MATH2004B — Test 4: 16:35–17:25, Nov 8

Surname ______________ **First Name** ______________**Student #** __________

Total: 15 points. No partial marks for Questions 1-3.
Closed book! Non-programmer calculators are allowed!

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1. (1 point) Find the constant k such that the vector field $\vec{F} = (\sin x + kxy, x^2 - e^y)$ is conservative.

   (a) -1 (b) 1 (c) -3 (d) 2 (e) 5

   Solution: (d)

   Let $P = \sin x + kxy$, $Q = x^2 - e^y$. Then $P_y = kx$, $Q_x = 2x$. Since $P_y = Q_x$, $k = 2$.

2. (1 point) Let $\vec{F}(x, y, z) = \ln(2x + z)\vec{i} + xy^3\vec{j} + xyz\vec{k}$. Find div $\vec{F}(1, 1, -1)$.

   (a) 1 (b) -3 (c) 5 (d) 6 (e) -5

   Solution: (d).

$$div\vec{F} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{2}{2x + z} + 3xy^2 + xy.$$

3. (1 point) Let $\vec{F}(x, y, z) = xz\vec{i} + xy^3\vec{j} + xyz\vec{k}$. Find curl $\vec{F}(1, 2, -3)$.

   (a) $-3\vec{i}+7\vec{j}+8\vec{k}$ (b) $-3\vec{i}+5\vec{j}+8\vec{k}$ (c) $-3\vec{i}+5\vec{j}+4\vec{k}$ (d) $3\vec{i}+5\vec{j}+6\vec{k}$ (e) $3\vec{i}+5\vec{j}+8\vec{k}$

   Solution: (a).

$$curl\vec{F} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

$$= (xz)\,\vec{i} + (x - yz)\,\vec{j} + \left(y^3\right)\vec{k}.$$

4. (5 points) Find the potential function $f(x, y, z)$ of the vector field
$\vec{F} = (e^{3z}, 2yz^3 + 1, 3xe^{3z} + 3y^2z^2)$.

Solution: Let $f(x, y, z)$ be a potential function. Then $F = (f_x, f_y, f_z) = (e^{3z}, 2yz^3 + 1, 3xe^{3z} + 3y^2z^2)$,

$$f_x = e^{3z}, f_y = 2yz^3 + 1, f_z = 3xe^{3z} + 3y^2z^2.$$

(1 point)

$$\begin{gathered} f_x = e^{3z} \Rightarrow f(x, y, z) = xe^{3z} + g(y, z) \Rightarrow \\ f_y = g_y = 2yz^3 + 1 \Rightarrow g(y, z) = y^2z^3 + y + h(z), \Rightarrow \\ f(x, y, z) = xe^{3z} + y^2z^3 + y + h(z) \Rightarrow \\ f_z = 3xe^{3z} + 3y^2z^2 + y + h'(z) = 3xe^{3z} + 3y^2z^2, \Rightarrow \\ h'(z) = 0 \Rightarrow h(z) = constant, say, C. \end{gathered}$$

(3 points)

Hence,

$$f = xe^{3z} + y^2z^3 + y + C.$$

(1 point)

5. (7 points) Let $f(x, y, z) = x + 3y + 2z,\ g(x, y, z) = 2x^2 + 2y^2 + z^2$.
(i) Calculate $\bigtriangledown f(x, y, z)$ and $\bigtriangledown g(x, y, z)$.
(ii) Use the method of Lagrange multipliers to find the maximum and minimum values of $f(x, y, z)$ subject to the constraint $g(x, y, z) = 2x^2 + 2y^2 + z^2 = 36$.

Solution: (i)

$$\bigtriangledown f(x, y, z) = (1, 3, 2), \quad \bigtriangledown g(x, y, z) = (4x, 4y, 2z).$$

(2 points)

(ii) We solve

$$\begin{aligned} &\bigtriangledown f - \lambda \bigtriangledown g = 0 \qquad (\textbf{1 point for writing this equation.}) \\ &g(x, y) = 36. \end{aligned}$$

We imply that

$$\begin{aligned} &1 - 4\lambda x = 0 \\ &3 - 4\lambda y = 0 \\ &2 - 2\lambda z = 0 \\ &2x^2 + 2y^2 + z^2 = 36. \end{aligned}$$

So we have

$$x = \frac{1}{4\lambda}, \qquad y = \frac{3}{4\lambda}, \qquad z = \frac{1}{\lambda}.$$

Hence

$$\frac{1}{8\lambda^2} + \frac{9}{8\lambda^2} + \frac{1}{\lambda^2} = 36,$$

which gives $\lambda = \pm\frac{1}{4}$.

When $\lambda = \frac{1}{4}$, we have $x = 1$, $y = 3$, and $z = 4$.

When $\lambda = -\frac{1}{4}$, we have $x = -1$, $y = -3$, and $z = -4$.

(3 points)

The maximum is $f(1, 3, 4) = 18$.

The minimum is $f(-1, -3, -4) = -18$. (1 point)