MATH2004B — Test 2: 16:35–17:25, Oct 11

Surname ______________ **First Name** ______________**Student #** __________

Total: 15 points. No partial marks for Questions 1-4.
Closed book! Non-programmer calculators are allowed!

1. (1 point) Converting the rectangular coordinates $(x, y) = (-3, \sqrt{3})$ to polar coordinates (r, θ), then $\theta =$

   (a) $\frac{2\pi}{3}$ (b) $\frac{5\pi}{6}$ (c) $\frac{3\pi}{4}$ (d) $\frac{4\pi}{5}$ (e) $\frac{6\pi}{7}$

   Solution: (b).

$$\tan\theta = \frac{y}{x} = -\frac{\sqrt{3}}{3}, \Rightarrow \theta = \frac{5\pi}{6}.$$

2. (1 point) Given the polar equation $r = 4\sin\theta$. Find the rectangular equation.

   (a) $x = 2y$ (b) $x^2 = 4y$ (c) $y^2 = 4x$ (d) $x^2 + y^2 = 4y$ (e) $x^2 + y^2 = 4x$

   Solution: (d)
   $x = r\cos\theta = 4\cos\theta\sin\theta, y = r\sin\theta = 4\sin^2\theta \Rightarrow$

$$x^2 = 16\cos^2\theta\sin^2\theta = 4y(\cos^2\theta) = 4y(1 - \sin^2\theta) = 4y(1 - \frac{y}{4}) = 4y - y^2.$$

3. (1 point) Converting the polar coordinates $(r, \theta) = (2, \frac{5\pi}{3})$ to the rectangular coordinates (x, y), then $y =$

   (a) $-2\sqrt{3}$ (b) $-\sqrt{3}$ (c) -1 (d) $2\sqrt{3}$ (e) -0.5

   Solution: (b).

$$y = r\sin\theta = 2\sin\frac{5\pi}{3} = -\sqrt{3}.$$

4. (1 point) Which of the following gives the length of the curve $r = 3\theta$, $0 \le \theta \le 2\pi$?

(a) $3\int_0^{2\pi} \sqrt{1+\theta^2}d\theta$ (b) $2\int_0^{2\pi} \sqrt{9+\theta^2}d\theta$ (c) $\int_0^{2\pi} \sqrt{1+9\theta^2}d\theta$ (d) $2\int_0^{2\pi} \sqrt{1+\theta^2}d\theta$

Solution: (a).

$$L = \int_\alpha^\beta \sqrt{(r'_\theta)^2 + r^2}d\theta = \int_0^{2\pi} \sqrt{9+9\theta^2}d\theta = 3\int_0^{2\pi} \sqrt{1+\theta^2}d\theta.$$

5. (4 points) Given the polar curve C: $r = 2 + \sin\theta$, where $0 \le \theta \le \pi/2$. Find the area of the region enclosed by the curve C and the rays: $\theta = 0$ and $\theta = \pi/2$.

Solution: We have

$$\begin{aligned} A &= \int_0^{\pi/2} \frac{1}{2}r^2 d\theta \qquad \textit{(1 point)} \\ &= \int_0^{\pi/2} \frac{1}{2}(2+\sin\theta)^2 d\theta = \int_0^{\pi/2} (2 + 2\sin\theta + \frac{1}{2}\sin^2\theta)d\theta \\ &= (2\theta - 2\cos\theta)|_0^{\pi/2} + \int_0^{\pi/2} \frac{1-\cos 2\theta}{4}d\theta \qquad \textit{(2 points)} \\ &= \pi + 2 + \frac{\pi}{8} = \frac{16+9\pi}{8}. \qquad \textit{(1 point)} \end{aligned}$$

6. (4 points) Find the equation of the tangent line to $r = 4\sin\theta$ at $\theta = \frac{\pi}{3}$.

Solution:

$$\frac{dy}{dx} = \frac{r'_\theta \sin\theta + r\cos\theta}{r'_\theta \cos\theta - r\sin\theta} = \frac{2\sin\theta\cos\theta}{\cos^2\theta - \sin^2\theta}.$$

At $\theta = \frac{\pi}{3}$,

$$\frac{dy}{dx} = -\sqrt{3}, \qquad \textit{(2 points)}$$

$$x = r\cos\theta = 4\sin\theta\cos\theta = \sqrt{3}, y = r\sin\theta = 4\sin\theta\sin\theta = 3. \qquad \textit{(1 point)}$$

Let

$$y = -\sqrt{3}x + b, \quad 3 = -\sqrt{3}(\sqrt{3}) + b, \Rightarrow b = 6, \Rightarrow$$
$$y = -\sqrt{3}x + 6. \qquad \textit{(1 point)}$$

7. (3 points) Find the length of the curve given by the parametric equation: $x = 1 + 9t^2$, $y = 4 + 2t^3$, $0 \le t \le 4$.

Solution:

$$L = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2}dt = \int_0^4 \sqrt{[18t]^2 + [6t^2]^2}dt = 6\int_0^4 t\sqrt{9+t^2}dt \qquad (1\ point)$$

$$= 3\int_9^{25} \sqrt{u}du \qquad u = 9 + t^2 \qquad (1\ point)$$

$$= 2(5^3 - 3^3) = 196. \qquad (1\ point)$$