarrow$ for $y = \delta$

$\frac{v}{U}$ Vs. $\frac{y}{\delta}$



#5.73

flow is represented by

$$\vec{V} = \left(x^7 - 21x^5y^2 + 35x^3y^4 - 7xy^6\right)\vec{i} + \left(7x^6y - 35x^4y^3 + 21x^2y^5 - y^7\right)\vec{j}$$

a) Incompressible ?

b) Irrotational ?

a). 2D flow

if the flow is incompressible $\Rightarrow \nabla \cdot \vec{V} = 0 \Rightarrow$ for a 2D flow $\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$?

$$\frac{\partial u}{\partial x} = 7x^6 - 21y^2 \times 5x^4 + 35y^4 \times 3x^2 - 7y^6 = 7x^6 - 105y^2x^4 + 105y^4x^2 - 7y^6$$

$$\frac{\partial v}{\partial y} = 7x^6 - 35 \times 3y^2 \times x^4 + 21x^2 \times 5y^4 - 7y^6 = 7x^6 - 105x^4y^2 + 105x^2y^4 - 7y^6$$

$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \neq 0 \Rightarrow$ the flow is not incompressible $\Rightarrow$ compressible

b) For an irrotational flow $\nabla \times \vec{V} = 0$ 2D flow (1)

$$\Rightarrow \nabla \times \vec{V} = \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ u & v & w \end{vmatrix} = \vec{i}\left(\overset{(1)}{\cancel{\frac{\partial w}{\partial y}}} - \overset{(1)}{\cancel{\frac{\partial v}{\partial z}}}\right) - \hat{j}\left(\overset{(1)}{\cancel{\frac{\partial w}{\partial x}}} - \overset{(1)}{\cancel{\frac{\partial u}{\partial z}}}\right) + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\vec{k} =$$

$\Rightarrow$ if irrotational $\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0$

$$\frac{\partial v}{\partial x} = 42x^5y - 35 \times 4x^3y^3 + 21 \times 2 \times y^5 = 42x^5y - 140x^3y^3 + 42xy^5$$

$$\frac{\partial u}{\partial y} = -2 \times 21 y \times x^5 + 35 \times 4x^3y^3 - 7 \times 6xy^5 = -42x^5y + 140x^3y^3 - 42xy^5$$

$\Rightarrow \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \neq 0 \Rightarrow$ the flow is not irrotational

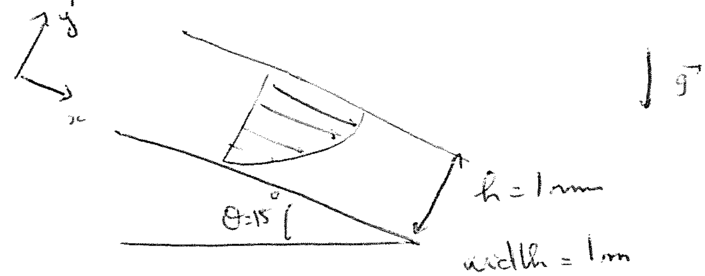
5.90

liquid film in example 5.9 not isothermal but

$$T(y) = T_0 + (T_w - T_0)\left(1 - \frac{y}{h}\right)$$ where T_0 and T_w ambient temperat and wall temperatu

$$\mu = \frac{\mu_0}{1 + a(T - T_0)} \quad \text{with } a > 0$$

Derive an expression for the velocity profile ?



+ Assumptions :
  - Incompressible (1)
  - 2D (2)
  - steady state (3)
  - Fully developed $\Rightarrow \frac{\partial}{\partial x} = 0$ (4)

Continuity equation : $\cancel{\frac{\partial \rho}{\partial t}}^{(1,3)} + \cancel{\frac{\partial(\rho u)}{\partial x}}^{(4)} + \frac{\partial(\rho v)}{\partial y} + \cancel{\frac{\partial(\rho w)}{\partial z}}^{(2)} = 0$

N-S :
$$\rho\left(\cancel{\frac{\partial u}{\partial t}}^{(3)} + \cancel{u\frac{\partial u}{\partial x}}^{(4)} + \cancel{v\frac{\partial u}{\partial y}}^{(5)} + \cancel{w\frac{\partial u}{\partial z}}^{(2)}\right) = \rho g_x - \cancel{\frac{\partial P}{\partial x}}^{(4)} + \frac{\partial}{\partial x}\left[\mu\left(2\cancel{\frac{\partial u}{\partial x}} - \cancel{\frac{2}{3}(\nabla\cdot\vec{V})}\right)\right]$$
$$+ \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \cancel{\frac{\partial v}{\partial x}}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\cancel{\frac{\partial w}{\partial x}}^{(2)} + \cancel{\frac{\partial u}{\partial z}}^{(2)}\right)\right] \quad \text{I}$$

$$\rho\left(\cancel{\frac{\partial v}{\partial t}}^{(3)} + u\cancel{\frac{\partial(\rho v)}{\partial x}}^{(5)} + v\cancel{\frac{\partial(\rho v)}{\partial y}}^{(5)} + w\cancel{\frac{\partial(\rho w)}{\partial z}}^{(5)}\right) = \rho g_y - \frac{\partial P}{\partial y} + \frac{\partial}{\partial y}\left[\mu\left(2\cancel{\frac{\partial v}{\partial y}}^{(5)} - \cancel{\frac{2}{3}\nabla\cdot\vec{V}}^{(1)}\right)\right]$$
$$+ \frac{\partial}{\partial z}\left[\mu\left(\cancel{\frac{\partial w}{\partial z}}^{(4)} + \cancel{\frac{\partial v}{\partial y}}^{(2)}\right)\right] + \frac{\partial}{\partial x}\left[\cancel{\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)}^{(4)}\right] \quad \text{II}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial(\rho w)}{\partial x} + v\frac{\partial(\rho w)}{\partial y} + w\frac{\partial(\rho w)}{\partial z}\right) = \rho g_z - \frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial w}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$
$$+ \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right]^{(4)} \quad \text{III}$$

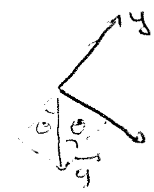
* From continuity equation

$$\frac{\partial(\rho v)}{\partial y} = 0 \Rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow v = C_1$$

we know that for $y = 0 \Rightarrow \left.\begin{array}{l} u = 0 \\ v = 0 \end{array}\right\}$ no-slip at the wall

if $v = 0$ for $y = 0$ and v is a constant $\Rightarrow v = 0$ everywhere (5)

* From (I) $\Rightarrow \rho g_x + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) = 0$ $\qquad g_x = g\sin\theta$
* From (II) $\Rightarrow \rho g_y - \frac{\partial P}{\partial y} = 0$ $\qquad g_y = -g\cos\theta$



$$\rho g \sin\theta = -\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) \Rightarrow \text{integrate}$$

$$\Rightarrow \int \rho g \sin\theta \, dy = \int -\frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y}\right) dy$$

$$\Rightarrow -\rho g \sin\theta \, y = \mu \frac{\partial u}{\partial y} + C_2$$

we know that at $y = h$ (free surface) $\Rightarrow \frac{\partial u}{\partial y} = 0 \Rightarrow C_2 = -\rho g \sin\theta \times h$

$$\Rightarrow -\rho g \sin\theta \, y + \rho g \sin\theta \, h = \mu \frac{\partial u}{\partial y}$$

$$\Rightarrow \rho g \sin\theta (h - y) = \mu \frac{\partial u}{\partial y} \Rightarrow \frac{\partial u}{\partial y} = \frac{\rho g \sin\theta}{\mu}(h - y)$$

$$\mu = f(y) \Rightarrow \mu = \frac{\mu_0}{1 + a(T - T_0)} = \frac{\mu_0}{1 + a(T_w - T_0)\left(1 - \frac{y}{h}\right)}$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\rho g \sin\theta}{\mu_0}(h - y)\left(1 + a(T_w - T_0)\left(1 - \frac{y}{h}\right)\right)$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\rho g h \sin\theta}{\mu_0}\left(1 - \frac{y}{h}\right)\left(1 + a(T_w - T_0)\left(1 - \frac{y}{h}\right)\right)$$

$$\Rightarrow \frac{\partial u}{\partial y} = \frac{\rho g h \sin\theta}{\mu_0}\left(\left(1 - \frac{y}{h}\right) + a(T_w - T_0)\left(1 - \frac{y}{h}\right)^2\right) \Rightarrow \text{integrate}$$

$$u(y) = \int \frac{\rho g h \sin\theta}{\mu_0}\left(\left(1 - \frac{y}{h}\right) + a(T_w - T_0)\left(1 - \frac{y}{h}\right)^2\right) dy + C_3$$

$$\Rightarrow u(y) = \frac{\rho g h \sin\theta}{\mu_0}\left[y - \frac{1}{2}\frac{y^2}{h} + a(T_w - T_0)\left(y - \frac{y^2}{h} + \frac{y^3}{3h^2}\right)\right] + C_3$$

3.c $\Rightarrow$ for $y = 0$ $u = 0 \Rightarrow C_3 = 0$

$$\Rightarrow \boxed{u(y) = \frac{\rho g h \sin\theta}{\mu_0}\left(y - \frac{1}{2}\frac{y^2}{h} + a(T_w - T_0)\left(y - \frac{y^2}{h} + \frac{y^3}{3h^2}\right)\right)}$$

5.94

6

Liquid Film in example 5.9 is horizontal $\Rightarrow \theta = 0°$

The flow is driven by a constant shear stress $y = h \Rightarrow \tau_{yx} = C$

Fully developed and zero net flow rate $\Rightarrow Q = 0$

a) velocity profile $u(y)$

b) pressure gradient $\frac{dP}{dx}$

Assumptions:
* Incompressible (1)
* steady (2)
* horizontal $\theta = 0°$ (3)
* Fully developed $\Rightarrow \frac{\partial}{\partial x} = 0$ (4)
* zero net flow rate (5)
* no gravity effect (6)
* 2D flow

$$\cancel{\frac{\partial u}{\partial x}}^{(4)} + \frac{\partial v}{\partial y} + \cancel{\frac{\partial w}{\partial z}}^{(7)} = 0 \quad \text{I}$$

$$\rho\left(\cancel{\frac{\partial u}{\partial t}}^{(2)} + u\cancel{\frac{\partial u}{\partial x}}^{(4)} + v\cancel{\frac{\partial u}{\partial y}}^{(8)} + w\cancel{\frac{\partial u}{\partial z}}^{(7)}\right) = \cancel{\rho g_x}^{(6)} - \frac{\partial P}{\partial x} + \mu\left(\cancel{\frac{\partial^2 u}{\partial x^2}}^{(4)} + \frac{\partial^2 u}{\partial y^2} + \cancel{\frac{\partial^2 u}{\partial z^2}}^{(7)}\right) \quad \text{II}$$

$$\rho\left(\cancel{\frac{\partial v}{\partial t}}^{(2)} + u\cancel{\frac{\partial v}{\partial x}}^{(8)} + v\cancel{\frac{\partial v}{\partial y}}^{(8)} + w\cancel{\frac{\partial v}{\partial z}}^{(7)}\right) = \cancel{\rho g_y}^{(6)} - \frac{\partial P}{\partial y} + \mu\left(\cancel{\frac{\partial^2 v}{\partial x^2}}^{(4)} + \cancel{\frac{\partial^2 v}{\partial y^2}}^{(8)} + \cancel{\frac{\partial^2 v}{\partial z^2}}^{(7)}\right) \quad \text{III}$$

$$\cancel{\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial P}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)}^{(7)} \quad \text{IV}$$

I $\rightarrow \frac{\partial v}{\partial y} = 0 \Rightarrow v = \text{constant} \Rightarrow$ we know for $y = 0 \Rightarrow \left.\begin{array}{l} v = 0 \\ u = 0 \end{array}\right\}$ no-slip at the wa

$\Rightarrow v = 0$ everywhere (8)

II $\Rightarrow -\frac{\partial P}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} = 0 \Rightarrow \mu\frac{\partial^2 u}{\partial y^2} = \frac{\partial P}{\partial x} \Rightarrow \int\frac{\partial^2 u}{\partial y^2} = \int\frac{1}{\mu}\frac{\partial P}{\partial x} \Rightarrow$ integrate

$\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{\partial P}{\partial x}y + C_1$ integrate again $\Rightarrow$ V

$u(y) = \frac{1}{2\mu}\frac{\partial P}{\partial x}y^2 + C_1 y + C_2$ VI

B.C $\Rightarrow y = 0 \Rightarrow u = 0 \Rightarrow C_2 = 0$

$\Rightarrow y = h \Rightarrow \tau_{xy} = C \Rightarrow \mu\frac{\partial u}{\partial y} = C \Rightarrow \frac{\partial u}{\partial y} = \frac{C}{\mu}$

V $\frac{c}{\mu} = \frac{1}{\mu}\frac{\partial P}{\partial x} h + C_1$

$\Rightarrow C_1 = \frac{c}{\mu} - \frac{h}{\mu}\frac{\partial P}{\partial x}$

$\Rightarrow u(y) = \frac{1}{2\mu}\frac{\partial P}{\partial x} y^2 + \frac{cy}{\mu} - \frac{hy}{\mu}\frac{\partial P}{\partial x}$

$$\boxed{u(y) = \frac{\partial P}{\partial x}\left(\frac{y^2}{2\mu} - \frac{hy}{\mu}\right) + \frac{cy}{\mu}}$$

b) $\frac{\partial P}{\partial x}$?

zero net flow rate $\Rightarrow Q = 0$

$\Rightarrow Q = \int_{\mu} u \, dA = \int_0^h \int_0^w u(y)\, dz\, dy = 0$

$\Rightarrow Q = \int_0^h \int_0^w \frac{\partial P}{\partial x} \times \frac{1}{\mu}\left(\frac{y^2}{2} - hy\right) + \frac{cy}{\mu} \; dz\, dy = 0$

$\Rightarrow \frac{\partial P}{\partial x}\frac{1}{\mu}\int_0^h w\left(\frac{y^2}{2} - hy\right) dy + \frac{c}{\mu}\int_0^h y\, dy = 0$

$\Rightarrow \frac{\partial P}{\partial x} \times \frac{1}{\mu} \times w \left[\frac{y^3}{6} - \frac{hy^2}{2}\right]_0^h + \frac{c}{\mu}\left[\frac{1}{2}y^2\right]_0^h = 0$

$\Rightarrow \frac{w}{\mu}\frac{\partial P}{\partial x}\left(\frac{h^3}{6} - \frac{3h^3}{\cancel{2}}\right) + \frac{c}{\mu} \times \frac{1}{2} h^2 = 0$

$\Rightarrow \frac{\partial P}{\partial x} = \left(-\frac{c}{\cancel{\mu}} \times \frac{1}{2} h^{\cancel{2}}\right) \times \frac{\cancel{\mu}}{w} \times \left(\frac{-3}{h^{\cancel{3}}}\right)$

$\Rightarrow \frac{\partial P}{\partial x} = \frac{c}{2} \times \frac{1}{w} \times \frac{3}{h} = \boxed{\frac{3}{2}\frac{c}{wh} = \frac{\partial P}{\partial x}}$

for $h = 1\,mm$ and ~~to~~ $w = 1\,m$

$\Rightarrow \frac{\partial P}{\partial x} = \frac{3}{2} \times \frac{c}{0.001}$

Simplify N-S

General equations: eqs 5.28 in the book

$$\frac{\partial \rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0 \qquad \mathrm{I}$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}\left[\mu\left(2\frac{\partial u}{\partial x} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] \qquad \mathrm{II}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial p}{\partial y} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] \qquad \mathrm{III}$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial p}{\partial z} + \frac{\partial}{\partial x}\left[\mu\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\mu\left(2\frac{\partial w}{\partial z} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right] \qquad \mathrm{IV}$$

a) 2D Stokes flow in y and z.

Stokes flow ⇒ Re << 1 ⇒ viscous effect dominates
⇒ drop the inertial terms.

Assumption:
* 2D in y and z (1)
* Stokes flow ⇒ drop the convective terms. (2)
* Stokes flow ⇒ incompressible (3)

⇒ I ⇒ $\cancel{\frac{\partial \rho}{\partial t}}^{(3)} + \cancel{\frac{\partial (\rho u)}{\partial x}}^{(1)} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial}{\partial z}(\rho w) = 0$

II ⇒ canceled because of (1)

III $\rho\left(\frac{\partial v}{\partial t} + \cancel{u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}}^{(2)}\right) = \rho g_y - \frac{\partial P}{\partial y} + \cancel{\frac{\partial}{\partial x}\left[\eta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]}^{(1)}$

$+ \frac{\partial}{\partial y}\left[\eta\left(2\frac{\partial v}{\partial y} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right] + \frac{\partial}{\partial z}\left[\eta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$?

$\vec{\nabla}\cdot\vec{V} = \cancel{\frac{\partial u}{\partial x}}^{(1)} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}$ V

⇒ III ⇒ $\rho\frac{\partial v}{\partial t} = \rho g_y - \frac{\partial P}{\partial y} + \frac{\partial}{\partial y}\left[\eta\left(\frac{2\partial v}{\partial y} - \frac{2}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[\eta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$

IV $\rho\left(\frac{\partial w}{\partial t} + \cancel{u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}}^{(2)}\right) = \rho g_z - \frac{\partial P}{\partial z} + \cancel{\frac{\partial}{\partial x}\left[\eta\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]}^{(1)} + \frac{\partial}{\partial y}\left[\eta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$

$+ \frac{\partial}{\partial z}\left[\eta\left(\frac{2\partial w}{\partial z} - \frac{2}{3}\nabla\cdot\vec{V}\right)\right]$ replace with V

⇒ $\rho\frac{\partial w}{\partial t} = \rho g_z - \frac{\partial P}{\partial z} + \frac{\partial}{\partial y}\left[\eta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\eta\left(\frac{4\partial w}{\partial z} - \frac{2}{3}\left(\frac{\partial v}{\partial y} - \frac{\partial w}{\partial z}\right)\right)\right]$

<u>In summary:</u> ~~fringe~~

$\cancel{\frac{\partial \rho}{\partial t}} + \frac{\partial (\rho v)}{\partial y} + \frac{\partial (\rho w)}{\partial z} = 0$

$\rho\frac{\partial v}{\partial t} = \rho g_y - \frac{\partial P}{\partial y} + \frac{\partial}{\partial y}\left[\eta\left(\frac{4}{3}\frac{\partial v}{\partial y} - \frac{2}{3}\frac{\partial w}{\partial z}\right)\right] + \frac{\partial}{\partial z}\left[\eta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$

$\rho\frac{\partial w}{\partial t} = \rho g_z - \frac{\partial P}{\partial z} + \frac{\partial}{\partial y}\left[\eta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right] + \frac{\partial}{\partial z}\left[\eta\left(\frac{4}{3}\frac{\partial w}{\partial z} - \frac{2}{3}\frac{\partial v}{\partial y}\right)\right]$

If incompressible
⇒ $\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$ ⇒ $\nabla\cdot\vec{V} = 0$ more simplified

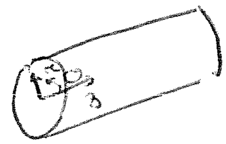
⇒

b). ~~Ebo~~ fully developed circular pipe flow

b

N-S equations in cylindrical coordinates

Newtonian



$$\frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\rho r V_r\right) + \frac{1}{r}\cancel{\frac{\partial \rho V_\theta}{\partial \theta}}^{(2)} + \cancel{\frac{\partial \rho V_z}{\partial z}}^{(1)} = 0 \qquad \text{(I)}$$

$$\cancel{\frac{\partial \rho}{\partial t}} \quad \rho\left(\frac{\partial V_r}{\partial t} + V_r\frac{\partial V_r}{\partial r} + \cancel{\frac{V_\theta}{r}\frac{\partial V_r}{\partial \theta}}^{(2)} - \cancel{\frac{V_\theta^2}{r}}^{(2)} + \cancel{V_z\frac{\partial V_r}{\partial z}}^{(1)}\right) = \rho g_r - \frac{\partial p}{\partial r}$$
$$+ \mu\left\{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r V_r)\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 V_r}{\partial \theta^2}}^{(2)} - \cancel{\frac{2}{r^2}\frac{\partial V_\theta}{\partial \theta}}^{(2)} + \cancel{\frac{\partial^2 V_r}{\partial z^2}}^{(1)}\right\} \qquad \text{(II)}$$

$$\rho\left(\cancel{\frac{\partial V_\theta}{\partial t}}^{(2)} + \cancel{V_r\frac{\partial V_\theta}{\partial r}}^{(2)} + \cancel{\frac{V_\theta}{r}\frac{\partial V_\theta}{\partial \theta}}^{(2)} + \cancel{\frac{V_r V_\theta}{r}}^{(2)} + \cancel{V_z\frac{\partial V_\theta}{\partial z}}^{(1)}\right) = \cancel{\rho g_\theta} - \cancel{\frac{1}{r}\frac{\partial p}{\partial \theta}}^{(1)}$$
$$+ \mu\left\{\cancel{\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r V_\theta)\right)}^{(2)} + \cancel{\frac{1}{r^2}\frac{\partial^2 V_\theta}{\partial \theta^2}}^{(2)} + \cancel{\frac{2}{r^2}\frac{\partial V_r}{\partial \theta}}^{(2)} + \cancel{\frac{\partial^2 V_\theta}{\partial z^2}}\right\} \qquad \text{(III)}$$

$$\rho\left(\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r} + \cancel{\frac{V_\theta}{r}\frac{\partial V_z}{\partial \theta}}^{(2)} + \cancel{V_z\frac{\partial V_z}{\partial z}}^{(1)}\right) = \rho g_z - \frac{\partial p}{\partial z} \quad \text{keep}$$
$$+ \mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right) + \cancel{\frac{1}{r^2}\frac{\partial^2 V_z}{\partial \theta^2}}^{(2)} + \cancel{\frac{\partial^2 V_z}{\partial z^2}}^{(1)}\right\} \qquad \text{(IV)}$$

Assumptions:

* fully developed $\Rightarrow \frac{\partial}{\partial z} = 0$ ~~and~~ (1)
* circular pipe $\Rightarrow$ axisymmetric $\Rightarrow$ no variation in $\theta \Rightarrow \frac{\partial}{\partial \theta} = 0$ (2)
  - if steady $\Rightarrow$ further simplifies $\Rightarrow \frac{\partial}{\partial t} = 0$
  - if pipe is horizontal $\Rightarrow$ drops gravity terms.
  - if incompressible $\Rightarrow \rho = c^{te}$

(I) $\Rightarrow \frac{\partial \rho}{\partial t} + \frac{1}{r}\frac{\partial}{\partial r}\left(\rho r V_r\right) = 0$

(II) $\Rightarrow \rho\left(\frac{\partial (V_r)}{\partial t} + V_r\frac{\partial V_r}{\partial r}\right) = \rho g_r - \frac{\partial p}{\partial r} + \mu\left(\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial}{\partial r}(r V_r)\right)\right)$

(III) cancel out

(IV) $\rho\left(\frac{\partial V_z}{\partial t} + V_r\frac{\partial V_z}{\partial r}\right) = \rho g_z - \frac{\partial p}{\partial z} + \mu\left\{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V_z}{\partial r}\right)\right\}$

c) Flow pass a fish :

⇒ Flow in water ⇒ incompressible fluid. fish in 3D

(11)

Assumptions ⇒

* incompressible (1) ⇒ $\nabla . \vec{V} = 0$

$$\text{I} \Rightarrow \cancel{\frac{\partial \rho}{\partial t}}^{(1)} + \frac{\partial (\rho u)}{\partial x} + \frac{\partial}{\partial y}(\rho v) + \frac{\partial}{\partial z}(\rho w) = 0$$

$$\Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \nabla . \vec{V} = 0$$

$$\text{II} \Rightarrow \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial P}{\partial x} + \frac{\partial}{\partial x}\left[\eta\, \frac{2\partial u}{\partial x} - \cancel{\frac{2}{3}\nabla . \vec{V}}^{(1)\,0}\right]$$

$$+ \frac{\partial}{\partial y}\left[\eta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right] + \frac{\partial}{\partial z}\left[\eta\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right]$$

$$\text{II} \Rightarrow \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial P}{\partial y} + \frac{\partial}{\partial x}\left[\eta\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\right]$$

$$+ \frac{\partial}{\partial y}\left[\eta\left(\frac{2\partial v}{\partial y} - \cancel{\frac{2}{3}\vec{\nabla} . \vec{V}}^{0\,(1)}\right)\right] + \frac{\partial}{\partial z}\left[\eta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$

$$\text{III}\ \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial P}{\partial z} + \frac{\partial}{\partial x}\left[\eta\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right)\right] + \frac{\partial}{\partial y}\left[\eta\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\right]$$

$$+ \frac{\partial}{\partial z}\left[\eta\left(\frac{2\partial w}{\partial z} - \cancel{\frac{2}{3}\nabla . \vec{V}}^{0\,(1)}\right)\right]$$

We could assume that the water is newtonian and η doesn't change with temperature for example ⇒

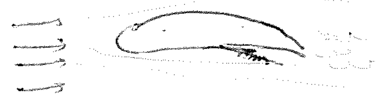
$$\text{I}\ \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \Rightarrow \nabla . \vec{V} = 0$$

$$\Rightarrow \text{II}\ \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial u}{\partial z}\right) = \rho g_x - \frac{\partial P}{\partial x} + \eta\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right)$$

$$\text{III}\ \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = \rho g_y - \frac{\partial P}{\partial y} + \eta\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right)$$

$$\text{IV}\ \rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = \rho g_z - \frac{\partial P}{\partial z} + \eta\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right)$$

d) flow around a ~~g~~ glider airplane wing cross section.



Assumption:

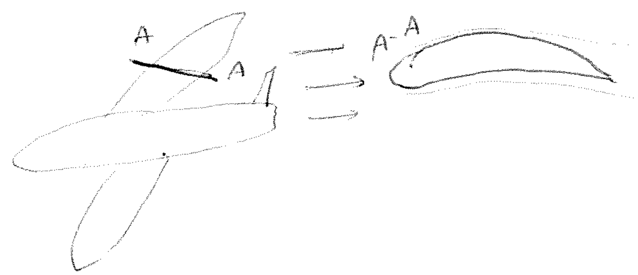
* 2D (1)
* could assume inviscid (in air and far from the wing) (2) → drop viscous terms
* ignore g for air (3)
* Check Mach Number ~~for~~ To check incompressible flow → assume incompressible (4)

$$\text{I}\quad \cancel{\frac{\partial \rho}{\partial t}}^{(4)} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial}{\partial y}(\rho v) + \cancel{\frac{\partial}{\partial z}(\rho w)}^{(1)} = 0 \Rightarrow \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\text{II}\quad \rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y} + w\cancel{\frac{\partial u}{\partial z}}^{(1)}\right) = \cancel{\rho g_x}^{(3)} - \frac{\partial p}{\partial x} + \cancel{\frac{\partial}{\partial x}\cdots}^{(2)}$$

$$\text{III}\quad \rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\cancel{\frac{\partial v}{\partial z}}^{(1)}\right) = \cancel{\rho g_y}^{(3)} - \frac{\partial p}{\partial y} + \cancel{\frac{\partial}{\partial x}\cdots}^{(2)}$$

IV → cancels out because of (1)



In Summary:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial x}$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y}\right) = -\frac{\partial p}{\partial y}$$

→ Since for the glider airplane, viscous effect are important → leave the viscous terms. could assume it is newtonian → simplifies further