MATH2004B — Test 3: 16:35–17:25, Nov 1

Surname ______________ **First Name** ______________ **Student #** __________

Total: 15 points. No partial marks for Questions 1-4.
Closed book! Non-programmer calculators are allowed!

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1. (1 point) Find $\dfrac{\partial z}{\partial x}(1,-1,1)$, if $x^2y^3+z^4+5xyz=3-x^4$.

   (a) $-1/3$ (b) $-7/9$ (c) -3 (d) -2 (e) -7

   Solution: (c)
   Let $f(x,y,z)=x^2y^3+z^4+5xyz-3+x^4$. Then

$$\frac{\partial z}{\partial x}=-\frac{f_x(x,y,z)}{f_z(x,y,z)}=-\frac{2xy^3+5yz+4x^3}{4z^3+5xy}\Rightarrow$$

$$\frac{\partial z}{\partial x}(1,-1,1)=-3.$$

2. (1 point) Let $f(x,y)=x^2-xy$. Find the maximum rate of change of $f(x,y)$ at $(5,-2)$.

   (a) $\sqrt{89}$ (b) 3 (c) 11 (d) 13 (e) $\sqrt{79}$

   Solution: (d).
   $\nabla f(x,y)=(2x-y,-x),\Rightarrow \nabla f(5,-2)=(12,-5),\Rightarrow |\nabla f(5,-2)|=13.$

3. (1 point) Let $z=f(x,y)=x+2y$, where $x=r\cos t$, $y=r\sin t$. Find $\frac{\partial f}{\partial t}$ at $(r,t)=(2,\frac{\pi}{4})$.

   (a) $\sqrt{2}$ (b) $2\sqrt{2}$ (c) $3\sqrt{2}$ (d) $4\sqrt{2}$ (e) $5\sqrt{2}$

   Solution: (a). At $(r,t)=(2,\frac{\pi}{4})$, $x=r\cos t=\sqrt{2}$, $y=r\sin t=\sqrt{2}$.

$$\frac{\partial f}{\partial t}=\frac{\partial f}{\partial x}\frac{\partial x}{\partial t}+\frac{\partial f}{\partial y}\frac{\partial y}{\partial t}$$

$$=(1)(-r\sin t)+(2)r\cos t=\sqrt{2}.$$

4. (1 point) Let the curve C be $\vec{r}(t) = (t^2, -t^2, t)$. Find the unit tangent vector at $t = 1$.

(a) $(\frac{2}{9}, -\frac{2}{9}, \frac{1}{9})$ (b) $(-\frac{2}{9}, -\frac{2}{9}, \frac{1}{9})$ (c) $(-\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})$ (d) $(\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3})$ (e) $(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3})$

Solution: (e).
$\vec{r}\,'(t) = (2t, -2t, 1)$, $\vec{r}\,'(1) = (2, -2, 1)$, $|\vec{r}\,'(1)| = 3$. Thus

$$\vec{T}(1) = (\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}).$$

5. (3 points) Find the directional derivative of the function $f(x, y, z) = x^3y^6 + 3\ln z$ at the point $(2, 1, 1)$ in the direction $(2, -1, -2)$.

Solution: $\vec{u} = \frac{(2,-1,-2)}{|(2,-1,-2)|} = (\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3})$. (0.5 points)
$f_x = 3x^2y^6$, $f_y = 6x^3y^5$, $f_z = \frac{3}{z}, \Rightarrow \nabla f(2, 1, 1) = (12, 48, 3)$. (1 point)

$$f_{\vec{u}}(2, 1, 1) = D_{\vec{u}}f(2, 1, 1) = \nabla f(2, 1, 1) \cdot \vec{u} = (12, 48, 3) \cdot (\frac{2}{3}, -\frac{1}{3}, -\frac{2}{3}) = 10.$$

(1.5 points)

6. (3 points) Find the equation of the tangent plane of the surface $z - 2xy = e^{x+y}$ at the point $(1, -1, -1)$.

Solution: Let $F(x, y, z) = z - 2xy - e^{x+y}$. Then

$$F_x = -e^{x+y} - 2y, F_y = -e^{x+y} - 2x, F_z = 1., \Rightarrow \nabla F(1, -1, -1) = (1, -3, 1).$$

(1 points)

Thus the equation of the tangent plane at the point $(1, -1, -1)$ is

$$\nabla F(1, -1, -1) \cdot (x - 1, y + 1, z + 1) = 0,$$

(1 points)

$$x - 3y + z = 3.$$

(1 points)

7. (5 points) Find and classify the critical points of $f(x,y) = 2x^3 + y^2 - 6xy + 4y$.

Solution: $f_x(x,y) = 6x^2 - 6y,\ f_y(x,y) = 2y - 6x + 4.$

(1 point)

Setting $f_x = 0$ and $f_y = 0$:

(0.5 points)

$6x^2 - 6y = 0,\ 2y - 6x + 4 = 0,\ \Rightarrow x^2 = 3x - 2$. We imply that $x = 1, 2$. Thus critical points are:

$$(1,1), \quad (2,4).$$

(1 points)

To test all of them, we use the Second-Partials Test.

$$f_{xx} = 12x, f_{yy} = 2, f_{xy} = -6.$$

(1 points)

$$d(x,y) = f_{xx}(x,y)f_{yy}(x,y) - (f_{xy}(x,y))^2.$$

(0.5 points)

- $d(1,1) = -12 < 0$, so $(1,1)$ is a saddle point.
- $d(2,4) = 12 < 0,\ f_{xx}(2,4) = 24 > 0$, so $(2,4)$ is a minimum point.

(1 point)