CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	203	All
Examination	**Date**	**Pages**
Final	December 2007	3

Instructors	**Course Examiner**
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Special Instructions

▷ **Only approved calculators are allowed.**

MARKS

[9] **1.** (a) Sketch the graph of the function $f(x) = |(x-1)^2 - 1|$ starting from the graph of the standard parabola and using appropriate transformations.

(b) Suppose $f(x) = \dfrac{2x+1}{x+1}$ and $g(x) = \dfrac{x-1}{2-x}$. Find $f \circ g$ and $g \circ f$.

(c) Solve for x:

$$3^{\log_3 (x^2)} = 2e^{\ln x} + 4 \cdot 10^{\log_{10}(2)}$$

[12] **2.** Evaluate the limits:

(a) $\displaystyle\lim_{t \to -1} \frac{t^2 + 3t + 2}{t^2 - t - 2}$ (b) $\displaystyle\lim_{x \to 9} \frac{9x - x^2}{3 - \sqrt{x}}$ (c) $\displaystyle\lim_{x \to -\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}}$

Do not use l'Hopital's rule.

[10] **3.** (a) Consider the function $f(x) = \dfrac{x^2 - x - 6}{|x-3|}$.

Calculate both one-sided limits at the point(s) where the function is undefined.

3. (b) Find parameters a and b such that the function

$$f(x) = \begin{cases} \cos x & \text{if} \quad x \le 0 \\ ax + b & \text{if} \quad 0 < x \le 3 \\ x^2 - 2 & \text{if} \quad x > 3 \end{cases}$$

will be continuous at every point. Sketch the graph of this function.

[15] **4.** Find derivatives of the functions (do not simplify the answer):

(a) $f(x) = (x + x^{-1})^2 \cos 2x$

(b) $f(x) = \dfrac{\sin^{-1}(\sqrt{1 - x^2})}{\sqrt{1 - x^2}}$

(c) $f(x) = (x^2)^\pi + \pi^{x^2}$

(d) $f(x) = 4x\sqrt{x + \sqrt{x}}$

(e) $f(x) = (\tan^{-1}(2x))^{\ln x}$ (use logarithmic differentiation).

[12] **5.** (a) If $f(x) = (1 + x)^n$, find the linearization $L(x)$ of $f(x)$ at $a = 0$ and use $L(x)$ to estimate $(1.003)^{50}$.

(b) Answer part (a) using differentials, that is, identify dx and calculate df.

(c) If $g(x) = (x - 1)^2$, use the definition of the derivative to find $g'(3)$.

(d) Use the appropriate differentiation rule(s) to verify your answer to part (c).

[18] **6.** (a) The equation of a curve defined implicitly is $x - 2y^2 + 5 = 3e^{x/y}$. Verify that the point $(0, 1)$ belongs to the curve. Find an equation of the tangent line to the curve at this point.

(b) Let $f(x) = x^2 + 2x - 1$. Find a number c that satisfies the Mean Value Theorem for the function $f(x)$ on $[0, 1]$.

(c) Use l'Hopital's rule to evaluate $\lim\limits_{x \to 0} \dfrac{\tan x - x}{x^3}$.

[10] **7.** (a) A plane is located at $x = 40$ km (horizontally) away from an airport at an altitude of h km. At time $t = t_0$ radar at the airport detects the distance $s(t)$ between the plane and the airport decreasing at the rate $s'(t_0) = -400$ km/h (the plane is flying towards the airport). If the plane is maintaining a constant altitude of $h = 4$ km what is the speed $x'(t_0)$ of the aircraft at time t_0?

(b) A rectangular plot of land is to be bounded on one side by a river and on the other three sides by a fence. If there are 800 m of fencing available, what is the largest area that can be enclosed, and what are its dimensions?

[14] **8.** Given the function $f(x) = 4x^3 - x^4$,

(a) Find the domain and check for symmetry. Find asymptotes (if any).

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function.

[5] **Bonus Question**

Let

$$f(x) = \begin{cases} -x^3 & \text{if } x \geq 0 \\ x^3 & \text{if } x < 0 \end{cases}$$

Use the definition of the derivative to show that f is differentiable at $x = 0$ and find $f'(0)$.

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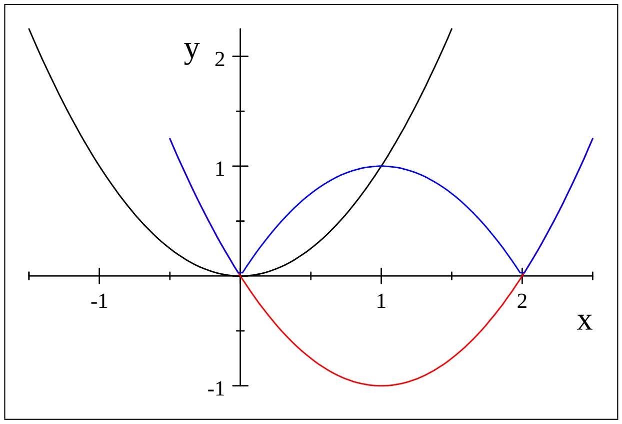
Solutions

1. (a) Sketch the graph of the function $f(x) = |(x-1)^2 - 1|$, starting from the graph

of the standard parabola and using appropriate transformations.

Solution



(b) Suppose $f(x) = \dfrac{2x+1}{x+1}$ and $g(x) = \dfrac{x-1}{2-x}$. Find and $g \circ f$

Solution $f \circ g\,(x) = \dfrac{2\dfrac{x-1}{2-x}+1}{\dfrac{x-1}{2-x}+1} = \dfrac{2x-2+2-x}{x-1+2-x} = x$ and $g \circ f\,(x) =$
$\dfrac{\dfrac{2x+1}{x+1}-1}{2-\dfrac{2x+1}{x+1}} = \dfrac{2x+1-x-1}{2x+2-2x-1} = x$

(c) Solve for x:

$$3^{\log_3 x^2} = 2e^{\ln x} + 4 \cdot 10^{\log_{10} 2}$$

Solution

$$\begin{aligned} x^2 &= 2x + 4 \cdot 2 \\ x^2 - 2x - 8 &= 0 \\ (x-4)(x+2) &= 0 \\ x &= 4 \end{aligned}$$

because x cannot be negative in the expression $e^{\ln x}$.

2. (a)

Solution

$$\begin{aligned}\lim_{t\to -1}\frac{t^2+3t+2}{t^2-t-2} &= \lim_{t\to -1}\frac{(t+1)(t+2)}{(t+1)(t-2)}\\ &= \lim_{t\to -1}\frac{(t+2)}{(t-2)}\\ &= \frac{\lim_{t\to -1}(t+2)}{\lim_{t\to -1}(t-2)} = \frac{1}{-3} = -\frac{1}{3}\end{aligned}$$

(b)

Solution

$$\begin{aligned}\lim_{x\to 9}\frac{9x-x^2}{3-\sqrt{x}} &= \lim_{x\to 9}\frac{x(9-x)}{3-\sqrt{x}}\cdot\frac{3+\sqrt{x}}{3+\sqrt{x}}\\ &= \lim_{x\to 9}\frac{x(9-x)}{9-x}\cdot\left(3+\sqrt{x}\right)\\ &= \lim_{x\to 9}x\cdot\left(3+\sqrt{x}\right)\\ &= 9\cdot(3+3) = 54\end{aligned}$$

(c)

Solution

$$\begin{aligned}\lim_{x\to\infty}\frac{e^x-e^{-x}}{e^x+e^{-x}} &= \lim_{x\to\infty}\frac{1-e^{-2x}}{1+e^{-2x}}\\ &= \frac{1}{1} = 1\end{aligned}$$

3. (a)

Solution

$$\begin{aligned}f(x) &= \frac{x^2-x-6}{|x-3|} = \frac{(x+2)(x-3)}{|x-3|}\\ &= \frac{(x-3)}{|x-3|}\cdot(x+2)\end{aligned}$$

is undefined only at $x = 3$ and we see that

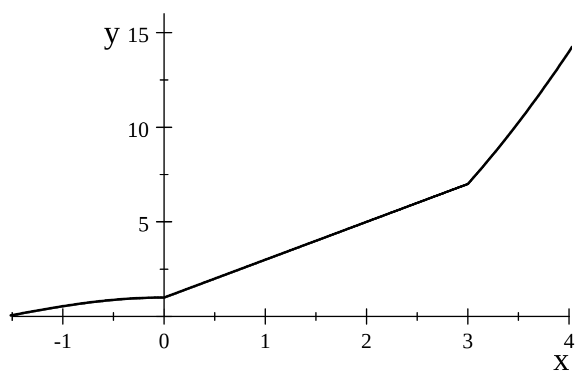
$$\begin{aligned}
\lim_{x\to 3^-} f(x) &= \lim_{x\to 3^-} \frac{(x-3)}{|x-3|} \lim_{x\to 3^-} (x+2) \\
&= (-1)\cdot(3+2) = -5 \\
\lim_{x\to 3^+} f(x) &= \lim_{x\to 3^+} \frac{(x-3)}{|x-3|} \lim_{x\to 3^+} (x+2) \\
&= (+1)\cdot(3+2) = +5
\end{aligned}$$

(b)

Solution

$$\begin{aligned}
f(x) &= \cos x \text{ if } x < 0 \text{ and} \\
f(x) &= ax + b \text{ if } 0 \le x < 3 \text{ and so} \\
\lim_{x\to 0^-} f(x) &= 1 = \lim_{x\to 0^+} f(x) = f(0) = b
\end{aligned}$$

for continuity. So $b = 1$. Similarly, $\lim_{x\to 3^-} f(x) = 3a + 1 = \lim_{x\to 3^+} f(x) = 3^2 - 2 = f(3)$ and so $3a + 1 = 7 \Longrightarrow a = 2$.



4. (a) $f(x) = \left(x + x^{-1}\right)^2 \cos 2x$

Solution $f'(x) = 2\left(x + x^{-1}\right)\left(1 - x^{-2}\right)\cos 2x - 2(\sin 2x)\left(x + x^{-1}\right)^2$

(b) $f(x) = \dfrac{\sin^{-1}\left(\sqrt{1-x^2}\right)}{\sqrt{1-x^2}}$

$$f'(x) = \frac{\sqrt{1-x^2}\cdot\frac{1}{\sqrt{1-(1-x^2)}}\cdot\frac{1}{2\sqrt{1-x^2}}\cdot(-2x) - \sin^{-1}\left(\sqrt{1-x^2}\right)\cdot\frac{1}{2\sqrt{1-x^2}}\cdot(-2x)}{1-x^2} =$$

$$-\frac{x}{(1-x^2)\sqrt{x^2}} + x\frac{\sin^{-1}\sqrt{1-x^2}}{(1-x^2)^{\frac{3}{2}}}$$

(c) $f(x) = (x^2)^{\pi} + \pi^{x^2} \quad = x^{2\pi} + \pi^{x^2}$

$$f'(x) = 2\pi x^{2\pi-1} + \pi^{x^2}\cdot\ln\pi\cdot 2x$$

(d) $f(x) = 4x\sqrt{x+\sqrt{x}}$

Solution $\quad f'(x) = 4\sqrt{x+\sqrt{x}} + 2\frac{x}{\sqrt{x+\sqrt{x}}}\left(1+\frac{1}{2\sqrt{x}}\right)$

(e) $f(x) = \left(\tan^{-1}(2x)\right)^{\ln x}$

Solution We take logarithms first:

$$\begin{aligned}
\ln f &= \ln x\cdot\ln\left(\tan^{-1}(2x)\right) \\
\frac{f'}{f} &= \frac{1}{x}\ln\left(\tan^{-1}(2x)\right) + \ln x\cdot\frac{1}{\tan^{-1}(2x)}\frac{1}{1+4x^2}\cdot 2 \\
f'(x) &= \left(\tan^{-1}(2x)\right)^{\ln x}\left(\frac{1}{x}\ln\left(\tan^{-1}(2x)\right) + \frac{1}{\tan^{-1}(2x)}\frac{2\ln x}{1+4x^2}\right)
\end{aligned}$$

5. (a) $f(x) = (1+x)^n$

Solution $f'(x) = n(1+x)^{n-1}$ and $f'(0) = n$ so $L(x) = f(0) + xf'(0) = 1+nx$. For $n = 50$ and $x_0 = .003$ we get $L(.003) = 1 + 50(.003) = 1.15$. (We note that the actual value of 1.003^{50} is 1.162)

(b)

Solution Write

$$\begin{aligned}
df &= f'(a)\,dx \text{ where } a = 0 \text{ and } dx = .003, \text{ so} \\
df &= f'(0)(.003) \\
&= (50)(.003) = 0.15 \\
\text{So } f(0+dx) &\approx f(0) + df = 1 + 0.15 = 1.15
\end{aligned}$$

(c) $g(x) = (x-1)^2$

Solution

$$
\begin{aligned}
g'(x) &= \lim_{h\to 0}\frac{g(x+h)-g(x)}{h} = \lim_{h\to 0}\frac{(x+h-1)^2-(x-1)^2}{h} \\
&= \lim_{h\to 0}\frac{x^2+2xh+h^2-2x-2h+1-x^2+2x-1}{h} \\
&= \lim_{h\to 0}\frac{2xh+h^2-2h}{h} = \lim_{h\to 0}\frac{h}{h}(2x+h-2) = 2x-2
\end{aligned}
$$

(d) $g(x) = (x-1)^2$

Solution

$$
\begin{aligned}
g'(x) &= 2(x-1)\cdot 1 \text{ (chain rule)} \\
&= 2x-2
\end{aligned}
$$

6. $x - 2y^2 + 5 = 3e^{x/y}$

(a)

Solution

$$
\begin{aligned}
0 - 2\cdot 1 + 5 &= 3e^0 \\
3 &= 3
\end{aligned}
$$

so the point $(0,1)$ is on the curve. Now differentiate:

$$
\begin{aligned}
1 - 4yy' &= 3e^{x/y}\left(1/y - x/y^2 y'\right) \text{ now substitute } (0,1) \\
1 - 4y' &= 3(1-0) \\
y' &= -\frac{1}{2} \text{ and a tangent line is} \\
y - 1 &= -\frac{1}{2}(x-0) \text{ or} \\
y &= -\frac{1}{2}x + 1
\end{aligned}
$$

(b) $f(x) = x^2 + 2x - 1$ on $[0,1]$.

Solution $f'(x) = 2x + 2$ and so we need

$$
\begin{aligned}
f'(c) &= \frac{f(1)-f(0)}{1-0} = \frac{2-(-1)}{1} = 3 \\
2c+2 &= 3 \\
2c &= 1
\end{aligned}
$$

So the solution is $c = \frac{1}{2}$.

(c)

$$\lim_{x\to 0}\frac{\tan x - x}{x^3}$$

Solution Check: a $\frac{0}{0}$ type expression, so L'Hopital's rule applies

$$\begin{aligned}\lim_{x\to 0}\frac{\tan x - x}{x^3} &= \lim_{x\to 0}\frac{\sec^2 x - 1}{3x^2} \text{ repeat}\\ &= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{6x} \text{ repeat again}\\ &= \lim_{x\to 0}\frac{4\sec^2 x\tan^2 x + 2\sec^4 x}{6} = \frac{2}{6}\\ &= \frac{1}{3}\end{aligned}$$

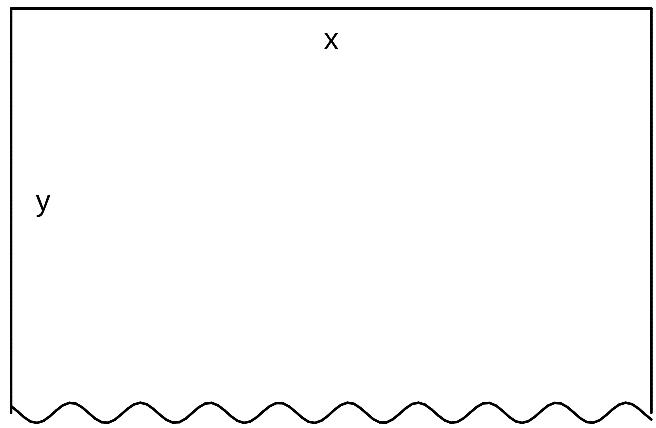
7. (a)

Solution We have the relationship

$$\begin{aligned}s^2 &= x^2 + h^2 \text{ and also at time } t_0\\ s^2 &= 40^2 + 4^2 = 1616 \Longrightarrow s = 40.2\\ \text{Differentiate} & \quad \text{with respect to time } t \text{ to get}\\ 2ss'(t) &= 2xx'(t) \text{ since } h \text{ is constant}\\ x'(t) &= \frac{s(t)\,s'(t)}{x(t)} \text{ and at } t_0 \text{ we get}\\ x'(t_0) &= \frac{40.2\cdot(-400)}{40} = -402 \text{ km/h}\end{aligned}$$

(b)

Solution Draw a sketch:



We want to maximize

$$A = xy$$

subject to the condition that

$$x + 2y = 800$$

So we eliminate y by expressing it in terms of $x : y = \frac{1}{2}(800 - x)$ and substitute to get

$$\begin{aligned} A &= \frac{1}{2}x(800 - x) = 400x - \frac{1}{2}x^2 \\ A' &= 400 - x = 0 \text{ for a critical point, so} \\ x &= 400 \text{ and consequently } y = 200 \end{aligned}$$

To check that this is the maximum area, we calculate $A'' = -1 < 0$, which means the graph is concave down and it is indeed a maximum.

8. **(a)** Find the domain and check for symmetry. Find asymptotes (if any).

Solution

$$y = f(x) = 4x^3 - x^4$$

Since this is a polynomial with odd and even powers, there is no symmetry and no asymptotes.

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

Solution

$$f'(x) = 12x^2 - 4x^3 = 4x^2(3-x)$$

We see that $x = 0, 3$ are critical points. Since $x^2 \geq 0$, the sign of $f'(x)$ changes only when x passes through $x = 3$. Here is what is happening.

	$x < 3$	$x = 3$	$x > 3$
	$+$	0	$-$
$f'(x)$	$\nearrow$	l. max	$\searrow$

So there is a local maximum at $(3, 27)$ which is also an absolute maximum.

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

Solution

$$f''(x) = 24x - 12x^2 = 12x(2-x)$$

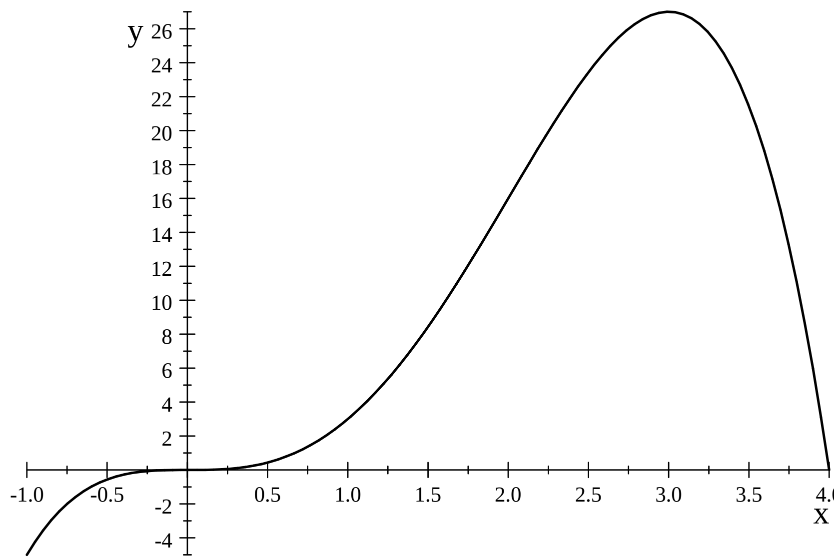
We see that $x = 0, 2$ are possible inflection points. Here is the analysis.

	$x < 0$	$x = 0$	$0 < x < 2$	$x = 2$	$x > 2$
$12x$	$-$	0	$+$	$+$	$+$
$2 - x$	$+$	$+$	$+$	0	$-$
$f''(x)$	$\cap$	i.p.	$\cup$	i.p.	$\cap$

So there are inflection points at $(0, 0)$ and $(2, 16)$

(d) Sketch the graph of the function.

Solution



Bonus

$$f(x) = \begin{cases} -x^3 & \text{if} \quad x \geq 0 \\ x^3 & \text{if} \quad x < 0 \end{cases}$$

Also, if $x > 0$ then $f'(x) = -3x^2$. If $x < 0$ then $f'(x) = 3x^2$. So the only question is what happens at $x = 0$. We can't just substitute $x = 0$ into the equation and get the answer 0, because we don't know ahead of time that $f'(0)$ even exists!. What we have to do is go back to the definition:

$$\begin{aligned} f'(0) &= \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\pm h^3 - 0}{h} \\ &= \lim_{h \to 0} \pm h^2 = 0 \end{aligned}$$

where we put $\pm$ depending on whether $h > 0$ (it would be $-$) or $h < 0$ (it would be $+$). Either way, the limit is zero, so the derivative does indeed exist at 0 (and everywhere).

Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	203	All
Examination	**Date**	**Pages**
Final	April 2007	3

Instructors	**Course Examiner**
B. Brown, J. Hayes, R. Mearns, M. Mei, D. Sevilla-Gonzalez	H. Proppe

Special Instructions

▷ Calculators are **not** allowed.

MARKS

[8] **1.** (a) Let $f(x) = x^2 + 1$ and $g(x) = \sin x$. Find $f \circ g$ and $g \circ f$.

(b) Find the inverse of the function $f(x) = e^{2x} - 1$. Determine the domain and range of f and f^{-1}.

[10] **2.** Evaluate the limits:

(a) $\displaystyle\lim_{x\to 7} \frac{x^2 - 6x - 7}{\sqrt{x+2} - 3}$ (b) $\displaystyle\lim_{x\to -\infty} \frac{4x^2\sqrt{9x^6 + 2x^2}}{x^5 + 3}$

Do not use l'Hopital's rule.

[12] **3.** (a) Consider the function $f(x) = \dfrac{|x+1|}{x^2 - 1}$.

Calculate both one-sided limits at the point(s) where the function is undefined.

(b) Find parameters a and b such that the function

$$f(x) = \begin{cases} 1 - \dfrac{x^2 - x}{x^2 - 1}, & \text{if} \quad 0 \le x < 1 \\ a, & \text{if} \quad x = 1 \\ x + b, & \text{if} \quad x > 1 \end{cases}$$

will be continuous at every point. Sketch the graph of this function.

[12] **4.** Find derivatives of the functions (do not simplify the answer):

(a) $f(x) = e^{-x^2}\,(\cos x + \tan x)^2$;

(b) $f(x) = \dfrac{\ln(1+x^2)}{1+\arctan(2x)}$;

(c) $f(x) = \sec^3(\sqrt{1+x^2})$;

(d) $f(x) = (\sin(5x))^{x^3}$ (use logarithmic differentiation).

[12] **5.** Given the function $f(x) = \sqrt{3x+1}$,

(a) Use appropriate differentiation rules to find the derivative of the function.

(b) Use the definition of derivative to verify (a).

(c) Find the differential of the function at $x = 1$.

(d) Use the differential above with the appropriate choice of $dx = \Delta x$ to estimate $\sqrt{5}$.

[10] **6.** (a) A curve called a lemniscate is defined implicitly by the equation $2(x^2+y^2)^2 = 25(x^2-y^2)$. Verify that the point $(3,1)$ belongs to the curve. Find an equation of the tangent line to the curve at this point.

(b) Use l'Hopital's rule to evaluate $\lim\limits_{x\to 0} \dfrac{1-\cos x}{(\ln(x+1))^2}$.

[8] **7.** (a) Let $f(x) = \arcsin(x^2)$. Find $f''(x)$.

(b) Let $f(x) = 3x^2 + 2x + 5$. Find a number c that satisfies the Mean Value Theorem for the function $f(x)$ on $[-1, 1]$.

[12] **8.** (a) A particle is moving along the curve $y = 4\sqrt{x}$. As the particle passes the point $(9, 12)$ its x-coordinate is increasing at a rate of 3 cm/sec. How fast is the distance from the particle to the origin changing at this instant?

(b) Find the area of the largest rectangle that has its base on the x - axis and its other two vertices above the x - axis and lying on the parabola $y = 12 - x^2$.

[16] **9.** Given the function $f(x) = \dfrac{e^x}{x}$,

(a) Find the domain and check for symmetry. Find asymptotes (if any).

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function.

[5] **Bonus Question**

Let $f(x) = x|x|$. Use the definition of the derivative to show that $f'(0)$ exists, then find $f'(x)$ for any x.

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Solutions

1. (a) $f(x) = x^2 + 1$ and $g(x) = \sin x$ so $f \circ g(x) = f(\sin x) = \sin^2 x + 1$ and $g \circ f(x) = g(x^2 + 1) = \sin(x^2 + 1)$

(b) $f(x) = e^{2x} - 1 = y$. We first note that this equation makes sense for all x, so the domain is $\mathbb{R}$ or . Since $e^{2x} > 0$ it follows that $e^{2x} - 1 > -1$ so the domain is $(-1, \infty)$. Next, we solve for x :

$$\begin{aligned} e^{2x} &= 1 + y \\ 2x &= \ln(1 + y) \\ x &= \frac{1}{2}\ln(1 + y) \text{ interchange } x \text{ and } y \\ y &= \frac{1}{2}\ln(1 + x) \end{aligned}$$

is the inverse function f^{-1}. Also, the domain of f = the range of $f^{-1} = (-\infty, \infty)$; and the range of f = the domain of $f^{-1} = (-1, \infty)$.

2. (a)

$$\begin{aligned} \lim_{x\to 7} \frac{x^2 - 6x - 7}{\sqrt{x+2} - 3} &= \lim_{x\to 7} \frac{(x-7)(x+1)}{\sqrt{x+2} - 3} \cdot \frac{\sqrt{x+2} + 3}{\sqrt{x+2} + 3} \\ &= \lim_{x\to 7} \frac{(x-7)(x+1)}{x-7} \left(\sqrt{x+2} + 3\right) \\ &= \lim_{x\to 7} (x+1)\left(\sqrt{x+2} + 3\right) = 8 \cdot 6 = 48 \end{aligned}$$

(b)

$$\begin{aligned} \lim_{x\to -\infty} \frac{4x^2\sqrt{9x^6 + 2x^2}}{x^5 + 3} &= \lim_{x\to -\infty} \frac{4x^2 \left|x^3\right| \sqrt{9 + 2/x^4}}{x^5 + 3} \\ &= \lim_{x\to -\infty} 4\frac{\left|x^5\right|}{x^5} \frac{\sqrt{9 + 2/x^4}}{1 + 3/x^5} = -4 \cdot \frac{3}{1} \\ &= -12 \end{aligned}$$

3. (a)

$$f(x) = \frac{|x+1|}{x^2 - 1} = \frac{|x+1|}{x+1} \cdot \frac{1}{x-1}$$

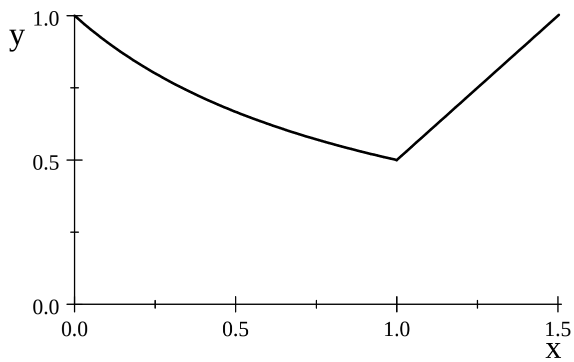
is undefined at ± 1 and we see that

$$\begin{aligned}
\lim_{x\to -1^-} f(x) &= \lim_{x\to -1^-} \frac{|x+1|}{x+1} \lim_{x\to -1^-} \frac{1}{x-1} \\
&= (-1)\cdot \frac{1}{-1-1} = \frac{1}{2} \\
\lim_{x\to -1^+} f(x) &= \lim_{x\to -1^+} \frac{|x+1|}{x+1} \lim_{x\to -1^+} \frac{1}{x-1} \\
&= (+1)\cdot \frac{1}{-1-1} = -\frac{1}{2} \\
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \frac{|x+1|}{x+1} \lim_{x\to 1^-} \frac{1}{x-1} \\
&= \frac{2}{2}\cdot(-\infty) = -\infty \\
\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} \frac{|x+1|}{x+1} \lim_{x\to 1^+} \frac{1}{x-1} \\
&= \frac{2}{2}\cdot(+\infty) = +\infty
\end{aligned}$$

(b)

$$\begin{aligned}
f(x) &= 1 - x\frac{x-1}{(x-1)(x+1)} \text{ if } 0 \le x < 1 \\
&= 1 - \frac{x}{x+1} \text{ if } 0 \le x < 1 \text{ and so} \\
\lim_{x\to 1^-} f(x) &= 1 - \frac{1}{2} = \frac{1}{2} = f(1)
\end{aligned}$$

for continuity. So $a = 1/2$. Similarly, $\lim_{x\to 1^+} f(x) = 1+b = a = 1/2$ so $b = -1/2$.



4. (a) $f(x) = e^{-x^2}(\cos x + \tan x)^2$; $f'(x) = -2xe^{-x^2}(\cos x + \tan x)^2 + e^{-x^2} \cdot 2(\cos x + \tan x) \cdot (-\sin x + \sec^2 x)$

(b) $f(x) = \frac{\ln(1+x^2)}{1+\arctan(2x)}$; $f'(x) = \frac{(1+\arctan(2x))\left(\frac{2x}{1+x^2}\right) - \ln(1+x^2)\left(\frac{1}{1+4x^2}\right)\cdot 2}{(1+\arctan(2x))^2}$

(c) $f(x) = \sec^3\left(\sqrt{1+x^2}\right)$; $f'(x) = 3\sec^2\left(\sqrt{1+x^2}\right) \cdot \sec\left(\sqrt{1+x^2}\right) \cdot \tan\left(\sqrt{1+x^2}\right) \cdot \frac{x}{\sqrt{1+x^2}}$

(d) $f(x) = (\sin(5x))^{x^3}$; $\ln(f(x)) = x^3 \ln(\sin(5x))$ and so $\frac{f'(x)}{f(x)} = 3x^2 \ln(\sin(5x)) + x^3 \frac{\cos(5x)}{\sin(5x)} \cdot 5$

and finally $f'(x) = (\sin(5x))^{x^3}\left(3x^2 \ln(\sin(5x)) + 5x^3 \cot(5x)\right)$

5. (a) $f(x) = \sqrt{3x+1} = (3x+1)^{1/2} \implies f'(x) = \frac{1}{2}(3x+1)^{-1/2} \cdot 3 = \frac{3}{2\sqrt{3x+1}}$

(b)

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \\ &= \lim_{h\to 0} \frac{\sqrt{3(x+h)+1} - \sqrt{3x+1}}{h} \cdot \frac{\sqrt{3(x+h)+1} + \sqrt{3x+1}}{\sqrt{3(x+h)+1} + \sqrt{3x+1}} \\ &= \lim_{h\to 0} \frac{3h}{h\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} \\ &= \lim_{h\to 0} \frac{3}{\left(\sqrt{3(x+h)+1} + \sqrt{3x+1}\right)} = \frac{3}{2\sqrt{3x+1}} \end{aligned}$$

(c) $= \dfrac{3}{2\sqrt{3x+1}}dx = \dfrac{3}{4}dx$ at $x = 1$

(d) To estimate $\sqrt{5}$ we first take $x = 1$ and get $f(1) = \sqrt{4} = 2$. If we increase x so that $3x + 1 = 5$, the new x is $4/3$ so $dx = 1/3$. Substituting in $df = \dfrac{3}{4}dx$ at $x = 1$ we get

$$df = \frac{3}{4} \cdot \frac{1}{3} = \frac{1}{4}$$

So $\sqrt{5}$ is approx. $2 + \frac{1}{4} = 2.25$

6. (a)

$$\begin{aligned} 2\left(x^2+y^2\right)^2 &= 25\left(x^2-y^2\right) \\ 2\left(3^2+1^2\right)^2 &= 200 \\ 25\left(3^2-1^2\right) &= 200 \end{aligned}$$

so the point $(3, 1)$ is on the curve. Now differentiate:

$$\begin{aligned} 4\left(x^2+y^2\right)(2x+2yy') &= 25\left(2x-2yy'\right) \\ 4(10)(6+2y') &= 25(6-2y') \\ 130y' &= 10 \\ y' &= -\frac{9}{13} \text{ and a tangent line is} \\ y-1 &= -\frac{9}{13}(x-3) \end{aligned}$$

(b)

$$\lim_{x\to 0} \frac{1-\cos x}{\left(\ln(x+1)\right)^2}$$

Check $\frac{0}{0}$ type expression so L'Hopital's rule applies

$$\begin{aligned} &= \lim_{x\to 0} \frac{\sin x}{2\ln(x+1)\cdot\frac{1}{x+1}} = \lim_{x\to 0} \frac{(x+1)\sin x}{2\ln(x+1)} \text{ repeat} \\ \lim_{x\to 0} \frac{\sin x + (x+1)\cos x}{\frac{2}{x+1}} &= \lim_{x\to 0} \frac{(x+1)(\sin x + (x+1)\cos x)}{2} = \frac{1\cdot(0+1\cdot 1)}{2} \\ &= \frac{1}{2} \end{aligned}$$

7. (a) $f(x) = \arcsin(x^2)$; $f'(x) = 2\frac{x}{\sqrt{1-x^4}}$; $f''(x) = 2\frac{1-x^4+2x^4}{\left(\sqrt{1-x^4}\right)^3}$

(b) $f(x) = 3x^2 + 2x + 5$ on $[-1, 1]$. $f(1) = 10$; $f(-1) = 6$ and

$$\frac{f(1) - f(-1)}{1-(-1)} = \frac{10-6}{2} = 3$$

so we are looking for a number c so that $f'(c) = 6c + 2 = 3$. Clearly the solution is $c = 1/6$.

8. (a)

$$\begin{aligned} y &= 4\sqrt{x} \\ \frac{dy}{dt} &= \frac{2}{\sqrt{x}}\frac{dx}{dt} \end{aligned}$$

We are given that $\frac{dx}{dt} = 3$ at $(9, 12)$. So

$$\frac{dy}{dt} = \frac{2}{\sqrt{9}}3 = 2$$

at this point. Also, if we denote the distance to the origin by s we have that $s = \sqrt{9^2 + 12^2} = 15$ at $(9, 12)$ and

$$\begin{aligned} s^2 &= x^2 + y^2 \\ 2s\frac{ds}{dt} &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt} \text{ so} \\ \frac{ds}{dt} &= \frac{9 \cdot 3 + 12 \cdot 2}{15} = \frac{51}{15} \end{aligned}$$

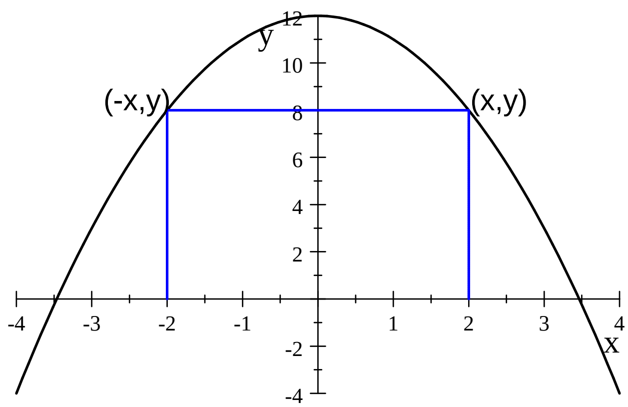
(b) Let the point on the parabola in quadrant 1 that gives the rectangle of maximum arwea be $P(x, y)$. Then the area is

$$A = 2xy$$

since the base is $2x$. To eliminate y we note $y - 12 - x^2$ and so

$$\begin{aligned} A &= 2x\left(12 - x^2\right) = 24x - 2x^3 \\ A' &= 24 - 6x^2 = 0 \Longleftrightarrow x^2 = 4 \\ A'' &= -12x < 0 \end{aligned}$$

So there is a maximum (absolute) at $x = 2$; the maximum area is $A_{\text{max}} = 32$.



9. $f(x) = \dfrac{e^x}{x}$

(a) Domain is all reals except $x = 0$ i.e. $(-\infty, 0) \cup (0, \infty)$. There is no symmetry since $f(-x)$ is neither $f(x)$ nor $-f(x)$. We see that

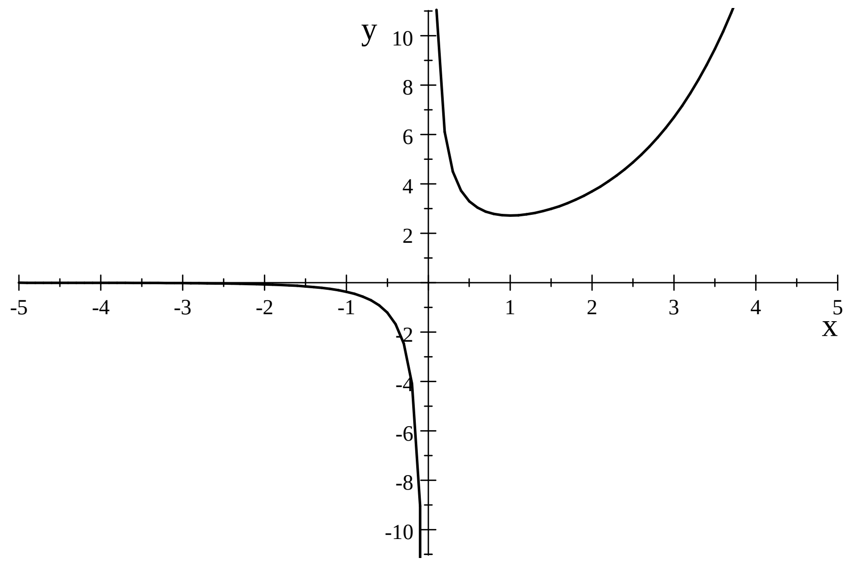
$$\begin{aligned} \lim_{x \to 0^-} \frac{e^x}{x} &= -\infty, \lim_{x \to 0^+} \frac{e^x}{x} = \infty \text{ and} \\ \lim_{x \to -\infty} \frac{e^x}{x} &= 0 \text{ but } \lim_{x \to \infty} \frac{e^x}{x} = \infty \text{ by L'H's rule} \end{aligned}$$

so $x = 0$ is a vertical asymptote and $y = 0$ is a horizontal asymptote.

(b) $f'(x) = \frac{1}{x}e^x - \frac{1}{x^2}e^x = e^x \frac{x-1}{x^2}$. Since both e^x and x^2 are never negative, we see that $f'(x) < 0$ if $x < 1 (x \neq 0)$ and > 0 if $x > 1$. So f is $\downarrow$ when $x < 0$ and again when $0 < x < 1$ and $\uparrow$ when $x > 1$. So there is a local minimum at $(1, e)$.

(c) $f''(x) = \frac{1}{x}e^x - \frac{2}{x^2}e^x + \frac{2}{x^3}e^x = e^x \cdot \frac{-2x+x^2+2}{x^3} = e^x \frac{(x-1)^2+1}{x^3}$. Since only the denominator x^3 can change sign, this shows that f is concave down when $x < 0$ and concave up when $x > 0$.

(d) Here is the graph



Bonus

$$\begin{aligned} f(x) &= x\,|x| \\ f'(0) &= \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h\,|h|-0}{h} \\ &= \lim_{h\to 0}|h| = 0 \end{aligned}$$

Also, if $x > 0$ then $f(x) = x^2$ and $f'(x) = 2x$. If $x < 0$ then $f(x) = -x^2$ because $|x| = -x$ when $x < 0$ and so $f'(x) = -2x$. So $f'(x)$ exists for every real number x.

CONCORDIA UNIVERSITY

Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	203	All
Examination	**Date**	**Pages**
Final	December 2008	3
Instructors		**Course Examiner**
A. Boyarsky, J. Brody, H. Greenspan, H. Proppe, N. Rossokhata		H. Proppe

Special Instructions

▹ **Only approved calculators are allowed.**

MARKS

[9] **1.** (a) Sketch the graph of the function $f(x) = |(x+1)^2 - 2|$ starting from the graph of the standard parabola $y = x^2$ and using appropriate transformations.

(b) Suppose $f(x) = x^2 - 2x + 1; x \geq 1$. Find $f^{-1}(x)$ and its domain and range.

(c) Solve for x:

$$5^{\log_5 (x^2)} = 2 \cdot 4^{\log_4 (x)} + e^{\ln 3}$$

[12] **2.** Evaluate the limits:

(a) $\lim_{t \to 1} \dfrac{t^2 + t - 2}{t^2 + 2t - 3}$ (b) $\lim_{x \to 4} \dfrac{x^2 - 4x}{2 - \sqrt{x}}$ (c) $\lim_{x \to -\infty} \dfrac{2x^3 - 7x + 4\sin x}{\sqrt{x^6 + 5x^2 + 1000}}$

Do not use l'Hopital's rule.

[10] **3.** (a) Consider the function $f(x) = \dfrac{x^2 + 2x - 8}{|x - 2|}$.

Calculate both one-sided limits at the point where the function is undefined.

3. (b) Find the numbers a and b that make the function

$$f(x) = \begin{cases} 1 - 2x & \text{if} \quad x \leq 0 \\ ax + b & \text{if} \quad 0 < x \leq 2 \\ x^2 + 1 & \text{if} \quad x > 2 \end{cases}$$

continuous at every point. Sketch the graph of this function.

[15] **4.** Find derivatives of the functions (do not simplify the answer):

(a) $f(x) = (x + x^2)^{-1} \sin 3x$

(b) $f(x) = \dfrac{\arctan(x^2)}{\sqrt{x^2 + 1}}$

(c) $f(x) = (x^3)^2 + 2^x$

(d) $f(x) = 2x\sqrt{x^2 + \sqrt{1 + x^2}}$

(e) $f(x) = (\ln x)^{\arctan(x)}$ (use logarithmic differentiation).

[12] **5.** (a) If $f(x) = \sqrt[3]{x}$, find the linearization $L(x)$ of $f(x)$ at $a = 8$ and use $L(x)$ to estimate $\sqrt[3]{8.5}$ (**Do not** use a calculator to answer this question).

(b) Answer part (a) using differentials, that is, identify dx and calculate df. (**Do not** use a calculator to answer this question).

(c) If $g(x) = x + x^{-1}$, use the definition of the derivative to find $g'(1)$.

(d) Use the appropriate differentiation rule(s) to verify your answer to part (c).

[18] **6.** (a) The equation of a curve defined implicitly is $x^2 e^y + x + \ln(1 + y) = 2$. Verify that the point $(1, 0)$ belongs to the curve. Find an equation of the tangent line to the curve at this point.

(b) Let $f(x) = x^{1/3}$. Find a number c that satisfies the Mean Value Theorem for the function $f(x)$ on $[0, 1]$.

(c) Use l'Hopital's rule to evaluate $\displaystyle\lim_{x \to 0} \frac{\sin x - x}{x^3}$.

[10] **7.** (a) The equation of a curve in the plane is $e^{xy} = x^2 + y^2$. A particle is moving along this curve at a certain velocity. At the instant that it moves through the point $(0, 1)$ the $y-$coordinate is decreasing at the rate of 3 cm/ sec. How fast is the $x-$coordinate changing at this instant and is it increasing or decreasing?

(b) A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

[14] **8.** Given the function $f(x) = xe^{-x}$,

(a) Find the domain and check for symmetry. Find asymptotes (if any).

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function.

[5] **Bonus Question**

Let $f(x) = x\sin(|x|)$.

Use the definition of the derivative to show that f is differentiable at $x = 0$ and find $f'(0)$.

Course	Number	Section(s)
Mathematics	203	All
Examination	**Date**	**Pages**
Final	December 2008	3
Instructors		**Course Examiner**
A. Boyarsky, J. Brody, H. Greenspan, H. Proppe, N. Rossokhata		H. Proppe

Special Instructions

▷ **Only approved calculators are allowed.**

Solutions

1. (9 Marks)

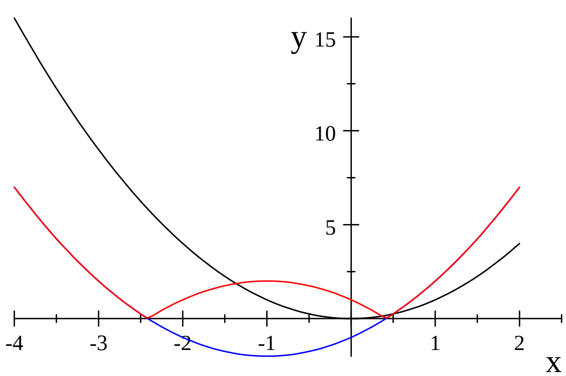
(a) Sketch the graph of the function $f(x) = |(x+1)^2 - 2|$ starting from the graph of the standard parabola $y = x^2$ and using appropriate transformations.

(b) Suppose $f(x) = x^2 - 2x + 1; x \geq 1$. Find $f^{-1}(x)$ and its domain and range.

(c) Solve for x:

$$5^{\log_5(x^2)} = 2 \cdot 4^{\log_4(x)} + e^{\ln 3}$$

Solutions (a)



(b) $f(x) = y = x^2 - 2x + 1 = (x-1)^2$ and since $x \geq 1$ we have $x = 1 + \sqrt{y}$ so the inverse function is $y = f^{-1}(x) = 1 + \sqrt{x}$. The domain is, of course, $[0, \infty)$ and the range $[1, \infty)$. We note, incidentally, that $f \circ f^{-1}(x) = (1+\sqrt{x})^2 - 2(1+\sqrt{x}) + 1 = x = f^{-1} \circ f(x)$.

(c)

$$\begin{aligned} 5^{\log_5(x^2)} &= 2 \cdot 4^{\log_4(x)} + e^{\ln 3} \Leftrightarrow \\ x^2 &= 2x + 3 \Leftrightarrow \\ x^2 - 2x - 3 &= (x-3)(x+1) = 0 \end{aligned}$$

by using the fact that the logarithm and exponential functions are inverse to each other. Now x can't be negative, otherwise $\log_4(x)$ in $4^{\log_4(x)}$ is not defined, so the solution must be $x = 3$.

2. (12 Marks) Evaluate the limits:

(a) $\displaystyle\lim_{t \to 1} \frac{t^2 + t - 2}{t^2 + 2t - 3}$ (b) $\displaystyle\lim_{x \to 4} \frac{x^2 - 4x}{2 - \sqrt{x}}$ (c) $\displaystyle\lim_{x \to -\infty} \frac{2x^3 - 7x + 4\sin x}{\sqrt{x^6 + 5x^2 + 1000}}$

Do not use l'Hopital's rule.

Solution (a)

$$\begin{aligned} \lim_{t \to 1} \frac{t^2 + t - 2}{t^2 + 2t - 3} &= \lim_{t \to 1} \frac{(t-1)(t+2)}{(t-1)(t+3)} \\ &= \lim_{t \to 1} \frac{(t+2)}{(t+3)} = \frac{3}{4} \end{aligned}$$

(b)

$$\begin{aligned} \lim_{x \to 4} \frac{x^2 - 4x}{2 - \sqrt{x}} &= \lim_{x \to 4} \frac{x(x-4)}{2 - \sqrt{x}} \cdot \frac{2 + \sqrt{x}}{2 + \sqrt{x}} \\ &= \lim_{x \to 4} \frac{x(x-4)}{4 - x} \cdot \left(2 + \sqrt{x}\right) \\ &= -\lim_{x \to 4} x \cdot \left(2 + \sqrt{x}\right) = -16 \end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to-\infty}\frac{2x^3-7x+4\sin x}{\sqrt{x^6+5x^2+1000}} &= \lim_{x\to-\infty}\frac{x^3\left(2-7/x^2+4\sin x/x^3\right)}{|x|^3\sqrt{1+5/x+1000/x^3}}\\ &= \lim_{x\to-\infty}\frac{2x^3}{|x|^3}=-2\end{aligned}$$

The reason for the $-$ sign is that when the x^6 comes out from under the square root sign it must have a $+$ sign, even though $x\to-\infty$ because x^6 was a positive number. So we have to write $|x|^3$ (or $-x^3$), not x^3.

3. (10 Marks)

(a) Consider the function $f(x)=\dfrac{x^2+2x-8}{|x-2|}$. Calculate both one-sided limits at the point where the function is undefined.

(b) Find the numbers a and b that make the function

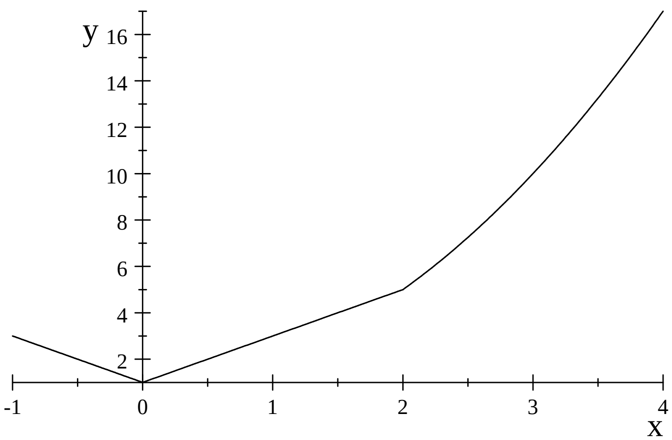
$$f(x)=\begin{cases}1-2x & \text{if} \quad x\le 0\\ ax+b & \text{if} \quad 0<x\le 2\\ x^2+1 & \text{if} \quad x>2\end{cases}$$

continuous at every point. Sketch the graph of this function.

Solution **(a)** $f(x)=\dfrac{x^2+2x-8}{|x-2|}=\dfrac{(x-2)(x+4)}{|x-2|}$ and so the point $x=2$ is where f is undefined.

$$\begin{aligned}\lim_{x\to2^-}\frac{(x-2)(x+4)}{|x-2|} &= \lim_{x\to2^-}\frac{(x-2)}{|x-2|}\lim_{x\to2^-}(x+4)\\ &= (-1)\cdot 6=-6\\ \lim_{x\to2^+}\frac{(x-2)(x+4)}{|x-2|} &= \lim_{x\to2^+}\frac{(x-2)}{|x-2|}\lim_{x\to2^+}(x+4)\\ &= (+1)\cdot 6=6\end{aligned}$$

(b) Since $f(0)=1$ and $\lim\limits_{x\to0^+}f(x)=b$ the two have to be equal so $b=1$. Also, $f(2)=2a+b=2a+1$ and $\lim\limits_{x\to2^+}f(x)=5$ so $2a+1=5\Longrightarrow a=2$. Here is the graph



4. (15 Marks) Find derivatives of the functions (do not simplify the answer):

(a) $f(x) = (x + x^2)^{-1} \sin 3x$

Solution $f'(x) = -(\sin 3x)\left(x^2 + x\right)^{-2} \cdot (1 + 2x) + 3\cos 3x(x + x^2)^{-1}$

(b) $f(x) = \dfrac{\arctan(x^2)}{\sqrt{x^2+1}}$

Solution $f'(x) = \dfrac{\sqrt{x^2+1}\left(\frac{2x}{(1+x^4)}\right) - \arctan x^2 \left(\frac{x}{\sqrt{x^2+1}}\right)}{x^2+1}$

(c) $f(x) = (x^3)^2 + 2^x$

Solution $f'(x) = 6x^5 + 2^x \ln 2$

(d) $f(x) = 2x\sqrt{x^2 + \sqrt{1+x^2}}$

Solution $f'(x) = 2\sqrt{\sqrt{x^2+1} + x^2} + x\dfrac{2x + \frac{x}{\sqrt{x^2+1}}}{\sqrt{\sqrt{x^2+1}+x^2}}$

(e) $f(x) = (\ln x)^{\arctan(x)}$ (use logarithmic differentiation).

Solution $\ln f(x) = \arctan x \cdot \ln(\ln x)$ so $\dfrac{f'(x)}{f(x)} = \dfrac{1}{1+x^2}\ln(\ln x) + \arctan x \dfrac{1}{\ln x}\dfrac{1}{x}$ and

$$\begin{aligned} f'(x) &= f(x)\left(\frac{1}{1+x^2}\ln(\ln x) + \arctan x\frac{1}{\ln x}\frac{1}{x}\right) \\ &= (\ln x)^{\arctan(x)}\left(\frac{1}{1+x^2}\ln(\ln x) + \arctan x\frac{1}{\ln x}\frac{1}{x}\right) \end{aligned}$$

5. (12 Marks)

(a) If $f(x) = \sqrt[3]{x}$, find the linearization $L(x)$ of $f(x)$ at $a = 8$ and use $L(x)$ to estimate $\sqrt[3]{8.5}$ (**Do not** use a calculator to answer this question).

(b) Answer part (a) using differentials, that is, identify dx and calculate df . (**Do not** use a calculator to answer this question).

(c) If $g(x) = x + x^{-1}$, use the definition of the derivative to find $g'(1)$.

(d) Use the appropriate differentiation rule(s) to verify your answer to part (c).

Solutions (a) $L(x) = f(a) + f'(a)(x-a)$ so substituting we have, since $f'(x) = 1/3x^{-2/3}$ $L(x) = f(8) + f'(8)(x-8) = 2 + \frac{1}{12}(x-8)$. We have $L(8.5) = 2 + \frac{1}{12}(8.5 - 8) = 2 + \frac{1}{24} = 2.0417$ (note the exact value is $\sqrt[3]{8.5} = 2.0408$)

(b) Using differentials we write

$$\begin{aligned} df &= f'(8)\,dx \\ &= \frac{1}{12}dx \\ &= \frac{1}{12} \cdot \frac{1}{2} = \frac{1}{24} \end{aligned}$$

where $dx = 8.5 - 8 = .5$ and $f'(8) = \frac{1}{12}$. So f changes by $\frac{1}{24}$ from its value at 8 (which is 2) to $2 + \frac{1}{24}$ which is 2.0417 as before.

(c)

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\left(x + h + \frac{1}{x+h}\right) - \left(x + \frac{1}{x}\right)}{h} \\ &= \lim_{h\to 0} \frac{h + \frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{h}{h} + \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= 1 + \lim_{h\to 0} \frac{x - (x+h)}{h \cdot x \cdot (x+h)} \\ &= 1 - \lim_{h\to 0} \frac{1}{x \cdot (x+h)} = 1 - \frac{1}{x^2}; \text{ so } g'(1) = 0 \end{aligned}$$

(d) $g'(x) = 1 + (-1)x^{-2} = 1 - \frac{1}{x^2}$ using the power rule and $g'(1) = 0$ as before.

6. (18 Marks)

(a) The equation of a curve defined implicitly is $x^2e^y + x + \ln(1+y) = 2$. Verify that the point $(1,0)$ belongs to the curve. Find an equation of the tangent line to the curve at this point.

(b) Let $f(x) = x^{1/3}$. Find a number c that satisfies the Mean Value Theorem for the function $f(x)$ on $[0,1]$.

(c) Use l'Hopital's rule to evaluate

$$\lim_{x \to 0} \frac{\sin x - x}{x^3}$$

Solutions (a) Substitute: $1 \cdot e^0 + 1 + \ln(1+0) = 2$ so the equation is satisfied and the point is on the curve. Next, differentiate:

$$\begin{aligned} x^2e^y + x + \ln(1+y) &= 2 \\ 2xe^y + x^2e^y y' + 1 + \frac{y'}{y+1} &= 0 \text{ now substitute} \\ 2 + y' + 1 + y' &= 0 \\ 2y' &= -3 \\ y' &= -\frac{3}{2} \end{aligned}$$

So an equation of the tangent line is

$$y = -\frac{3}{2}(x-1)$$

(b) $f'(x) = 1/3x^{-2/3}$ which is defined on $(0,1)$ but not on $[0,1]$ and we want a number c so that

$$\begin{aligned} \frac{f(1) - f(0)}{1-0} &= f'(c), \text{ i.e.} \\ 1 &= \frac{1}{3}c^{-2/3} \Longrightarrow c^{2/3} = \frac{1}{3} \text{ so} \\ c &= \left(\frac{1}{3}\right)^{3/2} \approx 0.192\,45 \end{aligned}$$

(c)

$$\begin{aligned}
\lim_{x\to 0}\frac{\sin x - x}{x^3} &= \lim_{x\to 0}\frac{\cos x - 1}{3x^2} \text{ l'Hopital's rule still applies:}\\
&= \lim_{x\to 0}\frac{-\sin x}{6x} \text{ l'Hopital's rule still applies:}\\
&= \lim_{x\to 0}\frac{-\cos x}{6} = -\frac{1}{6}
\end{aligned}$$

7. (10 Marks)

(a) The equation of a curve in the plane is $e^{xy} = x^2 + y^2$. A particle is moving along this curve at a certain velocity. At the instant that it moves through the point $(0, 1)$ the $y-$coordinate is decreasing at the rate of 3 cm/sec. How fast is the $x-$coordinate changing at this instant and is it increasing or decreasing?

(b) A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Solutions (a) Differentiate with respect to time and then substitute:

$$\begin{aligned}
e^{xy}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right) &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt}\\
1\left(0\cdot\frac{dy}{dt} + 1\cdot\frac{dx}{dt}\right) &= 0\cdot\frac{dx}{dt} + 2\cdot\frac{dy}{dt}\\
\frac{dx}{dt} &= 2\frac{dy}{dt} = 2\cdot 3 = 6 \text{ cm/sec}
\end{aligned}$$

and since the sign is + it is increasing.

(b) The semicircle (assume it is the upper one) has equation

$$y = \sqrt{4 - x^2}$$

and a rectangle would have vertices $(\pm x, 0)$ and $\left(\pm x, \sqrt{4 - x^2}\right)$ so the area is

$$A = 2x\sqrt{4 - x^2}; x \geq 0$$

differentiate to find local or global extrema:

$$\begin{aligned} A' &= 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}} = 4 \cdot \frac{2-x^2}{\sqrt{4-x^2}} \\ &= 0 \Leftrightarrow x = \sqrt{2} \text{ and } y = \sqrt{2} \text{ so} \\ A &= 4 \end{aligned}$$

To check that this is a maximum we can either calculate the second derivative which is

$$A'' = 4x \frac{x^2 - 6}{\left(\sqrt{4-x^2}\right)^3} < 0 \text{ at } x = \sqrt{2}$$

or (easier) note that A' changes sign from positive to negative as x passes through the critical point from left to right. So it is a global maximum since there is only one and $A = 0$ at the endpoints.

8. (14 Marks) Given the function $f(x) = xe^{-x}$,

(a) Find the domain and check for symmetry. Find asymptotes (if any).

(b) Calculate$f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

(c) Calculate$f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function.

Solutions (a) The equation shows that the domain is all of $\mathbb{R}$. Replacing x by $-x$ gives a different function - i.e. $f(-x) = -xe^x \neq f(x)$ and also $\neq -f(x)$ so there is no even or odd symmetry. Since the domain is $\mathbb{R}$ there are no vertical asymptotes but we see (using L'Hopital's rule, if necessary) that

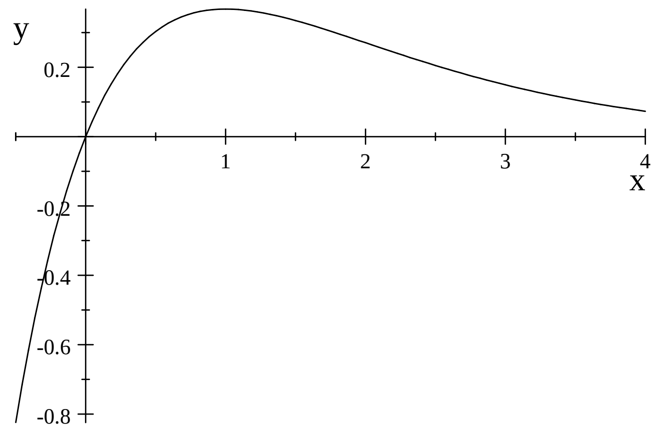
$$\lim_{x \to \infty} xe^{-x} = 0$$

so $y = 0$ is a horizontal asymptote.

(b) $f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x}$ so $x = 1$ is a critical point. Since the derivative is positive for $x < 1$ and negative for $x > 1$ it means the function is increasing on $(-\infty, 1)$, decreasing on $(1, \infty)$ and has a maximum (which must be global) at $x = 1$, $y = e^{-1}$.

(c) $f''(x) = xe^{-x} - 2e^{-x} = (x-2)e^{-x}$. Since it changes sign at $x = 2$ this must be an inflection point. The second derivative is positive for $x > 2$ and negative for $x < 2$ so the function is concave down on $(-\infty, 2)$, and concave up on $(2, \infty)$.

(d)



Bonus Question (5 Marks) Let $f(x) = x\sin(|x|)$.

Use the definition of the derivative to show that f is differentiable at $x = 0$ and

find $f'(0)$.

Solution We can't use differentiation rules here to calculate $f'(0)$ because $|x|$ is not differentiable. Also, if we calculate $f'(x)$ for $x \neq 0$ we get

$$\begin{aligned} f'(x) &= \sin x + x\cos x \text{ if } x > 0 \\ f'(x) &= -\sin x - x\cos x \text{ if } x < 0 \end{aligned}$$

and so substituting $x = 0$ in either equation gives the answer 0. But this is incorrect, because neither one is valid when $x = 0$. We have to use the the definition of the derivative:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f\left(0+h\right)-f\left(0\right)}{h} \\ &= \lim_{h\to 0} \frac{\left(0+h\right)\sin\left(\left|0+h\right|\right)-0}{h} \\ &= \lim_{h\to 0} \frac{h\sin\left(\left|h\right|\right)}{h} = \lim_{h\to 0} \sin\left(\left|h\right|\right) = 0 \end{aligned}$$

and **that** is why the answer is zero!.

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	203	All
Examination	**Date**	**Pages**
Final	April 2008	3
Instructors		**Course Examiner**
A. Djerrahian, C. Grabowski, H. Greenspan, J. Li, H. Proppe		H. Proppe

Special Instructions

▷ **Only Sharp EL 531 or Casio FX 300 MS calculators are allowed.**

MARKS

[9] **1.** (a) Sketch the graph of the function $f(x) = |(x+2)^2 - 4|$ starting from the graph of the standard parabola and using appropriate transformations.

(b) Suppose $f(x) = \sqrt[3]{1+e^x}$, and $g(x) = \ln(x^3 - 1)$. Find $f \circ g$ and $g \circ f$. Determine the domain and range of $f \circ g$ and $g \circ f$.

(c) Evaluate $\cos(\sin^{-1}(t))$.

[8] **2.** Evaluate the limits:

(a) $\displaystyle\lim_{x\to 4} \frac{\sqrt{2x+1}-3}{x^3-64}$ (b) $\displaystyle\lim_{x\to -\infty} \frac{(x^4+1)(3-2x)^3}{(x+1)^5(2-x^2)}$

Do not use l'Hopital's rule.

[10] **3.** (a) Consider the function $f(x) = \dfrac{|x+1|}{x^2-1}$.

Calculate both one-sided limits at the point(s) where the function is undefined.

(b) Find parameters a and b such that the function

$$f(x) = \begin{cases} x+5 & \text{if} \quad x < 0 \\ (x-a)^2 + b, & \text{if} \quad 0 \le x < 2 \\ 1, & \text{if} \quad x \ge 2 \end{cases}$$

will be continuous at every point. Sketch the graph of this function.

[15] **4.** Find derivatives of the functions (do not simplify the answer):

(a) $f(x) = \dfrac{\sqrt[3]{x} - 2\sqrt[5]{x^3} + x^4}{\sqrt{x}}$;

(b) $f(x) = e^{-2x^3}(\sin x + \tan x)^2$;

(c) $f(x) = \sec(\arcsin 2x)$;

(d) $f(x) = \dfrac{\ln^2(\sqrt{x})}{1 + \sqrt{e^{2x}}}$;

(e) $f(x) = (\arctan x)^{1+x^2}$ (use logarithmic differentiation).

[6] **5.** Given the function $f(x) = x^2 + \dfrac{1}{x}$,

(a) Use the definition of derivative to find the derivative of the function.

(b) Use the appropriate differentiation rule(s) to verify (a).

[6] **6.** (a) Find the differential of the function $f(x) = \tan x$ at $a = \pi/4$.

(b) Use the differential above to estimate $\tan 0.8$ (to calculate dx, you may use 0.785 as an approximation of $\pi/4$) .

[10] **7.** (a) A curve called a "devil's curve" is defined implicitly by the equation $y^2(y^2 - 4) = x^2(x^2 - 5)$. Verify that the point $(0, -2)$ belongs to the curve. Find an equation of the tangent line to the curve at this point.

(b) Use l'Hopital's rule to evaluate $\lim\limits_{x \to 0} \dfrac{\sin x - x}{x \cos x - x}$.

[10] **8.** (a) Let $f(x) = e^{\ln(\arctan(2x))}$. Find $f''(x)$.

(b) Let $f(x) = x^3 + x - 1$. Find a number c that satisfies the conclusion of the Mean Value Theorem for the function $f(x)$ on $[0, 2]$.

[10] **9.** (a) At what rate is the area of an equilateral triangle increasing if its base is 10 cm long and increasing at 0.5 cm/s?

(b) If 1200 cm^2 of material is available to make a box with a square base and no top, find the largest possible volume of the box.

[16] **10.** Given the function $f(x) = \dfrac{2x^2}{x^2 - 1}$,

(a) Find the domain and check for symmetry. Find all asymptotes.

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function, and label the local extrema and inflection point(s) you have found (if any) in parts (b) and (c).

[5] **Bonus Question**

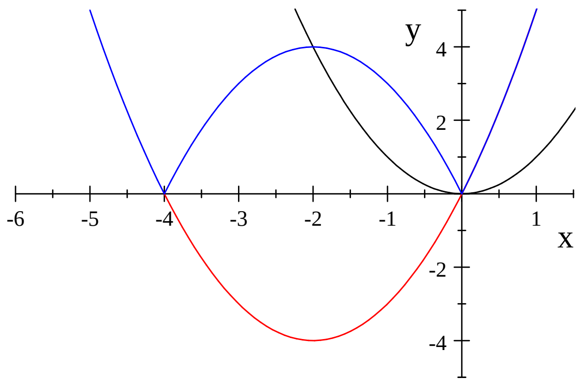
Two runners start a race at the same time and finish in a tie. Show that at some time during the race they have exactly the same speed. [*Hint:* Let $g(t)$ be the distance the first runner covers in time t and let $h(t)$ be the distance the second runner covers in time t. Now put $f(t) = g(t) - h(t)$ and explain how to apply Rolle's Theorem to this situation].

MATH 203 Final Exam April 2008
Solutions

1. (a) Sketch the graph of the function $f(x) = |(x+2)^2 - 4|$, starting from the graph

of the standard parabola and using appropriate transformations.

Solution



(b) Suppose $f(x) = \sqrt[3]{1+e^x}$ and $g(x) = \ln(x^3 - 1)$. Find $f \circ g$ and $g \circ f$. Determine the domain and range of $f \circ g$ and $g \circ f$.

Solution $f \circ g\,(x) = \sqrt[3]{1 + e^{\ln(x^3-1)}} = \sqrt[3]{1 + x^3 - 1} = x$ and $g \circ f\,(x) = \ln\left(\left(\sqrt[3]{1+e^x}\right)^3 - 1\right) = \ln(e^x) = x$. So f and g are inverse to each other which means Domain(f) = Range(g) and vice-versa. Now f is defined for all x, so Domain(f) = Range$(g) = \mathbb{R}$. On the other hand g is only defined if $x^3 - 1 > 0$, i.e. $x > 1$ so Domain(g) = Range$(f) = (1, \infty)$.

(c) Evaluate $\cos(\sin^{-1}(t))$.

Solution Let $u = \sin^{-1}(t)$. Then $\sin u = t$ and so since $\sin^2 u + \cos^2 u = 1$, $\cos^2 u = 1 - t^2$. So $\cos u = \cos(\sin^{-1}(t)) = \sqrt{1-t^2}$ with the + sign because $-\pi/2 \le u \le \pi/2$

2. Evaluate the limits:

(a) $\displaystyle\lim_{x\to 4} \frac{\sqrt{2x+1} - 3}{x^3 - 64}$; (b) $\displaystyle\lim_{x\to\infty} \frac{(x^4+1)(3-2x)^3}{(x+1)^5(2-x^2)}$

Do not use l'Hopital's rule.

Solution (a)

$$\begin{aligned}\lim_{x\to 4}\frac{\sqrt{2x+1}-3}{x^3-64} &= \lim_{x\to 4}\frac{\sqrt{2x+1}-3}{(x-4)(x^2+4x+16)}\cdot\frac{\sqrt{2x+1}+3}{\sqrt{2x+1}+3}\\ &= \lim_{x\to 4}\frac{2x-8}{(x-4)(x^2+4x+16)\left(\sqrt{2x+1}+3\right)}\\ &= \lim_{x\to 4}\frac{2}{(x^2+4x+16)\left(\sqrt{2x+1}+3\right)}\\ &= \frac{2}{(4^2+4^2+4^2)\left(\sqrt{9}+3\right)}=\frac{1}{144}\end{aligned}$$

(b)

Solution

$$\begin{aligned}\lim_{x\to -\infty}\frac{(x^4+1)(3-2x)^3}{(x+1)^5(2-x^2)} &= \lim_{x\to -\infty}\frac{\left[(x^4+1)/x^4\right]\left[(3-2x)^3/x^3\right]}{\left[(x+1)^5/x^5\right]\left[(2-x^2)/x^2\right]}\\ &= \lim_{x\to -\infty}\frac{(x^4/x^4+1/x^4)(3/x-2x/x)^3}{(x/x+1/x)^5(2/x^2-x^2/x)}\\ &= \lim_{x\to -\infty}\frac{(1+1/x^4)(3/x-2)^3}{(1+1/x)^5(2/x^2-1)}=\frac{1\cdot(-2)^3}{1\cdot(-1)}\\ &= 8\end{aligned}$$

3. (a)

Solution

$$f(x)=\frac{|x+1|}{x^2-1}=\frac{|x+1|}{(x-1)(x+1)}$$

is undefined only at $x = 1$ and -1, and we see that

$$\begin{aligned}
\lim_{x\to 1^-} f(x) &= \lim_{x\to 1^-} \frac{|x+1|}{(x+1)} \lim_{x\to 1^-} \frac{1}{(x-1)} \\
&= (+1)\cdot(-\infty) = -\infty \\
\lim_{x\to 1^+} f(x) &= \lim_{x\to 1^+} \frac{|x+1|}{(x+1)} \lim_{x\to 1^+} \frac{1}{(x-1)} \\
&= (+1)\cdot(+\infty) = +\infty \\
\lim_{x\to -1^-} f(x) &= \lim_{x\to 1^-} \frac{|x+1|}{(x+1)} \lim_{x\to -1^-} \frac{1}{(x-1)} \\
&= (-1)\cdot\left(-\frac{1}{2}\right) = \frac{1}{2} \\
\lim_{x\to -1^+} f(x) &= \lim_{x\to 1^+} \frac{|x+1|}{(x+1)} \lim_{x\to -1^+} \frac{1}{(x-1)} \\
&= (+1)\cdot\left(-\frac{1}{2}\right) = -\frac{1}{2}
\end{aligned}$$

(b)

Solution

$$\begin{aligned}
f(x) &= x+5 \text{ if } x<0 \text{ and} \\
f(x) &= (x-a)^2 + b \text{ if } 0 \le x < 2 \text{ and so} \\
\lim_{x\to 0^-} f(x) &= 5 = \lim_{x\to 0^+} f(x) = f(0) = a^2 + b
\end{aligned}$$

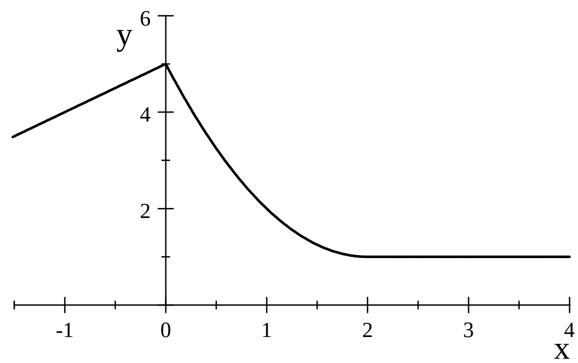
for continuity. Similarly,

$$\lim_{x\to 2^-} f(x) = 1 = 4 - 4a + a^2 + b$$

So we have the two equations

$$\begin{aligned}
a^2 + b &= 5 \\
a^2 + b - 4a &= -3
\end{aligned}$$

Subtracting gives $4a = 8$ and so $a = 2$ and finally $b = 1$. The graph is below.



4. (a) $f(x) = \dfrac{\sqrt[3]{x} - 2\sqrt[5]{x^3} + x^4}{\sqrt{x}} = \dfrac{x^{1/3} - 2x^{3/5} + x^4}{x^{1/2}} = x^{-1/6} - 2x^{1/10} + x^{3.5}$

Solution $f'(x) = -\dfrac{1}{6}x^{-7/6} - \dfrac{2}{10}x^{-9/10} + 3.5x^{2.5}$ (of course if you don't simplify before differentiating the answer will look different).

(b) $f(x) = e^{-2x^3}(\sin x + \tan x)^2$

Solution

$f'(x) = e^{-2x^3} \cdot 2(\sin x + \tan x)(\cos x + \sec^2 x) + (-6x^2)e^{-2x^3}(\sin x + \tan x)^2$ (product, chain rules)

(c) $f(x) = \sec(\arcsin 2x)$

Solution

$$f'(x) = \sec(\arcsin 2x)\tan(\arcsin 2x) \cdot \frac{1}{\sqrt{1-4x^2}} \cdot 2 \text{ (chain rule)}$$

(d) $f(x) = \dfrac{\ln^2(\sqrt{x})}{1+\sqrt{e^{2x}}} = \dfrac{\ln^2(x)}{4(1+e^x)}$ because $\ln(\sqrt{x}) = \frac{1}{2}\ln(x)$

Solution $f'(x) = \dfrac{1}{4}\dfrac{(1+e^x)\cdot 2\ln x \cdot 1/x - \ln^2(x)\cdot e^x}{(1+e^{x/2})^2}$

(e) $f(x) = (\arctan(x))^{1+x^2}$

Solution We take logarithms first:

$$\begin{aligned}
\ln f &= \left(1+x^2\right)\cdot\ln\left(\arctan\left(x\right)\right) \\
\frac{f'}{f} &= 2x\ln\left(\arctan\left(x\right)\right)+\left(1+x^2\right)\cdot\frac{1}{\arctan\left(x\right)}\frac{1}{1+x^2} \\
f'\left(x\right) &= \left(\arctan\left(x\right)\right)^{1+x^2}\left(2x\ln\left(\arctan\left(x\right)\right)+\frac{1}{\arctan\left(x\right)}\right)
\end{aligned}$$

5. (a) $f\left(x\right)=x^2+\dfrac{1}{x}$

Solution

$$\begin{aligned}
\frac{f\left(x+h\right)-f\left(x\right)}{h} &= \frac{\left(x+h\right)^2+\frac{1}{x+h}-x^2-\dfrac{1}{x}}{h} \\
&= \frac{2xh+h^2}{h}+\frac{x-\left(x+h\right)}{h\cdot x\cdot\left(x+h\right)} \\
&= 2x+h-\frac{1}{x\cdot\left(x+h\right)};h\neq 0
\end{aligned}$$

and so

$$\begin{aligned}
f'\left(x\right) &= \lim_{h\to 0}\frac{f\left(x+h\right)-f\left(x\right)}{h} \\
&= \lim_{h\to 0}2x+h-\frac{1}{x\cdot\left(x+h\right)} \\
&= 2x-\frac{1}{x^2}
\end{aligned}$$

(b)

Solution Since $f\left(x\right)=x^2+\dfrac{1}{x}=x^2+x^{-1}$ the power rule gives $f'\left(x\right)=2x-x^{-2}$ which is the same as above.

6. (a) $f\left(x\right)=\tan x$ and $a=\pi/4$

Solution

$$\begin{aligned}
df &= f'\left(a\right)dx \\
&= \sec^2\left(\pi/4\right)dx \\
&= 2dx \text{ since } \sec^2\left(\pi/4\right)=2
\end{aligned}$$

(b) Estimate $\tan 0.8$ using the above differential.

Solution

Assume $\pi/4 = .785$. Then

$$\begin{aligned} dx &= .8 - .785 = .015 \\ df &= 2dx = .030 \end{aligned}$$

Since $\tan \pi/4 = 1$ it follows that $\tan 0.8 \approx 1 + 0.030 = 1.030$ (note that the exact value is 1.0296

7.

$$y^2\left(y^2-4\right) = x^2\left(x^2-5\right)$$

(a)

Solution $4(4-4) = 0$ so the point $(0,-2)$ is on the curve. By implicit differentiation we have

$$\begin{aligned} 2yy'\left(y^2-4\right) + y^2(2yy') &= 2x\left(x^2-5\right) + x^2(2x) \text{ now substitute} \\ 0 + 4\cdot(-4)\,y' &= 0 \\ y' &= 0 \end{aligned}$$

So the tangent line has equation $y - (-2) = 0$ or $y = -2$

(b)

Solution Since $\sin 0 - 0 = 0 = 0\cos 0 - 0$ we have an indeterminate form of the $\frac{0}{0}$ type so L'Hopital's rule applies:

$$\begin{aligned} \lim_{x\to 0}\frac{\sin x - x}{x\cos x - x} &= \lim_{x\to 0}\frac{\cos x - 1}{\cos x - x\sin x - 1};\ \text{still } \frac{0}{0} \text{ type} \\ &= \lim_{x\to 0}\frac{-\sin x}{-\sin x - \sin x - x\cos x} \\ &= \lim_{x\to 0}\frac{\sin x}{2\sin x + x\cos x};\ \text{still } \frac{0}{0} \text{ type} \\ &= \lim_{x\to 0}\frac{\cos x}{2\cos x + \cos x - x\sin x} = \frac{1}{3} \end{aligned}$$

8. (a) $f(x) = e^{\ln(\arctan(2x))}$

Solution Simplifying first gives

$$\begin{aligned} f(x) &= \arctan(2x) \text{ and so} \\ f'(x) &= \frac{2}{1+4x^2} \\ f''(x) &= \frac{(1+4x^2)\cdot 0 - 2\cdot 8x}{(1+4x^2)^2} = \frac{-16x}{(1+4x^2)^2} \end{aligned}$$

(b) Let $f(x) = x^3 + x - 1$; find c satisfying MVT on $[0, 2]$.

Solution

$$\begin{aligned} f(0) &= -1;\ f(2) = 9 \\ \frac{f(2) - f(0)}{2-0} &= \frac{9+1}{2} = 5 \\ f'(x) &= 3x^2 + 1 \text{ so we want} \\ f'(c) &= 3c^2 + 1 = 5 \\ c^2 &= \frac{4}{3} \\ c &= \sqrt{\frac{4}{3}} \end{aligned}$$

9. (a) At what rate is the area of an equilateral triangle increasing if its base is

10 cm long and increasing at 0.5 cm/s?

Solution Let x be the length of a side of the triangle. The area (call it A) is then given by

$$A = \frac{1}{2}x \cdot h$$

where h is the altitude. But it is clear that $\frac{h}{x} = \sin(\pi/3) = \sqrt{3}/2$ for an equilateral triangle (with all angles equal to $\pi/3$). So

$$\begin{aligned} A &= \frac{\sqrt{3}}{4}x^2; \text{ now differentiate with respect to } t \\ \frac{dA}{dt} &= \frac{\sqrt{3}}{2}x\frac{dx}{dt} \\ &= \frac{\sqrt{3}}{2}(10)(0.5) = \frac{5\sqrt{3}}{2} \text{ cm}^2/\text{s} \end{aligned}$$

or approximately 4.33 cm^2/s.

(b) If 1200 cm^2 of material is available to make a box with a square base and

no top, find the largest possible volume of the box.

Solution Let x be the length of the base, h the height and V the total volume. We are given that

$$\begin{aligned} x^2 + 4xh &= 1200 \text{ and want to maximize} \\ V &= x^2 h \end{aligned}$$

So we eliminate h using the first equation: $h = \dfrac{1}{4x}\left(1200 - x^2\right)$ and so

$$\begin{aligned} V &= \frac{x}{4}\left(1200 - x^2\right) = -\frac{x^3}{4} + 300x \\ \frac{dV}{dx} &= -\frac{3x^2}{4} + 300 = 0 \Longrightarrow x = 20 \text{ and } h = \frac{1200}{400} = 3 \end{aligned}$$

To check that we have a maximum,

$$\frac{d^2V}{dx^2} = -\frac{3x}{2} + 300 = -30 + 300 < 0$$

so we do have a local and hence an absolute maximum.

10.

$$f(x) = \frac{2x^2}{x^2 - 1}$$

Solution

(a) Find the domain and check for symmetry. Find all asymptotes.

Solution The domain consists of all numbers for which the formula is defined, which means for which the denominator $\neq 0$. So $x^2 - 1 = (x-1)(x+1) \neq 0$ which means $x \neq 1$ and $x \neq -1$. The domain is $(-\infty, -1) \cup (-1, 1) \cup (1, \infty)$ and there are vertical asymptotoes at $x = \pm 1$ since the function has infinite one-sided limits at those points. Also,

$$\lim_{x \to \infty} \frac{2x^2}{x^2 - 1} = 2$$

so $y = 2$ is a horizontal asymptote. Since $f(-x) = f(x)$ the function is even (even symmetry).

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is

increasing, interval(s) where the function is decreasing, and local extrema

(if any).

Solution

$$\begin{aligned} f'(x) &= 4\frac{x}{x^2-1} - 4\frac{x^3}{(x^2-1)^2} \\ &= -4\frac{x}{(x-1)^2(x+1)^2} \end{aligned}$$

so $x = 0$ is a critical point. We have the following table (valid except for the points ± 1, which are not in the domain):

	$x < 0$	$x = 0$	$x > 0$
$f'(x)$	$+$	0	$-$
f	$\nearrow$	l. max	$\searrow$

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

Solution

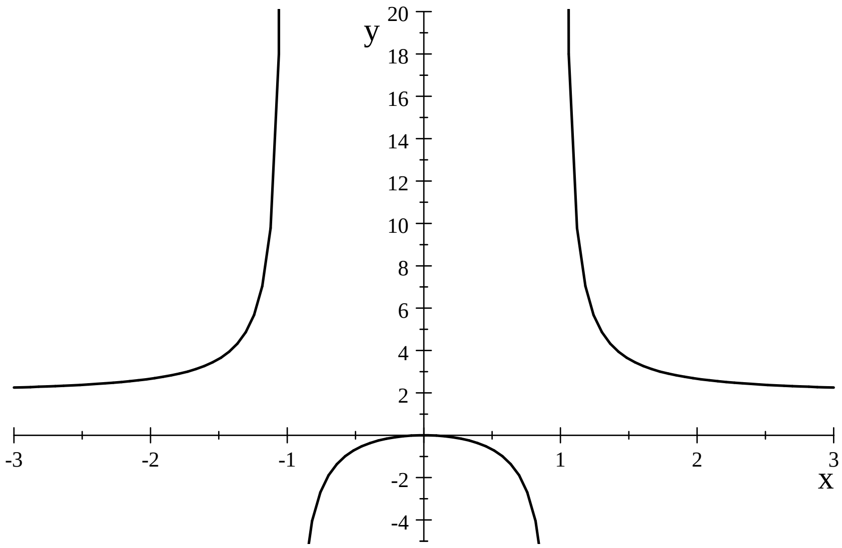
$$f''(x) = 4\frac{3x^2+1}{(x-1)^3(x+1)^3}$$

We see that there are no possible inflection points, but the concavity does change as we cross the vertical asymptotes. Here is the analysis:

	$x < -1$	$x = -1$	$-1 < x < 1$	$x = 1$	$x > 1$
$x - 1$	$-$		$-$		$+$
$x + 1$	$-$		$+$		$+$
$f''(x)$	$+$	undef.	$-$	undef.	$+$
	$\cup$		$\cap$		$\cup$

(d) Sketch the graph of the function.

Solution



Bonus Let $g(t)$ be the distance the first runner covers in time t and let $h(t)$ be the distance the second runner covers in time t. We know $g(0) = h(0)$ and also $g(T) = h(T)$ where T is the time is takes either one to run the race.Put $f(t) = g(t) - h(t)$. Now $f(0) = f(T) = 0$ and so, by Rolles' Theorem, there must be a time t_c at which

$$\begin{aligned} \frac{f(T) - f(0)}{T - 0} &= f'(t_c), \text{ i.e.} \\ 0 &= f'(t_c) = g'(t_c) - h'(t_c) \text{ and so} \\ g'(t_c) &= h'(t_c) \end{aligned}$$

Which means the speed of the two runners at time t_c is exactly the same.

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	203	All
Examination	**Date**	**Pages**
Final	December 2009	3
Instructors		**Course Examiner**
A. Boyarsky, J. Brody, N. Hardy, H. Proppe, R. Perez-Buendia, S.S. Roy		A. Atoyan

Special Instructions

▷ **Only Sharp EL 531 or Casio FX 300 MS calculators are allowed.**

MARKS

[10] **1.** (a) Suppose $f(x) = \sqrt{x-1}$, $g(x) = x^2 + 2$ and $h(x) = \ln(x+3)$. Find $f \circ g \circ h$ and the domain of $f \circ g \circ h$.

(b) Find the inverse of the function $f(x) = \sqrt[3]{2^x + 8}$. Determine the domain and range of f and f^{-1}.

[10] **2.** Evaluate the limits:

(a) $\displaystyle\lim_{x \to 7} \frac{\sqrt{x+2} - 3}{x^2 - 5x - 14}$ (b) $\displaystyle\lim_{x \to -\infty} \frac{x^2\,(2x+1)^3}{(x^2+2)^2\,(\sqrt{x^2+2x+3})}$

Do not use l'Hopital's rule.

[12] **3.** (a) Consider the function $f(x) = \dfrac{|x+2|}{x^2 - x - 6}$.

Calculate both one-sided limits at the point(s) where the function is undefined.

(b) Find parameters a and b such that the function

$$f(x) = \begin{cases} x^2 - a, & \text{if} \quad x \le 0 \\ bx + 1, & \text{if} \quad 0 < x \le 1 \\ \dfrac{1}{x} + 2, & \text{if} \quad x > 1 \end{cases}$$

will be continuous at every point. Sketch the graph of this function.

[15] **4.** Find derivatives of the functions (you do not need to simplify your answers):

(a) $f(x) = \dfrac{2\sqrt{x} - x^{3/2} - 5x^3}{\sqrt[4]{x}}$;

(b) $f(x) = \left(x - \dfrac{1}{x}\right)^2 \cos^3 x$;

(c) $f(x) = \tan(\sqrt{3x^2 + 5e^{-x}})$;

(d) $f(x) = \dfrac{\arctan(x/2)}{4 + x^2}$;

(e) $f(x) = (1 + x^2)^{2x}$ (use logarithmic differentiation).

[12] **5.** Given the function $f(x) = \sqrt{2x + 2}$,

(a) Use appropriate differentiation rules to find the derivative of the function.

(b) Use the definition of derivative to verify (a).

(c) Find the linear approximation of the function at $x_0 = 7$.

(d) Use the linear approximation above to approximate $\sqrt{18}$.

[15] **6.** (a) The equation of a curve is $\ln(x^2 + 3y) = xy - \dfrac{3}{x} + 3$.

Verify that the point $(1, 0)$ belongs to the curve. Find an equation of the tangent line to the curve at this point.

(b) Let $f(x) = x^3 - 3x + 1$. Find a number c such that $f'(c)$ is equal to the slope of the secant line joining the points $(0, f(0))$ and $(2, f(2))$.

(c) Use l'Hopital's rule to evaluate $\lim\limits_{x \to 0} \dfrac{\sin x + \sin(2x) - \sin(3x)}{x^3}$.

[10] **7.** (a) A particle is moving along the curve $y = \sqrt{1+x^3}$ in the $x-y$ plane. At the point (2,3) the y-coordinate is increasing at a rate of 4 cm/sec. How fast is the x-coordinate changing at that instant?

(b) Find the dimensions of the rectangle of largest area that can be inscribed in a circle of radius r.

[16] **8.** Given the function $f(x) = \dfrac{x^2-3}{x+2}$,

(a) Find the domain and check for symmetry. Find asymptotes (if any).

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function $f(x)$.

[5] **Bonus Question**

Let $f(x) = |x-1|$.

(a) Show that there is no value of c such that

$$f(3) - f(0) = f'(c)(3-0)$$

(b) Why does this not contradict the Mean Value Theorem?

MATH 203 Final Examination
December 2009

SOLUTIONS

MARKS

10

1 (a) $h(x) = f \circ g(x) \circ h(x) = f \circ (g(h(x))) = f\left(\ln^2(x+3)+2\right) = \sqrt{\left(\ln^2(x+3)+2\right)-1}$
$= \sqrt{\ln^2(x+3)+1}$. The domain of h is determined by the requirement

$$\begin{aligned} \ln^2(x+3)+1 &\geq 0 \\ \ln^2(x+3) &\geq -1 \text{ and so} \\ x+3 &> 0 \end{aligned}$$

since $\ln^2(x+3) \geq -1$ automatically if it exists. So the domain is $(-3, \infty)$.

1 (b) If $y = f(x) = \sqrt[3]{2^x+8}$ we see that that $2^x + 8 > 8$ for all x, so $f(x)$ is defined on all of $\mathbb{R}$. Also, the range is $(2, \infty)$ because $\lim_{x\to-\infty} \sqrt[3]{2^x+8} = 2$. To find the inverse function f^{-1} first solve for x. We do this by putting

$$\begin{aligned} y^3 &= 2^x + 8 \\ y^3 - 8 &= 2^x \\ x &= \log_2\left(y^3-8\right) \end{aligned}$$

Now interchange x and y to get

$$y = \log_2\left(x^3-8\right)$$

as the inverse function f^{-1}. It is defined for all $x > 2$, i.e. it has domain $(2, \infty)$ and of course has range $(-\infty, \infty)$, the domain of f.

10

2 (a)

$$\begin{aligned} \lim_{x\to 7} \frac{\sqrt{x+2}-3}{x^2-5x-14} &= \lim_{x\to 7} \frac{\sqrt{x+2}-3}{(x-7)(x+2)} \cdot \frac{\sqrt{x+2}+3}{\sqrt{x+2}+3} \\ &= \lim_{x\to 7} \frac{(x-7)}{(x-7)(x+2)} \cdot \frac{1}{\sqrt{x+2}+3} \\ &= \lim_{x\to 7} \frac{1}{(x+2)\left(\sqrt{x+2}+3\right)} = \frac{1}{54} \end{aligned}$$

(b)

$$\lim_{x\to\infty}\frac{x^2\left(2x+1\right)^3}{\left(x^2+2\right)^2\sqrt{x^2+2x+3}}=\lim_{x\to\infty}\frac{x^5\left(2+1/x\right)^3}{x^5\left(1+2/x\right)^2\sqrt{1+2/x+3/x^2}}$$
$$=\frac{\lim_{x\to\infty}\left(2+1/x\right)^3}{\lim_{x\to\infty}\left(1+2/x\right)^2\sqrt{1+2/x+3/x^2}}$$
$$=\frac{\left(2+0\right)^3}{\left(1+0\right)\sqrt{1+0+0}}=\frac{8}{1}=8$$

12

3 (a)

$$f\left(x\right)=\frac{|x+2|}{x^2-x-6}=\frac{|x+2|}{(x+2)}\cdot\frac{1}{(x-3)}$$

is undefined at $x=-2$ and $x=3$. We have

$$\lim_{x\to-2^-}f\left(x\right)=\lim_{x\to-2^-}\frac{|x+2|}{(x+2)}\lim_{x\to-2^-}\frac{1}{(x-3)}=(-1)\cdot\left(-\frac{1}{5}\right)=\frac{1}{5}$$
$$\lim_{x\to-2^+}f\left(x\right)=\lim_{x\to-2^+}\frac{|x+2|}{(x+2)}\lim_{x\to-2^-}\frac{1}{(x-3)}=(+1)\cdot\left(-\frac{1}{5}\right)=-\frac{1}{5}$$
$$\lim_{x\to3^-}f\left(x\right)=\lim_{x\to3^-}\frac{1}{(x-3)}\lim_{x\to3^-}\frac{|x+2|}{(x+2)}=(-\infty)\cdot(1)=-\infty$$
$$\lim_{x\to3^+}f\left(x\right)=\lim_{x\to3^+}\frac{1}{(x-3)}\lim_{x\to3^+}\frac{|x+2|}{(x+2)}=(+\infty)\cdot(1)=\infty$$

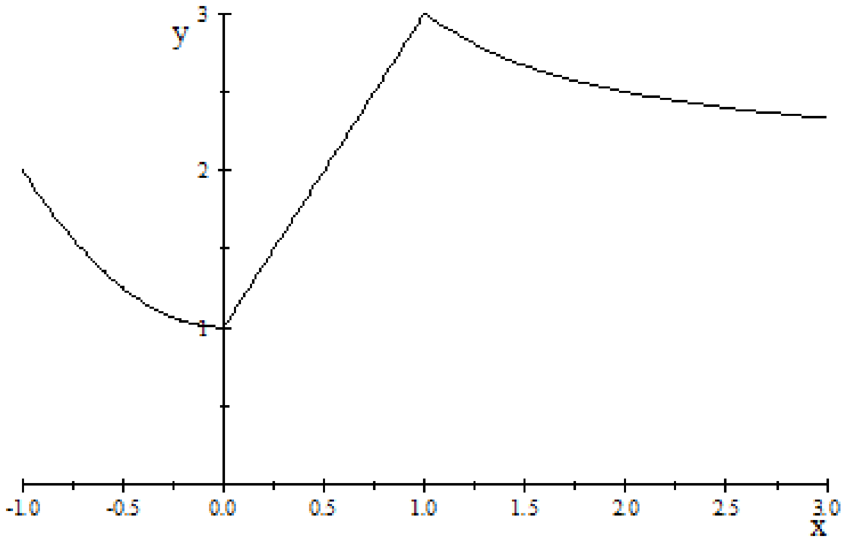
3 (b) We calculate

$$\lim_{x\to0^-}f\left(x\right)=-a=\lim_{x\to0^+}f\left(x\right)=1$$

so $a=-1$. Also

$$\lim_{x\to1^-}f\left(x\right)=b+1=\lim_{x\to1^+}f\left(x\right)=3$$

so $b+1=3$ and $b=2$. Here is the graph:



15

4 (a)

$$f\,(x) = \frac{2\sqrt{x} - x^{3/2} - 5x^3}{\sqrt[4]{x}} = 2x^{1/4} - x^{5/4} - 5x^{11/4}$$
$$f'\,(x) = 1/2x^{-3/4} - 5/4x^{1/4} - 55/4x^{7/4}$$

(b)

$$f\,(x) = \left(x - \frac{1}{x}\right)^2 \cos^3 x$$
$$f'\,(x) = 2\left(x - \frac{1}{x}\right)\left(1 + \frac{1}{x^2}\right)\cos^3 x + \left(x - \frac{1}{x}\right)^2 \cdot 3\cos^2 x \cdot (-\sin x)$$

(c)

$$f\,(x) = \tan\left(\sqrt{3x^2 + 5e^{-x}}\right)$$
$$f'\,(x) = \sec^2\left(\sqrt{3x^2 + 5e^{-x}}\right) \cdot \frac{6x - 5e^{-x}}{2\sqrt{3x^2 + 5e^{-x}}}$$

(d)

$$
\begin{aligned}
f(x) &= \frac{\arctan(x/2)}{4+x^2} \\
f'(x) &= \frac{\left(4+x^2\right)\cdot \dfrac{1/2}{1+x^2/4} - \arctan(x/2)\cdot 2x}{\left(4+x^2\right)^2} \\
&= \frac{2-\arctan(x/2)\cdot 2x}{\left(4+x^2\right)^2}
\end{aligned}
$$

(e)

$$
\begin{aligned}
f(x) &= \left(1+x^2\right)^{2x} \\
\ln(f(x)) &= 2x\ln\left(1+x^2\right) \\
\frac{f'(x)}{f(x)} &= 2\ln\left(1+x^2\right) + \frac{2x}{1+x^2}\cdot 2x \\
&= 2\ln\left(1+x^2\right) + \frac{4x^2}{1+x^2} \\
f'(x) &= \left(1+x^2\right)^{2x}\left(2\ln\left(1+x^2\right) + \frac{4x^2}{1+x^2}\right)
\end{aligned}
$$

12

5 (a) $f'(x) = \dfrac{1}{2\sqrt{2x+2}}\cdot 2 = \dfrac{1}{\sqrt{2x+2}}$

(b)

$$
\begin{aligned}
f(x) &= \sqrt{2x+2} \\
f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{\sqrt{2(x+h)+2}-\sqrt{2x+2}}{h} \\
&= \lim_{h\to 0}\frac{\sqrt{2x+2h+2}-\sqrt{2x+2}}{h}\cdot\frac{\sqrt{2x+2h+2}+\sqrt{2x+2}}{\sqrt{2x+2h+2}+\sqrt{2x+2}} \\
&= \lim_{h\to 0}\frac{(2x+2h+2)-(2x+2)}{h\left(\sqrt{2x+2h+2}+\sqrt{2x+2}\right)} \\
&= \frac{2}{\left(\lim_{h\to 0}\sqrt{2x+2h+2}+\sqrt{2x+2}\right)} = \frac{2}{2\sqrt{2x+2}} \\
&= \frac{1}{\sqrt{2x+2}}
\end{aligned}
$$

(c) The formula is

$$
\begin{aligned}
L(x) &= f(7) + f'(7)(x-7) \\
&= 4 + \frac{1}{4}(x-7)
\end{aligned}
$$

(d) If we put $x = 8$, we see $f(8) = \sqrt{18}$. So we use the linear approximation:

$$L(8) = 4 + \frac{1}{4}(8 - 7) = 4.25$$

The exact value of $\sqrt{18}$ to 20 decimal digits is 4. 242 640 687 119 285 146 4

15

6 (a) Substituting $(1, 0)$ into $\ln(x^2 + 3y) = xy - 3/x + 3$ gives $0 = -3 + 3$ which is correct. Differentiate implicitly:

$$\frac{2x + 3y'}{x^2 + 3y} = y + xy' + 3/x^2$$

now substitute before solving for y'

$$2 + 3y' = y' + 3$$
$$y' = 1/2$$

and an equation of the tangent line is $y = 1/2(x - 1)$.

(b) $f(x) = x^3 - 3x + 1$ so if we let m be the slope of the secant line,

$$m = \frac{f(2) - f(0)}{2 - 0} = \frac{3 - 1}{2}$$
$$= 1$$

Now $f'(x) = 3x^2 - 3$ so if $0 \le c \le 2$ and $f'(c) = m = 1$ we have

$$3c^2 - 3 = 1$$
$$3c^2 = 4$$
$$c = \sqrt{4/3} \text{ (since } c > 0)$$

(c) Since the expression $\dfrac{\sin x + \sin 2x - \sin(3x)}{x^3}$ is an indeterminate form of the type $\dfrac{0}{0}$ at 0, we have

$$\lim_{x \to 0} \frac{\sin x + \sin 2x - \sin(3x)}{x^3} = (\text{L}'\text{H}) \lim_{x \to 0} \frac{\cos x + 2\cos(2x) - 3\cos(3x)}{3x^2}$$
$$= (\text{L}'\text{H}) \lim_{x \to 0} \frac{-\sin x - 4\sin(2x) + 9\sin(3x)}{6x}$$
$$= (\text{L}'\text{H}) \lim_{x \to 0} \frac{-\cos x - 8\cos(2x) + 27\cos(3x)}{6}$$
$$= \frac{18}{6} = 3$$

where we use L'Hôpital's rule three times since the second and third expressions are also indeterminate forms of the type $\dfrac{0}{0}$ at 0.

7 (a) We have $y = \sqrt{1+x^3}$. Differentiate with respect to time t to get

$$\begin{aligned} \frac{dy}{dt} &= \frac{3x^2}{2\sqrt{1+x^3}} \frac{dx}{dt} \\ 4 &= \frac{3 \cdot 4}{2\sqrt{1+8}} \frac{dx}{dt} = 2\frac{dx}{dt} \end{aligned}$$

and so $\dfrac{dx}{dt} = 2$ cm/sec.

(b) Let the upper right coordinate of the rectangle be (x, y) in quadrant 1. Then the area A is given by

$$A = 4xy$$

with the condition that

$$\begin{aligned} y &= \sqrt{r^2 - x^2} \text{ and } x > 0, \text{ so} \\ A &= 2x\sqrt{r^2 - x^2} \text{ or} \\ A^2 &= 2x^2\left(r^2 - x^2\right) = 2x^2r^2 - 2x^4 \end{aligned}$$

Differentiate with respect to x to get

$$\begin{aligned} 2A\frac{dA}{dx} &= 4xr^2 - 8x^3 \\ &= 4x\left(r^2 - 2x^2\right) \end{aligned}$$

The critical value is $x = r/\sqrt{2}$ (no minus sign!) and so $y = x = r/\sqrt{2}$ also. A is a maximum at this point because the derivative goes from positive to zero to negative. So the largest area is $A_{\max} = 4 \cdot \left(r/\sqrt{2}\right)^2 = 2r^2$.

16

8 (a) $f(x) = \dfrac{x^2 - 3}{x + 2}$ is a rational function, defined at all points except where the denominator is zero, so the domain is $\mathbb{R} - \{-2\} = (-\infty, -2) \cup (-2, \infty)$. It is not symmetric since $f(-x) \neq f(x)$ and $f(-x) \neq -f(x)$ There is a vertical asymptote at $x = -2$ because $\lim_{x \to -2^-} f(x) = -\infty$ and $\lim_{x \to -2^+} f(x) = \infty$. There are no horizontal asymptotes because $\lim\limits_{x \to \pm\infty} f(x) = \pm\infty$.

(b)

$$f'(x) = 2\frac{x}{x+2} - \frac{x^2 - 3}{(x+2)^2} = \frac{(x+3)(x+1)}{(x+2)^2}$$

The critical numbers are -3 and -1. Constructing the table to see where it is increasing/decreasing we have

	$x<-3$	$x=-3$	$-1<x<-2$	$x=-2$	$-2<x<-1$	$x=-1$	$x>-1$
$x+3$	$-$	0	$+$	undefined	$+$	$+$	$+$
$x+1$	$-$	$-$	$-$	undefined	$-$	0	$+$
$f'(x)$	$+$	0	$-$	undefined	$-$	0	$+$
	$\nearrow$	Max.	$\searrow$	undefined	$\searrow$	min.	$\nearrow$

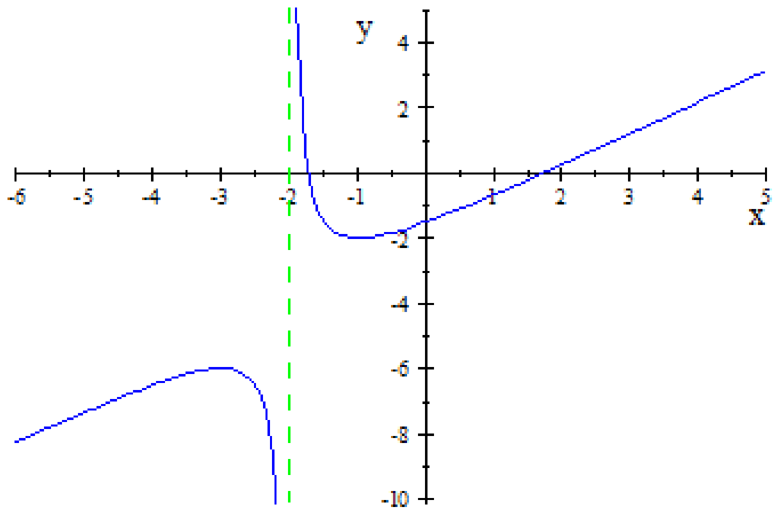
We can read off the intervals and the local extrema: local maximum at $(-3,-6)$ and a local minimum at $(-1,-2)$.

(c)

$$\begin{aligned} f''(x) &= \frac{2}{x+2} + 2\frac{x^2-3}{(x+2)^3} - 4\frac{x}{(x+2)^2} \\ &= \frac{2}{(x+2)^3} \end{aligned}$$

We see the concavity depends on the sign of $(x+2)^3$ which is the same as the sign of $x+2$. So the curve is concave up if $x>-2$ and concave down if $x<-2$. There are no inflection points ($x=-2$ is NOT a place where an inflection point can occur because it is a vertical asymptote).

(d) The graph: We can add the point $(0, -3/2)$ as the $y-$intercept and $\left(\pm\sqrt{3}, 0\right)$ as $x-$intercepts.



5

Bonus

$$f(x) = |x - 1|$$
$$f(3) - f(0) = 2 - 1 = 1$$
$$f'(c) = \begin{cases} 1 & \text{if} \quad c > 1 \\ \text{undefined} & \text{if} \quad c = 1 \\ -1 & \text{if} \quad c < 1 \end{cases}$$

and so

$$f'(c)(3 - 0) = \begin{cases} 3 & \text{if} \quad c > 1 \\ \text{undefined} & \text{if} \quad c = 1 \\ -3 & \text{if} \quad c < 1 \end{cases}$$

In other words there is no c, positive or negative, to make the equation

$$f(3) - f(0) = f'(c)(3 - 0)$$

hold because the LHS is 1 and the RHS is always ± 3 or undefined. The Mean Value Theorem does not apply in this case since $f(x)$ is not differentiable at $x = 1$, and 1 is in the interval $[0, 3]$.

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	203	All
Examination	**Date**	**Pages**
Final	April 2009	3
Instructors		**Course Examiner**
D. Dryanov, A. Laurin, J.R. Perez-Buendia, N. Rossokhata		H. Proppe

Special Instructions

▷ **Only Sharp EL 531 or Casio FX 300 MS calculators are allowed.**

MARKS

[10] **1.** (a) Sketch the graph of the function $f(x) = |1-(x-2)^2|$ starting from the graph of the standard parabola and using appropriate transformations.

(b) Suppose $f(x) = e^x - 1$. Find its domain and range, and then find $f^{-1}(x)$ and its domain and range.

(c) Suppose $f(x) = \sqrt{x}$, $g(x) = x/4$ and $h(x) = 4x - 8$. Find formulas for $h \circ g \circ f(x)$ and $f \circ g \circ h(x)$.

[9] **2.** Evaluate the limits:

(a) $\lim_{x\to 7} \dfrac{\sqrt{x+2}-3}{x-7}$ (b) $\lim_{x\to 0} \left(\dfrac{1}{x} - \dfrac{1}{x^2+x}\right)$ (c) $\lim_{x\to -\infty} \dfrac{\sqrt{4x^2+1}}{3x-5}$

Do not use l'Hôpital's rule.

[10] **3.** (a) Let $f(x) = \dfrac{|x^2-1|}{x^2-1}$.

Calculate both one-sided limits at the two points where the function is undefined.

(b) Find the numbers a and b that make the function

$$f(x) = \begin{cases} \sqrt{4-x^2} & \text{if} \quad -2 \le x < 0 \\ ax+b, & \text{if} \quad 0 \le x < 2 \\ 0, & \text{if} \quad x \ge 2 \end{cases}$$

continuous on its whole domain. Sketch the graph of this function.

[15] **4.** Find derivatives of the functions (do not simplify the answer):

(a) $f(x) = \dfrac{x^3 + 3x\sqrt{x} + x^{-1/2}}{x^2}$;

(b) $f(x) = e^{x^2}(x^2+1) + x^4(2^x+1)$;

(c) $f(x) = \dfrac{\ln(x^2 e^{2x})}{x + \ln x}$;

(d) $f(x) = \sin(3\tan(x^2))$;

(e) $f(x) = (\cos x)^x$ (use logarithmic differentiation).

[12] **5.** (a) Given that $f(x) = \sin x$, find the linearization $L(x)$ of $f(x)$ at $a = 1$ and use $L(x)$ to estimate $\sin(1.1)$. You may use the following data: $\sin(1) = 0.84$ and $\cos(1) = 0.54$ (**Do not** use a calculator to answer this question).

(b) Answer part (a) using differentials, that is, identify dx and calculate df. (**Do not** use a calculator to answer this question).

(c) If $g(x) = \dfrac{1}{\sqrt{2x+1}}$, use the definition of the derivative to find $g'(1)$.

(d) Use the appropriate differentiation rule(s) to verify your answer to part (c).

[18] **6.** (a) The equation of a curve defined implicitly is $x^3y^2 = -3xy$. Verify that the point $(-1, -3)$ belongs to the curve. Find an equation of the tangent line to the curve at this point.

(b) Show that $f(x) = x^3 - x^2 - x + 1$ satisfies the conditions of the Mean Value Theorem on $[0, 1]$. Then find a number c that satisfies the conclusions of the Mean Value Theorem for the function $f(x)$ on $[0, 1]$.

(c) Use l'Hôpital's rule to evaluate $\displaystyle\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$.

[10] **7.** (a) Let $f(x) = e^x \sin x$. Find $f''(x)$, $f^{(4)}(x)$ and $f^{(8)}(x)$.

(b) A forest fire spreads in a circle with radius increasing at a rate of 2 m/minute. When the radius reaches 200 m, at what rate is the area of the burning region increasing?

[16] **8.** Given the function $f(x) = \dfrac{x^2 - 3}{x^3}$,

(a) Find the domain and check for symmetry. Find all asymptotes.

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function, and label the local extrema and inflection point(s) you have found (if any) in parts (b) and (c).

[5] **Bonus Question**

Two runners start a race at time 0. At some time $t = a > 0$ one runner has pulled ahead, but later, at time $t = b$, the other runner has taken the lead. Show that at some instant $t = c$ during the race they were running at exactly the same speed. [*Hint:* Let $g(t)$ be the distance the first runner covers in time t and let $h(t)$ be the distance the second runner covers in time t. Now put $f(t) = g(t) - h(t)$ and notice that $f(0) = 0$, $f(a) > 0$ and $f(b) < 0$; explain how Rolle's Theorem applies to this situation and you will see the solution].

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	203	All
Examination	**Date**	**Pages**
Final	April 2009	3
Instructors		**Course Examiner**
D. Dryanov, A. Laurin, J.R. Perez-Buendia, N. Rossokhata		H. Proppe

Special Instructions

▷ **Only Sharp EL 531 or Casio FX 300 MS calculators are allowed.**

Solutions

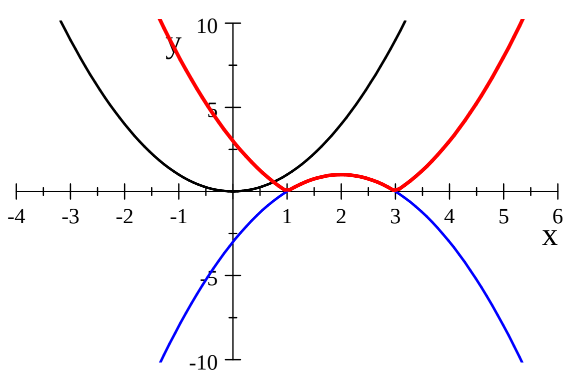
1. [10 marks]

(a) Sketch the graph of the function $f(x) = |1 - (x-2)^2|$ starting from the graph of the standard parabola and using appropriate transformations.

(b) Suppose $f(x) = e^x - 1$. Find its domain and range, and then find $f^{-1}(x)$ and its domain and range.

(c) Suppose $f(x) = \sqrt{x}$, $g(x) = x/4$ and $h(x) = 4x - 8$. Find formulas for $h \circ g \circ f(x)$ and $f \circ g \circ h(x)$

Solutions (a)



(b) $f(x) = y = e^x - 1$ which is defined for all x and has range $(-1, \infty)$. So $e^x = y + 1$ and $x = \ln(y+1)$; interchanging x and y we get $y = \ln(x+1)$ so this is the the inverse function $f^{-1}(x)$. The domain is, of course, $(-1, \infty)$ and the range $(-\infty, \infty)$. We note, incidentally, that $f \circ f^{-1}(x) = e^{(\ln(x+1))} - 1 = x + 1 - 1 = x = f^{-1} \circ f(x)$.

(c)

$$\begin{aligned} h \circ g \circ f(x) &= h\left(g\left(\sqrt{x}\right)\right) = h\left(\frac{\sqrt{x}}{4}\right) \\ &= 4\left(\frac{\sqrt{x}}{4}\right) - 8 = \sqrt{x} - 8 \\ f \circ g \circ h(x) &= f\left(g\left(4x - 8\right)\right) = f\left(x - 2\right) = \sqrt{x - 2} \end{aligned}$$

2. (12 Marks) Evaluate the limits:

$$\text{(a) } \lim_{x \to 7} \frac{\sqrt{x+2} - 3}{x - 7} \quad \text{(b) } \lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2 + x}\right) \quad \text{(c) } \lim_{x \to -\infty} \frac{\sqrt{4x^2 + 1}}{3x - 5}$$

Do not use l'Hopital's rule.

Solution (a)

$$\begin{aligned} \lim_{x \to 7} \frac{\sqrt{x+2} - 3}{x - 7} &= \lim_{x \to 7} \frac{\sqrt{x+2} - 3}{x - 7} \cdot \frac{\sqrt{x+2} + 3}{\sqrt{x+2} + 3} \\ &= \lim_{x \to 7} \frac{x - 7}{x - 7} \cdot \frac{1}{\sqrt{x+2} + 3} \\ &= \frac{1}{3 + 3} = \frac{1}{6} \end{aligned}$$

(b)

$$\begin{aligned} \lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x^2 + x}\right) &= \lim_{x \to 0} \frac{x^2 + x - x}{x\left(x^2 + x\right)} \\ &= \lim_{x \to 0} \frac{x^2}{x^2\left(x + 1\right)} \\ &= \lim_{x \to 0} \frac{1}{\left(x + 1\right)} = 1 \end{aligned}$$

(c)

$$\begin{aligned}
\lim_{x\to-\infty}\frac{\sqrt{4x^2+1}}{3x-5} &= \lim_{x\to-\infty}\frac{|x|\sqrt{4+1/x^2}}{x\,(3-5/x)} \\
&= \lim_{x\to-\infty}\frac{|x|}{x}\lim_{x\to-\infty}\frac{\sqrt{4+1/x^2}}{(3-5/x)} \\
&= -1\cdot\frac{\sqrt{4}}{3}=-\frac{2}{3}
\end{aligned}$$

The reason for the $|x|$ is that when the x^2 comes out from under the square root sign it must have a + sign; even though $x\to-\infty$, x^2 is a positive number and we are taking the positive square root. So we have to write $|x|$ (or $-x$), not x in the numerator.

3. (10 Marks)

(a) Consider the function $f(x)=\dfrac{|x^2-1|}{x^2-1}$. Calculate both one-sided limits at the two points where the function is undefined.

(b) Find the numbers a and b that make the function

$$f(x)=\begin{cases}\sqrt{4-x^2} & \text{if} \quad x<0\\ ax+b & \text{if} \quad 0\le x<2\\ 0 & \text{if} \quad x\ge 2\end{cases}$$

continuous on its whole domain. Sketch the graph of this function.

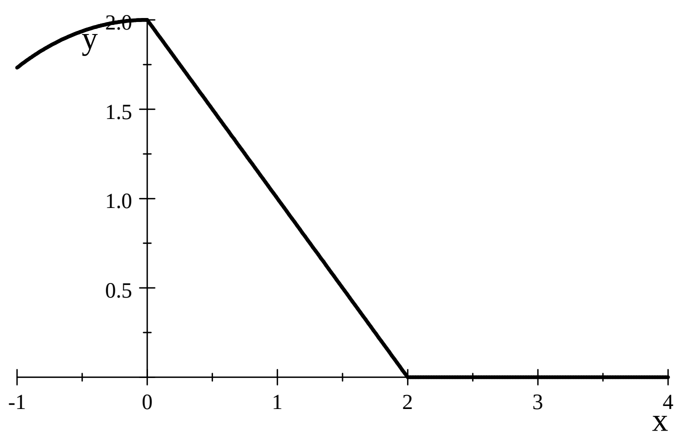
Solution (a) $f(x)=\dfrac{|x^2-1|}{x^2-1}=\dfrac{|x-1|\cdot|x+1|}{(x-1)\,(x+1)}$ and so f is undefined at the points $x=-1,1$. First consider -1. If x is -1 then $\dfrac{|x-1|}{x-1}$ $=-1$ and we have

$$\begin{aligned}
\lim_{x\to-1^-}\frac{|x-1|\cdot|x+1|}{(x-1)\,(x+1)} &= (-1)\cdot\lim_{x\to-1^-}\frac{|x+1|}{x+1}=(-1)\cdot(-1)=1 \\
\text{since } |x+1| &= -(x+1)\ \text{ when } x<-1 \\
\lim_{x\to-1^+}\frac{|x-1|\cdot|x+1|}{(x-1)\,(x+1)} &= (-1)\cdot\lim_{x\to-1^+}\frac{|x+1|}{x+1}=-1\cdot1=1 \\
\text{since } |x+1| &= x+1 \text{ when } x>-1
\end{aligned}$$

Now consider the point +1. If x is 1 then $\dfrac{|x+1|}{x+1} = 1$ and we have

$$\begin{aligned}
\lim_{x\to 1^-} \frac{|x-1|\cdot|x+1|}{(x-1)(x+1)} &= \lim_{x\to 1^-} \frac{|x-1|}{x-1} = -1 \\
\text{since } |x-1| &= -(x-1) \text{ when } x<1 \\
\lim_{x\to -1^+} \frac{|x-1|\cdot|x+1|}{(x-1)(x+1)} &= \lim_{x\to -1^+} \frac{|x-1|}{x-1} = 1 \\
\text{since } |x-1| &= x-1 \text{ when } x>1
\end{aligned}$$

(b) Since $f(0) = b$ and $\lim_{x\to 0^+} f(x) = 2$ the two have to be equal so $b = 2$. Also, $f(2) = 0$ and $\lim_{x\to 2^-} f(x) = 2a + b = 2a + 2$ so $2a + 2 = 0 \Longrightarrow a = -1$. Here is the graph



4. (15 Marks) Find derivatives of the functions (do not simplify the answer):

(a) $f(x) = \dfrac{x^3 + 3x\sqrt{x} + x^{-1/2}}{x^2}$

Solution Simplify **first -** $f(x) = x + 3x^{-1/2} + x^{-5/2}$ and now the differentiation is easy: $f'(x) = 1 - \dfrac{3}{2}x^{-3/2} - \dfrac{5}{2}x^{-7/2}$ (power rule)

(b) $f(x) = e^{x^2}(x^2+1) + x^4(2^x+1)$

Solution $f'(x) = e^{x^2}\cdot 2x(x^2+1) + e^{x^2}\cdot 2x + 4x^3(2^x+1) + x^4\cdot 2^x \ln 2$ (product, sum rules and derivative of exponential function)

(c) $f(x) = \dfrac{\ln(x^2 e^{2x})}{x + \ln x}$

Solution Simplify **first -** $f(x) = \dfrac{2\ln x + 2x}{x + \ln x} = 2$ and so $f'(x) = 0$

(d) $f(x) = \sin(3\tan(x^2))$

Solution $f'(x) = \cos(3\tan(x^2)) \cdot 3\sec^2(x^2) \cdot 2x$ (chain rule)

(e) $f(x) = (\cos x)^x$ (use logarithmic differentiation).

Solution $\ln f(x) = x \cdot \ln(\cos x)$ so $\dfrac{f'(x)}{f(x)} = \ln(\cos x) + x \cdot \dfrac{1}{\cos x} \cdot (-\sin x)$ and

$$\begin{aligned} f'(x) &= f(x)(\ln(\cos x) - x \cdot \tan x) \\ &= (\cos x)^x (\ln(\cos x) - x \cdot \tan x) \end{aligned}$$

5. (12 Marks)

(a) Given that $f(x) = \sin x$, find the linearization $L(x)$ of $f(x)$ at $a = 1$ and use $L(x)$ to estimate $\sin(1.1)$. You may use the following data: $\sin(1) = 0.84$ and $\cos(1) = 0.54$ (**Do not** use a calculator to answer this question).

(b) Answer part (a) using differentials, that is, identify dx and calculate df . (**Do not** use a calculator to answer this question).

(c) If $g(x) = \dfrac{1}{\sqrt{2x+1}}$, use the definition of the derivative to find $g'(1)$.

(d) Use the appropriate differentiation rule(s) to verify your answer to part (c).

Solutions (a) $L(x) = f(a) + f'(a)(x - a)$ so substituting we have, since $f'(x) = \cos x$ $L(x) = \sin(1) + \cos(1)(x-1) = .84 + .54(x-1)$. We have $L(1.1) = .84 + .54(1.1 - 1) = .84 + .054 = .894$ (note the exact value to 4 digits is $\sin 1.1 = 0.891\,2$)

(b) Using differentials we write

$$\begin{aligned} df &= f'(1)\,dx \\ &= .54 \cdot dx \\ &= .54 \cdot (.1) = .054 \end{aligned}$$

where $dx = 1.1 - 1 = .1$ and $f'(1) = \cos 1 = .54$. So f changes by .054 from its value at 1 (which is .84) to $.84 + .054$ which is .894 as before.

(c)

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{2(x+h)+1}} - \frac{1}{\sqrt{2x+1}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{2x+1} - \sqrt{2(x+h)+1}}{h\sqrt{2(x+h)+1}\sqrt{2x+1}} \\
&= \lim_{h\to 0} \frac{\sqrt{2x+1} - \sqrt{2(x+h)+1}}{h\sqrt{2(x+h)+1}\sqrt{2x+1}} \cdot \frac{\sqrt{2x+1} + \sqrt{2(x+h)+1}}{\sqrt{2x+1} + \sqrt{2(x+h)+1}} \\
&= \lim_{h\to 0} \frac{-2h}{h\cdot\sqrt{2(x+h)+1}\sqrt{2x+1}\left(\sqrt{2x+1} + \sqrt{2(x+h)+1}\right)} \\
&= \frac{-2}{(2x+1)\left(\sqrt{2x+1} + \sqrt{2x+1}\right)} = -\frac{1}{(2x+1)^{3/2}}; \text{ so } g'(1) = -\frac{1}{3^{3/2}}
\end{aligned}$$

If we substitute $x = 1$ **before** we go through all this we get a

somewhat less cumbersome procedure as follows. Either one is acceptable.

$$\begin{aligned}
g'(1) &= \lim_{h\to 0} \frac{g(1+h) - g(1)}{h} = \lim_{h\to 0} \frac{\frac{1}{\sqrt{2(1+h)+1}} - \frac{1}{\sqrt{3}}}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{3} - \sqrt{3+2h}}{h\sqrt{3+2h}\sqrt{3}} \\
&= \lim_{h\to 0} \frac{\sqrt{3} - \sqrt{3+2h}}{h\sqrt{3+2h}\sqrt{3}} \cdot \frac{\sqrt{3} + \sqrt{3+2h}}{\sqrt{3} + \sqrt{3+2h}} \\
&= \lim_{h\to 0} \frac{-2h}{h\cdot\sqrt{3+2h}\sqrt{3}\left(\sqrt{3} + \sqrt{3+2h}\right)} \\
&= \frac{-2}{3\cdot\left(\sqrt{3} + \sqrt{3}\right)} = -\frac{1}{(3)^{3/2}}
\end{aligned}$$

(d) $g(x) = (2x+1)^{-1/2}$ so $g'(x) = -\frac{1}{2}(2x+1)^{-3/2} \cdot 2 = -(2x+1)^{-3/2}$ using the power rule, and so $g'(1) = -3^{-3/2}$ as before.

6. (18 Marks)

(a) The equation of a curve defined implicitly is $x^3y^2 = -3xy$. Verify that the point $(-1, -3)$ belongs to the curve. Find an equation of the tangent line to the curve at this point.

(b) Show that $f(x) = x^3 - x^2 - x + 1$ satisfies the conditions of the Mean Value Theorem on $[0, 1]$. Then find a number c that satisfies the conclusions of the Mean Value Theorem for the function $f(x)$ on $[0, 1]$.

(c) Use l'Hopital's rule to evaluate

$$\lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x}$$

Solutions (a) Substitute: $-1 \cdot 9 = -3 \cdot (-1) \cdot (-3)$ so the equation is satisfied and the point is on the curve. Next, differentiate:

$$\begin{aligned} 3x^2y^2 + 2x^3yy' &= -3y - 3xy' \\ &\quad \text{now substitute} \\ 3 \cdot (27) + 2 \cdot (-1) \cdot (-3)\, y' &= 9 \\ 6y' &= 9 - 81 \\ y' &= -12 \end{aligned}$$

So an equation of the tangent line is

$$y + 3 = -12\,(x + 1)$$

(b) $f\,(x) = x^3 - x^2 - x + 1$ which is defined and differentiable on $[0, 1]$ and we want a number c so that

$$\begin{aligned} \frac{f\,(1) - f\,(0)}{1 - 0} &= f'\,(c)\,, \text{ i.e.} \\ \frac{0 - 1}{1 - 0} &= 3c^2 - 2c - 1 \text{ so} \\ 3c^2 - 2c - 1 &= -1 \\ 3c^2 - 2c &= 0 \end{aligned}$$

Since $3c^2 - 2c = c\,(3c - 2)$ the roots are 0 and $\frac{2}{3}$. The root 0 is not allowed because c must be in $(0, 1)$ so $c = \frac{2}{3}$.

(c)

$$\begin{aligned} \lim_{x\to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} &= \lim_{x\to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} \text{ l'Hopital's rule still applies:} \\ &= \lim_{x\to 0} \frac{e^x - e^{-x}}{\sin x} \text{ l'Hopital's rule still applies:} \\ &= \lim_{x\to 0} \frac{e^x + e^{-x}}{\cos x} = \frac{2}{1} = 2 \end{aligned}$$

7. (10 Marks)

(a) Let $f(x) = e^x \sin x$. Find $f''(x)$, $f^{(4)}(x)$ and $f^{(8)}(x)$.

(b) A forest fire spreads in a circle with radius increasing at a rate of 2 m/minute. When the radius reaches 200 m, at what rate is the area of the burning region increasing?

Solutions (a) Differentiate with respect to time and then substitute:

$$\begin{aligned} f'(x) &= e^x \cos x + e^x \sin x \\ f''(x) &= e^x \cos x - e^x \sin x + e^x \sin x + e^x \cos x \\ &= 2e^x \cos x \\ f'''(x) &= 2e^x \cos x - 2e^x \sin x \\ f^{(4)}(x) &= 2e^x \cos x - 2e^x \sin x - 2e^x \sin x - 2e^x \cos x \\ &= -4e^x \sin x \text{ and so from the pattern, clearly} \\ f^{(8)}(x) &= 16e^x \sin x \end{aligned}$$

(b) The circle has area A, say and radius r, both depending on time t. We know

$$A = \pi r^2$$

and we can differentiate both sides with respect to t. So we get

$$\frac{dA}{dt} = 2\pi r \frac{dr}{dt}$$

Given $r = 200$ and $\dfrac{dr}{dt} = 2$ this gives

$$\frac{dA}{dt} = 2\pi \cdot 200 \cdot 2 \simeq 2,513.3 \text{ m}^2/\text{min}$$

8. (16 Marks) Given the function $f(x) = \dfrac{x^2 - 3}{x^3}$,

(a) Find the domain and check for symmetry. Find all asymptotes.

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function, and label the local extrema and inflection point(s) you have found (if any) in parts (b) and (c).

Solutions (a) The equation shows that the domain is $(-\infty, 0) \cup (0, \infty)$, i.e. all real number except 0. Replacing x by $-x$ gives

$$f(-x) = \frac{(-x)^2 - 3}{(-x)^3} = -\frac{x^2 - 3}{x^3} = -f(x)$$

so there is **odd** symmetry. Since the f is not defined at 0 and becomes unbounded near 0, the vertical line $x = 0$ is a vertical asymptote. Also, since (using L'Hopital's rule, if necessary)

$$\lim_{x \to \infty} \frac{x^2 - 3}{x^3} = \lim_{x \to -\infty} \frac{x^2 - 3}{x^3} = 0$$

it follows that $y = 0$ is a horizontal asymptote.

(b) $f'(x) = \dfrac{2}{x^2} - \dfrac{3}{x^4}(x^2 - 3) = -\dfrac{(x+3)(x-3)}{x^4}$ so $x = -3$ and $x = 3$ are critical points. If we do a sign chart we get the following:

	$x < -3$	$x = -3$	$-3 < x < 3$	$x = 3$	$3 < x$
$x + 3$	$-$	0	$+$	$+$	$+$
$x - 3$	$-$	$-$	$-$	0	$+$
$-\dfrac{(x+3)(x-3)}{x^4}$	$-$	0	$+$	0	$-$
	↘	l. min	↗	l. max	↘

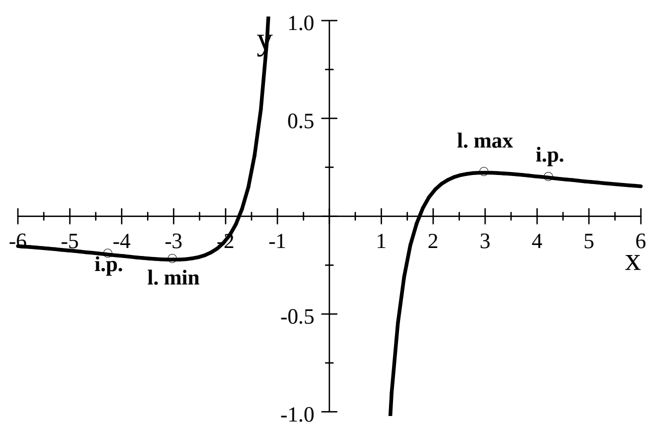
The chart indicates there is a local minimum at $\left(-3, -\frac{2}{9}\right)$ and a local maximum at $\left(3, \frac{2}{9}\right)$ - of course this is to be expected because of the symmetry. the intervals of increasing and decreasing are identified.

(c) $f''(x) = \frac{12}{x^5}(x^2-3) - \frac{10}{x^3} = 2\frac{x^2-18}{x^5} = 2\frac{(x-3\sqrt{2})(x+3\sqrt{2})}{x^5}$. Here is the sign chart for this expression, but this time we consider only $x > 0$ and use symmetry for $x < 0$:

	$x = 0$	$0 < x < 3\sqrt{2}$	$x = 3\sqrt{2}$	$3\sqrt{2} < x$
$x + 3\sqrt{2}$		$+$	$+$	$+$
$x - 3\sqrt{2}$		$-$	0	$+$
$2\frac{(x-3\sqrt{2})(x+3\sqrt{2})}{x^5}$	undef.	$-$	0	$+$
	not i.p.	$\cap$	i.point	$\cup$

We see the graph is concave down on $(0, 3\sqrt{2})$ and concave up on $(3\sqrt{2}, \infty)$. So $\left(3\sqrt{2}, \frac{5}{36}\sqrt{2}\right) \simeq (4.24, .196)$ is an inflection point. By symmetry, so is $\left(-3\sqrt{2}, -\frac{5}{36}\sqrt{2}\right)$ and the concavity changes accordingly - concave up on $(-3\sqrt{2}, 0)$ and concave down on $(-\infty, 3\sqrt{2})$. Note: there is no inflection point at 0 because f is not defined there.

(d)



Bonus Question (5 Marks) Two runners start a race at time 0. At some time $t = a > 0$ one runner has pulled ahead, but later, at time $t = b$, the other runner has taken the lead. Show that at some instant $t = c$ during the race they were running at exactly the same speed. [Hint: Let $g(t)$ be the distance the first runner covers in time t and let $h(t)$ be the distance the second runner covers in time t. Now put $f(t) = g(t) - h(t)$ and notice that $f(0) = 0$, $f(a) > 0$ and $f(b) < 0$; explain how Rolle's Theorem applies to this situation and you will see the solution].

Solution Using the hint, we have $f(0) = 0$. We also are told that $f(a) > 0$ and $f(b) < 0$ (or the other way around: $f(a) < 0$ and $f(b) > 0$). Either way, between a and b there has to be a time T where $f(T) = 0$ again, since the function $f(t)$ goes either from $-$ to $+$ or from $+$ to $-$ between a and b, so it has to cross 0 because it is continuous. Now we have $f(0) = f(T) = 0$, and we can assume the function f has a derivative for all values of t. By Rolle's Theorem there is a number c between 0 and T such that

$$\begin{aligned} f'(c) &= 0, \text{ and so} \\ g'(c) - h'(c) &= 0 \text{ because } f(t) = g(t) - h(t) \end{aligned}$$

Therefore $g'(c) = h'(c)$, which means the two runners have the same speed at time c.

Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	203	All
Examination	**Date**	**Pages**
Final	December 2010	3
Instructors		**Course Examiners**
A. Boyarsky, J. Brody, N. Hardy, Z. Li, H. Proppe, N. Rossokhata		A. Atoyan & H. Proppe

Special Instructions

▷ **Only Sharp EL 531 or Casio FX 300 MS calculators are allowed.**

MARKS

[11] **1.** (a) Sketch the graph of the function $f(x) = |x+2|^3 - 1$ starting from the graph of the standard cubic and using appropriate transformations.

(b) Suppose $f(x) = \sqrt{x}$ and and $g(x) = \sqrt[3]{1-x}$. Find $f \circ g$, $g \circ f$, and $f \circ f$.

(c) Find the inverse of the function $f(x) = \sqrt[3]{e^x - 1}$. Determine the domain and range of f and f^{-1}.

[8] **2.** Evaluate the limits:

(a) $\displaystyle \lim_{x \to 3} \frac{4 - \sqrt{x^2+7}}{2x^2 - 18}$ (b) $\displaystyle \lim_{x \to -\infty} \frac{\sqrt{4x^6 + x^2}}{x^3 + x}$

Do not use l'Hôpital's rule.

[11] **3.** (a) Consider the function $f(x) = \dfrac{|x-1|}{x^2 + x - 2}$.

Calculate both one-sided limits at the point(s) where the function is undefined.

(b) Find parameters a and b such that the function

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2}, & \text{if} \quad x < 2 \\ ax^2 - bx + 3, & \text{if} \quad 2 \le x < 3 \\ 2x - a + b, & \text{if} \quad x \ge 3 \end{cases}$$

will be continuous at every point. (It may help to sketch the graph).

[16] **4.** Find derivatives of the functions (you don't have to simplify the answers):

(a) $f(x) = \dfrac{x\sqrt{x^3} - x^{-1/2} + \sqrt{x}}{x\sqrt{x}}$;

(b) $f(x) = e^{\cos x}(\dfrac{1}{x} + \ln x)$

(c) $f(x) = \dfrac{(\arcsin x)^2}{\sqrt{1-x^2}}$;

(d) $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}}$;

(e) $f(x) = (\ln x)^{\sin(2x)}$ (use logarithmic differentiation).

[12] **5.** Given the function $f(x) = \sqrt{x^2 + 35}$,

(a) Use appropriate differentiation rules to find the derivative of the function.

(b) Use the definition of derivative to verify (a).

(c) Find the linear approximation of the function at $a = 1$.

(d) Use the linear approximation above to approximate $\sqrt{39}$.

[17] **6.** (a) The equation of a curve defined implicitly is $2(x^2 + y^2)^2 = 25(x^2 - y^2)$. Verify that the point $(3, 1)$ belongs to the curve. Find an equation of the tangent line to the curve at this point.

(b) The length of a rectangle is increasing at the rate of 8 cm/s and its width is increasing at the rate of 3 cm/s. When the length is 20 cm and the width is 10 cm, how fast is the area of the rectangle increasing?

(c) Use l'Hôpital's rule to evaluate $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$.

[6] **7.** Let $f(x) = x^3 - 2x$.

(a) Find the slope m of the secant line joining the points $(-3, f(-3))$ and $(3, f(3))$.

(b) Find the point(s) $x = c$ on the interval $[-3, 3]$ such that $f'(c) = m$.

[5] **8.** (a) A rectangle $ABCD$ is inscribed in a right triangle with legs of 3 cm and 4 cm along the x-axis and y-axis respectively. A is located at the origin, AB lies along the 3 cm leg and AD lies along the 4 cm leg (and C is on the hypotenuse). Find the rectangle $ABCD$ with the largest area.

[14] **9.** Given the function $f(x) = x\sqrt{2 - x^2}$,

(a) Find the domain and check for symmetry. Find asymptotes (if any).

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function.

[5] **Bonus Question**

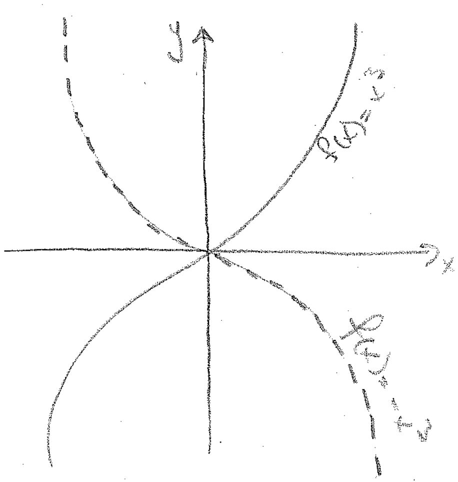
A number a is called a **fixed point** of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has no more than one fixed point.

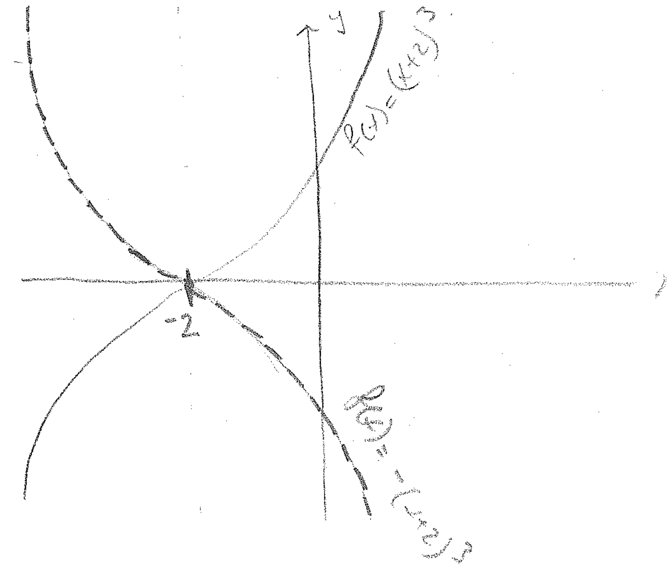
[Hint: Apply Rolle's Theorem to the function $g(x) = f(x) - x$.*]*

#1

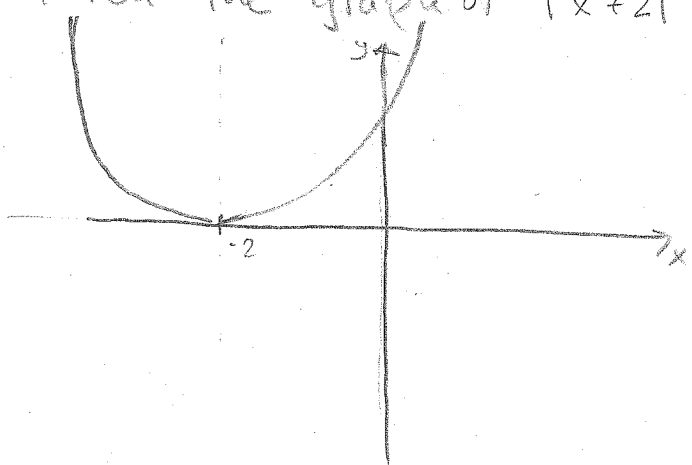
a) $f(x) = |x+2|^3 - 1$

$$|x+2|^3 = \begin{cases} -(x+2)^3 & \text{if } x+2 < 0 \\ (x+2)^3 & \text{if } x+2 \geq 0 \end{cases}$$

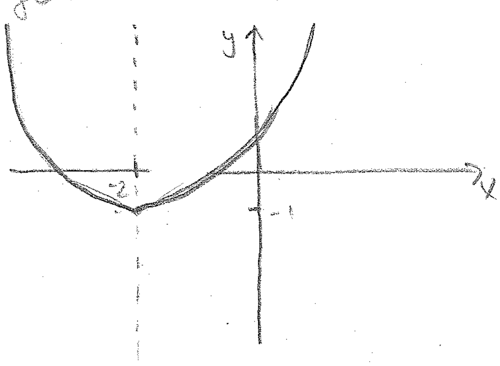




Then the graph of $|x+2|^3$ is:



Finally we shift the graph of $|x+2|^3$ for one unit down to get the of $f(x) = |x+2|^3 - 1$



#1 b) $f(x) = \sqrt{x}$; $g(x) = \sqrt[3]{1-x}$

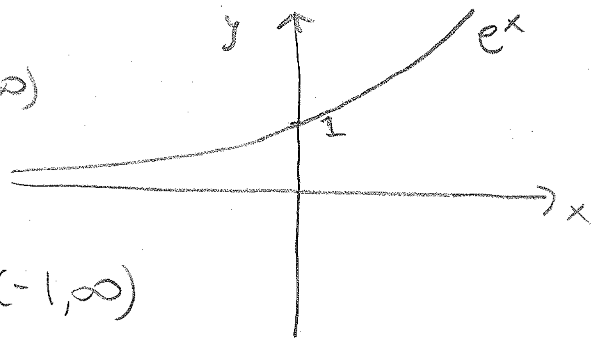
$f \circ g = f(g(x)) = \sqrt{\sqrt[3]{1-x}}$

$g \circ f = \sqrt[3]{1-\sqrt{x}}$

$f \circ f = \sqrt{\sqrt{x}}$

c) $f(x) = \sqrt[3]{e^x - 1}$

Domain: $D_f = \{x \mid x \in \mathbb{R}\}$ or $x \in (-\infty, \infty)$



Range: The range of e^x is $(0,\infty)$

$\Rightarrow$ The range of $e^x - 1$ is $(-1,\infty)$

So The range of $\sqrt[3]{e^x - 1}$ is $(-1,\infty)$

Inverse:

$$y = \sqrt[3]{e^x - 1}$$

$$y^3 = e^x - 1$$

$$y^3 + 1 = e^x \quad / \ln$$

$$\ln(y^3+1) = \ln e^x$$

$$\ln(y^3+1) = x$$ interchange x and y to obtain

$$f^{-1}(x) = \ln(x^3+1)$$

Domain of $f^{-1}(x)$: $x^3+1 > 0 \Rightarrow x^3 > -1 \Rightarrow x > -1 \Rightarrow x \in (-1,\infty)$

Range of $f^{-1}(x)$: Recall The graph of The natural logarithm to conclude That The range of $f^{-1}(x)$ is $y \in (-\infty,\infty)$.

#2

a) $$\lim_{x\to 3} \frac{4-\sqrt{x^2+7}}{2x^2-18} \cdot \frac{4+\sqrt{x^2+7}}{4+\sqrt{x^2+7}} = \lim_{x\to 3} \frac{(4)^2-\left(\sqrt{x^2+7}\right)^2}{2(x^2-9)\left[4+\sqrt{x^2+7}\right]} =$$

$$= \lim_{x\to 3} \frac{16-x^2-7}{2(x^2-9)(4+\sqrt{x^2+7})} = \lim_{x\to 3} \frac{9-x^2}{2(x^2-9)(4+\sqrt{x^2+7})} =$$

$$= \lim_{x\to 3} \frac{-(x^2-9)}{2(x^2-9)(4+\sqrt{x^2+7})} = \lim_{x\to 3} \frac{-1}{2(4+\sqrt{x^2+7})} = -\frac{1}{16}$$

b) $$\lim_{x\to -\infty} \frac{\sqrt{4x^6+x^2}}{x^3+x} = \lim_{x\to -\infty} \frac{\sqrt{x^6(4+\frac{1}{x^4})}}{x^3(1+\frac{1}{x^2})} =$$

$$= \lim_{x\to -\infty} \frac{-x^3\sqrt{4+\frac{1}{x^4}}}{x^3(1+\frac{1}{x^2})} = -\frac{\sqrt{4}}{1} = -2$$

($\frac{1}{x^4} \xrightarrow{x\to-\infty} 0$, $\frac{1}{x^2} \xrightarrow{x\to-\infty} 0$)

#3

a) $$f(x) = \frac{|x-1|}{x^2+x-2} = \frac{|x-1|}{(x-1)(x+2)}$$

$f(x)$ is not defined for $x_1 = 1$ and $x_2 = -2$

Note that $$|x-1| = \begin{cases} -(x-1) & \text{if } x-1<0 \\ x-1 & \text{if } x-1 \geq 0 \end{cases} = \begin{cases} -(x-1) & \text{if } x<1 \\ x-1 & \text{if } x\geq 1 \end{cases}$$

So, $$\lim_{x\to -2^-} \frac{|x-1|}{(x-1)(x+2)} = \lim_{x\to -2^-} \frac{-(x-1)}{(x-1)(x+2)} = \lim_{x\to -2^-} \frac{-1}{(x+2)} = +\infty$$

$$\lim_{x\to -2^+} \frac{|x-1|}{(x-1)(x+2)} = \lim_{x\to -2^+} \frac{-(x-1)}{(x-1)(x+2)} = \lim_{x\to -2^+} \frac{-1}{(x+2)} = -\infty$$

$$\lim_{x\to 1^-} \frac{|x-1|}{(x-1)(x+2)} = \lim_{x\to 1^-} \frac{-(x-1)}{(x-1)(x+2)} = \lim_{x\to 1^-} \frac{-1}{(x+2)} = -\frac{1}{3}$$

and

$$\lim_{x\to 1^+} \frac{|x-1|}{(x-1)(x+2)} = \lim_{x\to 1^+} \frac{(x-1)}{(x-1)(x+2)} = \lim_{x\to 1^+} \frac{1}{(x+2)} = \frac{1}{3}$$

#3 b)

$$f(x) = \begin{cases} \frac{x^2-4}{x-2}, & \text{if } x<2 \\ ax^2-bx+3, & \text{if } 2\le x<3 \\ 2x-a+b, & \text{if } x\ge 3 \end{cases}$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} \frac{x^2-4}{x-2} = \lim_{x\to 2^-} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x\to 2^-} (x+2) = 4$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} (ax^2-bx+3) = 4a-2b+3$$

In order $f(x)$ to be continuous at $x=2$ we need $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$ i.e. $4 = 4a-2b+3 \Rightarrow \boxed{1 = 4a-2b}$

Now we turn to the point $x=3$:

$$\left.\begin{array}{l} \lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} ax^2-bx+3 = 9a-3b+3 \\ \lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} 2x-a+b = 6-a+b \end{array}\right\} \Rightarrow 9a-3b+3 = 6-a+b \Rightarrow$$

$$\Rightarrow \boxed{10a-4b=3}$$

So we need to solve the system:

$$\begin{cases} 4a-2b=1 \\ 10a-4b=3 \end{cases} = \begin{cases} b = \frac{4a-1}{2} \\ 10a - 4\left(\frac{4a-1}{2}\right) = 3 \Rightarrow 10a-8a+2=3 \Rightarrow \boxed{a=\frac{1}{2}} \end{cases} \Longrightarrow$$

$$\Rightarrow \quad b = \frac{4a-1}{2} = \frac{4\cdot\frac{1}{2}-1}{2} = \frac{2-1}{2} = \frac{1}{2}$$

#4

a) $$f(x) = \frac{x\sqrt{x^3} - x^{-\frac{1}{2}} + \sqrt{x}}{x\sqrt{x}} = \frac{x\cdot x^{\frac{3}{2}} - x^{-\frac{1}{2}} + x^{\frac{1}{2}}}{x\cdot x^{\frac{1}{2}}} = \frac{x^{\frac{5}{2}} - x^{-\frac{1}{2}} + x^{\frac{1}{2}}}{x^{\frac{3}{2}}} =$$

$$= x^{\frac{5}{2}-\frac{3}{2}} - x^{-\frac{1}{2}-\frac{3}{2}} + x^{\frac{1}{2}-\frac{3}{2}} = x - x^{-2} + x^{-1}$$

Then $$f'(x) = \left(x - x^{-2} + x^{-1}\right)' = 1 - (-2)x^{-3} + (-1)x^{-2} =$$

$$= 1 + 2x^{-3} - x^{-2}$$

b) $$f(x) = e^{\cos x}\left(\frac{1}{x} + \ln x\right)$$

$$f'(x) = e^{\cos x}\cdot(-\sin x)\left(\frac{1}{x} + \ln x\right) + e^{\cos x}\left(-x^{-2} + \frac{1}{x}\right)$$

c) $$f(x) = \frac{(\arcsin x)^2}{\sqrt{1-x^2}}$$

$$f'(x) = \frac{2(\arcsin x)\cdot\frac{1}{\sqrt{1-x^2}}\cdot\sqrt{1-x^2} - (\arcsin x)^2\cdot\frac{1}{2\sqrt{1-x^2}}(-2x)}{\left(\sqrt{1-x^2}\right)^2} =$$

$$= \frac{2\arcsin x + \frac{x(\arcsin x)^2}{\sqrt{1-x^2}}}{1-x^2}$$

d) $f(x) = \sqrt{x + \sqrt{x + \sqrt{x}}} = \left[x + (x + x^{\frac{1}{2}})^{\frac{1}{2}}\right]^{\frac{1}{2}}$

$f'(x) = \frac{1}{2}\left[x + (x + x^{\frac{1}{2}})^{\frac{1}{2}}\right]^{-\frac{1}{2}} \cdot \frac{1}{2}(x + x^{\frac{1}{2}})^{-\frac{1}{2}} \cdot \frac{1}{2}x^{-\frac{1}{2}}$

e) $f(x) = (\ln x)^{\sin(2x)}$ $\Leftrightarrow$

$y = (\ln x)^{\sin(2x)}$ / ln

$\ln y = \ln\left[(\ln x)^{\sin 2x}\right] = \sin 2x \ln(\ln x)$

By implicit differentiation:

$\frac{1}{y} y' = 2\cos 2x \ln(\ln(x)) + \sin 2x \cdot \frac{1}{\ln x} \cdot \frac{1}{x}$

So $y' = y\left[2\cos(2x)\ln(\ln(x)) + \frac{\sin 2x}{x \ln x}\right]$

Finally:

$y' = (\ln x)^{\sin 2x} \cdot \left[2\cos(2x)\ln(\ln(x)) + \frac{\sin 2x}{x \ln x}\right]$

#5 a) $f(x) = \sqrt{x^2+35}$

$$f'(x) = y' = \frac{1}{2}(x^2+35)^{-\frac{1}{2}} \cdot 2x = \frac{2x}{2\sqrt{x^2+35}} = \frac{x}{\sqrt{x^2+35}}$$

b) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{(x+h)^2+35} - \sqrt{x^2+35}}{h} \cdot \frac{\sqrt{(x+h)^2+35} + \sqrt{x^2+35}}{\sqrt{(x+h)^2+35} + \sqrt{x^2+35}}$$

$$= \lim_{h \to 0} \frac{(x+h)^2+35-(x^2+35)}{h\left[\sqrt{(x+h)^2+35} + \sqrt{x^2+35}\right]} = \lim_{h \to 0} \frac{x^2+2xh+h^2+35-x^2-35}{h\left[\sqrt{(x+h)^2+35} + \sqrt{x^2+35}\right]} =$$

$$= \lim_{h \to 0} \frac{h(2x+h)}{h\left[\sqrt{(x+h)^2+35} + \sqrt{x^2+35}\right]} = \frac{2x}{\sqrt{x^2+35} + \sqrt{x^2+35}} = \frac{2x}{2\sqrt{x^2+35}} =$$

$$= \frac{x}{\sqrt{x^2+35}}$$

c) $L(x) = f(a) + f'(a)(x-a)$ where $a = 1$

So $$L(x) = \sqrt{(1)^2+35} + \frac{1}{\sqrt{(1)^2+35}}(x-1) = \sqrt{36} + \frac{1}{\sqrt{36}}(x-1) = 6 + \frac{1}{6}(x-1)$$

d) We know that $f(x) \approx L(x)$

$$\sqrt{39} = \sqrt{4+35} = \sqrt{2^2+35} = f(2) \approx L(2) = 6 + \frac{1}{6}(2-1) = 6 + \frac{1}{6} = 6 + 0.1667 =$$

$$= 6.1667$$

The exact value is $\sqrt{39} = 6.244998$

#6 a) $2(x^2+y^2)^2 = 25(x^2-y^2)$

Plug the point (3, 1) to verify that it belongs to the curve:

$$2(9+1)^2 = 25(9-1)$$

$$2\cdot 100 = 25\cdot 8$$

$$200 = 200$$

Differentiating implicitly we get:

$$2\cdot 2(x^2+y^2)\cdot(2x+2yy') = 25(2x-2yy')$$

At the point (3,1) we get:

$$4(9+1)(6+2y') = 25(6-2y')$$

$$240+40y' = 150-50y'$$

$$90y' = -90 \implies \boxed{y' = -1}$$

The equation of a line passing through a point (x_0, y_0) with slope m is: $y-y_0 = m(x-x_0)$. Hence, the equation of the tangent line is:

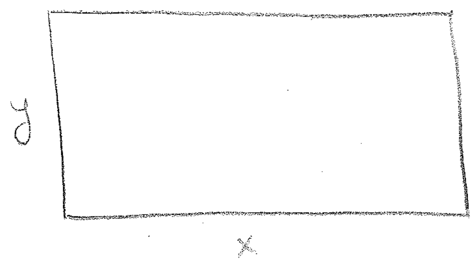
$$y-1 = -1(x-3)$$

$$y = -x+3+1$$

$$\boxed{y = -x+4}$$

#6

b)



$$\frac{dx}{dt} = 8$$

$$\frac{dy}{dt} = 3$$

$x = 20, \quad y = 10$

$$\frac{dA}{dt} = ?$$

$A = x \cdot y$

$$\frac{dA}{dt} = \frac{dx}{dt} y + x \cdot \frac{dy}{dt} = 8 \cdot 10 + 20 \cdot 3 = 80 + 60 = 140$$

So the area is increasing at the rate of 140 cm/s

c) $\lim\limits_{x \to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x}$ is of $\frac{0}{0}$ type. That means

that we can use the l'Hôpital's rule:

$$\lim_{x \to 0} \frac{e^x - e^{-x} - 2x}{x - \sin x} \overset{l'H}{=} \lim_{x \to 0} \frac{e^x - (-e^{-x}) - 2}{1 - \cos x} = \lim_{x \to 0} \frac{e^x + e^{-x} - 2}{1 - \cos x} \overset{l'H}{=}$$

$$= \lim_{x \to 0} \frac{e^x - e^{-x}}{\sin x} \overset{l'H}{=} \lim_{x \to 0} \frac{e^x + e^{-x}}{\cos x} = \frac{1 + 1}{1} = 2$$

#7 $f(x) = x^3 - 2x$

a) $P_1(-3, f(-3)) = P_1(-3, -21)$

$P_2(3, f(3)) = P_2(3, 21)$

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{21 - (-21)}{3 - (-3)} = \frac{42}{6} = 7$$

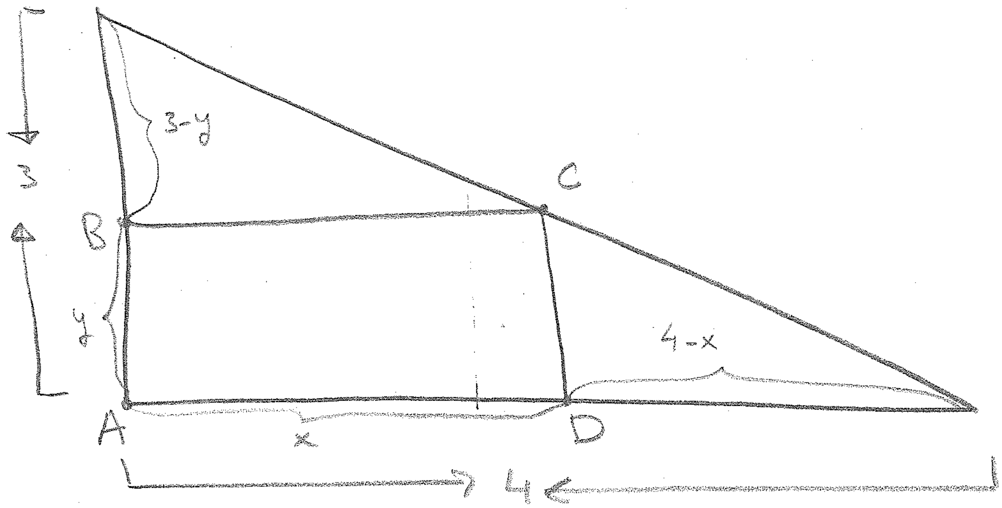
b) Note that $f(x)$ is continuous on $[-3, 3]$ and it is differentiable on $(-3, 3)$, then by the Mean Value Theorem, There is at least one $c \in (-3, 3)$ such that $f'(c) = \frac{f(3) - f(-3)}{3 - (-3)} = 7$.

$f'(c) = 3c^2 - 2$. So

$$3c^2 - 2 = 7$$
$$3c^2 = 9$$
$$c^2 = 3$$
$$c_1 = \sqrt{3} \qquad c_2 = -\sqrt{3}$$

Since $c_1 \in (-3, 3)$ and $c_2 \in (-3, 3)$ we have found two points $x = c$ such that $f'(c) = m$.

#8 a)



$A_{ABCD} = x \cdot y$

Note that: $\frac{3-y}{x} = \frac{3}{4} \Rightarrow$

$\Rightarrow 4(3-y) = 3x \Rightarrow \boxed{x = \frac{4}{3}(3-y)}$

Substituting in the formula for the area of the rectangle, we get:

$$A = \frac{4}{3}(3-y)y = \frac{4}{3}(3y - y^2)$$

$$A' = \frac{4}{3} \cdot (3 - 2y)$$

The critical values: $\frac{4}{3}(3-2y) = 0 \Rightarrow \boxed{y = \frac{3}{2}}$

$A'' = \frac{4}{3} \cdot (-2) < 0$ so by the second derivative test $y = \frac{3}{2}$ maximizes the area. Then $x = \frac{4}{3}(3 - \frac{3}{2}) = \frac{4}{3} \cdot \frac{3}{2} = 2$

So the larges area is $A = x \cdot y = 2 \cdot \frac{3}{2} = 3$

#9 $f(x) = x\sqrt{2-x^2}$

a) - Domain: $2-x^2 \geq 0$

$$(\sqrt{2}+x)(\sqrt{2}-x) \geq 0$$

For $x=\sqrt{2}$ or $x=-\sqrt{2}$ $2-x^2=0$.

By testing values in the intervals $(-\infty,-\sqrt{2})$, $(-\sqrt{2},\sqrt{2})$ and $(\sqrt{2},+\infty)$ we can conclude that only for $x\in(-\sqrt{2},\sqrt{2})$ $2-x^2>0$. Hence the domain of $f(x)$ is $[-\sqrt{2},\sqrt{2}]$

- Symmetry: $f(-x)=(-x)\sqrt{2-(-x)^2} = -x\sqrt{2-x^2} = -f(x) \Rightarrow$

$\Rightarrow$ $f(x)$ is an odd function and so the graph of $f(x)$ has rotational symmetry with respect to the origin

- Asymptotes:

No horizontal asymptotes.

Vertical asymptotes:

$$\left.\begin{array}{l} \lim\limits_{x\to -\sqrt{2}^+}(x\sqrt{2-x^2}) = 0 \\ \lim\limits_{x\to \sqrt{2}^-}(x\sqrt{2-x^2}) = 0 \end{array}\right\} \Rightarrow \text{No vertical asymptotes.}$$

- x intercept: $y=0 \Rightarrow 0=x\sqrt{2-x^2} \Rightarrow x_1=-\sqrt{2}$ $x_2=0$ $x_3=\sqrt{2}$

b) $f'(x) = \sqrt{2-x^2} + x\frac{(-2x)}{2\sqrt{2-x^2}} = \sqrt{2-x^2} - \frac{2x^2}{2\sqrt{2-x^2}}$

$= \frac{2-x^2-x^2}{\sqrt{2-x^2}} = \frac{2(1-x^2)}{\sqrt{2-x^2}} = \frac{2(1-x)(1+x)}{\sqrt{2-x^2}}$

The critical points are: $-\sqrt{2}, -1, 1, \sqrt{2}$

intervals	sign of $f'(x)$	$f(x)$	min or max
$(-\sqrt{2}, -1)$	$-$	$\searrow$	
			min at $x=-1$
$(-1, 1)$	$+$	$\nearrow$	
			max at $x=1$
$(1, \sqrt{2})$	$-$	$\searrow$	

$\Rightarrow$ $f(-1) = (-1)\sqrt{2-(-1)^2} = -\sqrt{2-1} = -1$ → minimum value of $f(x)$

$f(1) = \sqrt{2-1} = 1$ → maximum value of $f(x)$

$f(x)$ is increasing on $(-1,1)$ and $f(x)$ is decreasing on $(-\sqrt{2},-1)\cup(1,\sqrt{2})$

c) $f''(x) = \frac{-4x\sqrt{2-x^2} - \frac{2(1-x^2)}{2\sqrt{2-x^2}}(-2x)}{2-x^2} = \frac{-4x\sqrt{2-x^2} + \frac{2x(1-x^2)}{\sqrt{2-x^2}}}{2-x^2} =$

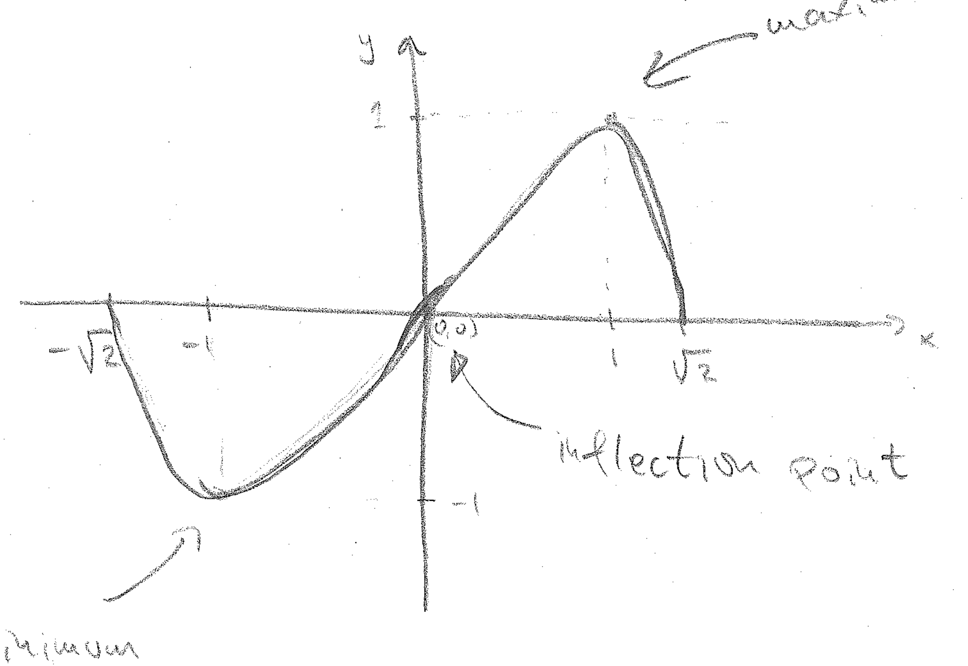
$= \frac{-4x(2-x^2)+2x-2x^3}{(2-x^2)\sqrt{2-x^2}} = \frac{-8x+4x^3+2x-2x^3}{(2-x^2)\sqrt{2-x^2}} = \frac{-6x+2x^3}{(2-x^2)\sqrt{2-x^2}} =$

$= \frac{2x(x^2-3)}{(2-x^2)\sqrt{2-x^2}}$

$f''(x)=0$ for $x=0$, $x=-\sqrt{3}$ and $x=\sqrt{3}$, but the last two numbers are out of our domain so the critical points for $f''(x)$ are $x=0$, $x=-\sqrt{2}$ and $x=\sqrt{2}$

intervals	Sign of $f''(x)$	concavity of $f(x)$	inflection points
$(-\sqrt{2}, 0)$	+	∪	inflection point at $x=0$
$(0, \sqrt{2})$	−	∩	$\Rightarrow f(0)=0$

d)



Bonus Question: Proof by contradiction:

Let us assume that $f(x)$ has two fixed points (more then one fixed points) i.e. there are numbers a and b such that $f(a)=a$ and $f(b)=b$. Let us define

$g(x)=f(x)-x \Rightarrow g(a)=f(a)-a=0$ and $g(b)=f(b)-b=0$

Since $g(a)=g(b)$, by the Rolle's theorem there is a number $c \in (a,b)$ such that $g'(c)=0$.

$g'(x)=f'(x)-1 \Rightarrow g'(c)=f'(c)-1=0 \Rightarrow \underline{f'(c)=1}$. But our condition says $f'(x) \neq 1$ for any x. Contradiction!

We conclude that if $f'(x) \neq 1$ then for $x \in \mathbb{R}$ then f has no more then one fixed point.

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	**Number**	**Sections**
Mathematics	203	All
Examination	**Date**	**Pages**
Final	April 2010	2
Instructors:	J. Bachrachas, D. Dryanov, W. Li, Z. Li, P. Zorin	**Course Examiner** A. Atoyan
Special Instructions:	Only Sharp EL 531 or Casio FX 300 MS calculators are allowed	

MARKS

[11] **1.** **(a)** Let $f(x) = (x-2)^2$ and $g(x) = \sqrt{4-x}$. Find $h = g \circ f$ and determine the domain and the range of h,

(b) Find the range of the function $f = e^{2x} + 2e^x$, the inverse function f^{-1}, and the range of f^{-1}. (HINT: assume $e^x = u$ to see how to find f^{-1})

[10] **2.** Evaluate the limits:

(a) $\lim_{x\to -2} \frac{x^2-x-6}{4-x^2}$ **(b)** $\lim_{x\to 0} \frac{\sqrt{x+a^2}-a}{ax}$ $(a > 0)$

Do not use l'Hôpital rule.

[6] **3.** Find all horizontal and vertical asymptotes of the function

$$f(x) = \frac{|x|\sqrt{4x^2+1} - 2x^2}{x^2-4}$$

.

[15] **4.** Find the derivatives of the following functions:

(a) $f(x) = \frac{2\sqrt{x^5} - x^{3/2}}{x^2}$

(b) $f(x) = \ln \frac{x^4}{\sqrt{x-3}}$

(c) $f(x) = e^3 + \arctan(e^x - e^{-x})$

(d) $f(x) = \frac{3^x}{1+\cos(x^2)}$

(e) $f(x) = (1+x^2)^{2x}$ (use logarithmic differentiation)

---

[15] **5.** **(a)** Verify that the point (2,0) belongs to the curve defined by the equation $y + x\sqrt{1+y^2} + 2 = x^2$, and find the equation of the tangent line to the curve at this point.

(b) A particle is moving along a circle with radius $r = 5\,\mathrm{m}$ described by the equation $x^2 + y^2 = 25$ in the (x, y) plane. At the point $(-4, 3)$ the x-coordinate changes at the rate $\frac{\mathrm{d}x}{\mathrm{d}t} = 15\,\frac{\mathrm{m}}{\mathrm{sec}}$. How fast is the y coordinate changing at that instant?

(c) Use the l'Hôpital's rule to evaluate the $\lim\limits_{x\to 0} \frac{e^x - x - 1}{x^2 + x^3}$.

[6] **6.** Let $f(x) = \frac{x}{3x-1}$.

(a) Find the slope m of the secant line joining the points $(1, f(1))$ and $(3, f(3))$.

(b) Find all points $x = c$ (if any) on the interval [1,3] such that $f'(c) = m$.

[9] **7.** The volume of a sphere with radius r is given by the formula $V(r) = \frac{4\pi}{3} r^3$.

(a) Use the **definition of the derivative** to show that $\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$.

(b) If a is a given fixed value for r, write the formula for the linearization of the volume function at a.

(c) Use this linearization to calculate the thickness Δr (in centimeters) of a layer of paint on the surface of a spherical ball with radius $r = 52\,\mathrm{cm}$ if the total volume of paint used is $340\,\mathrm{cm}^3$.

[12] **8.** **(a)** Find the absolute extrema of $f(x) = x\,e^{-x^2}$ on the interval $[-\frac{1}{2}, 1]$.

(b) Find the radius r and the height h of the a cylindrical can that is open at the top and has a volume $1000\,\mathrm{cm}^3$, but has the smallest possible surface area.

[16] **9.** Given the function $f(x) = 2x^2 - x^4$.

(a) Find the domain of f and check for symmetry. Find asymptotes of f (if any).

(b) Calculate $f'(x)$ and use it to determine intervals where the function is increasing, intervals where it is decreasing, and the local extrema (if any).

(c) Calculate $f'(x)$ and use it to determine intervals where the function is concave upward, intervals where the function is concave downward, and the inflection points (if any).

(d) Sketch the graph of the function $f(x)$ using the information obtained above.

[5] **Bonus Question**

Let $f(x) = \frac{\sin(ax)}{x-a}$ where a is a real number. Using l'Hôpital's rule, the following limit of f at $x \to a$ is calculated:

$\lim\limits_{x\to a} \frac{\sin(ax)}{x-a} = \lim\limits_{x\to a} \frac{a\cos(ax)}{1} = a\cos(a^2)$.

But if $a = 1$, this says $\lim\limits_{x\to 1} \frac{\sin(x)}{x-1} = \cos(1)$.

(a) Explain what is wrong with this calculation.

(b) Are there values of a for which the calculation is correct?

MATH 203 Final Examination
April 2010

SOLUTIONS

MARKS

11

1 (a) $h(x) = g \circ f(x) = g\left((x-2)^2\right) = \sqrt{4-(x-2)^2} = \sqrt{4x-x^2}$. The domain of h is determined by the requirement

$$4x - x^2 \geq 0$$
$$x(4-x) \geq 0 \Rightarrow 0 \leq x \leq 4$$

So the domain is $[0,4]$. If we put $y = h(x)$ we see that $(x-2)^2 + y^2 = 4$ so $h(x)$ is the graph of the upper semi-circle of radius 2 centered at $(2,0)$. The range is therefore $[0,2]$.

1 (b) If $y = f(x) = e^{2x} + 2e^x$ we see that $y > 0$ for every x but $\lim_{x\to-\infty} f(x) = 0$,and $\lim_{x\to\infty} f(x) = \infty$ So the range is $(0,\infty)$. To find the inverse function, first solve for x. : We do this by letting $u = e^x$ as per the hint and we get

$$u^2 + 2u - y = 0$$
$$u = -1 \pm \sqrt{1+y}$$

We take the positive root because $u = e^x > 0$ so $e^x = \sqrt{1+y} - 1$. Now interchange x and y to get

$$y = \ln\ \sqrt{1+x} - 1)$$

as the inverse function. It is defined for all $x > 0$ and has range $\mathbb{R}$.

10

2 (a)

$$\lim_{x\to-2} \frac{x^2-x-6}{4-x^2} = \lim_{x\to-2} \frac{(x+2)(x-3)}{(2+x)(2-x)}$$
$$= \lim_{x\to-2} \frac{(x-3)}{(2-x)} = \frac{-5}{4} = -\frac{5}{4}$$

(b)

$$\lim_{x\to0} \frac{\sqrt{x+a^2}-a}{ax} = \lim_{x\to0} \frac{\sqrt{x+a^2}-a}{ax} \cdot \frac{\sqrt{x+a^2}+a}{\sqrt{x+a^2}+a}$$
$$= \lim_{x\to0} \frac{x}{ax} \frac{1}{\sqrt{x+a^2}+a}$$
$$= \frac{1}{a} \lim_{x\to0} \frac{1}{\sqrt{x+a^2}+a} = \frac{1}{a} \cdot \frac{1}{2a} = \frac{1}{2a^2}$$

6

3

$$f(x) = \frac{|x|\sqrt{4x^2+1} - 2x^2}{x^2-4} = \frac{|x|\sqrt{4x^2+1} - 2x^2}{(x-2)(x+2)}$$

and since $\lim_{x\to\pm 2} |x|\sqrt{4x^2+1} - 2x^2) = 2\sqrt{17} - 8 \neq 0$, it follows that $x = \pm 2$ are vertical asymptotes. Also

$$\lim_{x\to\pm\infty} f(x) = \lim_{x\to\pm\infty} \left(\frac{\sqrt{4+1/x^2} - 2}{1-4/x^2} \right) = 0$$

so $y = 0$ is the only horizontal asymptote.

15

4 (a)

$$f(x) = \frac{2\sqrt{x^5} - x^{3/2}}{x^2} = 2x^{1/2} - x^{-1/2}$$
$$f'(x) = x^{-1/2} + 1/2x^{-3/2}$$

(b)

$$f(x) = \ln\left(\frac{x^4}{\sqrt{x-3}}\right) = 4\ln x - \frac{1}{2}\ln(x-3)$$
$$f'(x) = \frac{4}{x} - \frac{1}{2(x-3)}$$

(c)

$$f(x) = e^3 + \arctan\ e^x - e^{-x})$$
$$f'(x) = 0 + \frac{1}{1+(e^x - e^{-x})^2} \cdot\ e^x + e^{-x})$$

(d)

$$f(x) = \frac{3^x}{1+\cos(x^2)}$$
$$f'(x) = \frac{1+\cos\ x^2)) \cdot 3^x \cdot \ln 3 - 3^x \cdot\ -\sin\ x^2)) \cdot 2x}{(1+\cos(x^2))^2}$$

(e)

$$
\begin{aligned}
f(x) &= \ 1+x^2)^{2x} \\
\ln(f(x)) &= 2x\ln\ 1+x^2) \\
\frac{f'(x)}{f(x)} &= 2\ln\ 1+x^2) + \frac{2x}{1+x^2}\cdot 2x \\
&= 2\ln\ 1+x^2) + \frac{4x^2}{1+x^2} \\
f'(x) &= \ 1+x^2)^{2x}\left(2\ln\ 1+x^2) + \frac{4x^2}{1+x^2}\right)
\end{aligned}
$$

15

5 (a) Substituting $(2,0)$ into $y+x\sqrt{1+y^2}+2=x^2$ gives $2+2=2^2$ which is correct. Differentiate implicitly:

$$y' + \sqrt{1+y^2} + \frac{xy}{\sqrt{1+y^2}}y' = 2x$$

now substitute before solving for y'

$$
\begin{aligned}
y'+1 &= 4 \\
y' &= 3
\end{aligned}
$$

and an equation of the tangent line is $y = 3(x-2)$.

(b) We have $x^2+y^2=25$. Differentiate with respect to time t to get

$$
\begin{aligned}
2x\frac{dx}{dt} + 2y\frac{dy}{dt} &= 0 \\
\frac{dy}{dt} &= -\frac{x}{y}\frac{dx}{dt} \\
&= -\left(\frac{-4}{3}\right)\cdot 15 \\
&= 20 \text{ m/s}
\end{aligned}
$$

(c) Since the expression $\dfrac{e^x-x-1}{x^2+x^3}$ is an indeterminate form of the type $\dfrac{0}{0}$ at 0, we have

$$
\begin{aligned}
\lim_{x\to 0}\frac{e^x-x-1}{x^2+x^3} &= (\text{L'H})\ \lim_{x\to 0}\frac{e^x-1}{2x+3x^2} \\
&= (\text{L'H})\ \lim_{x\to 0}\frac{e^x}{2+6x} \\
&= \frac{1}{2}
\end{aligned}
$$

where we use L'Hôpital's rule a second time since the second expression is also an indeterminate form of the type $\dfrac{0}{0}$ at 0.

6

6 (a) $f(x) = \dfrac{x}{3x-1}$ so

$$\begin{aligned} m &= \frac{f(3) - f(1)}{3-1} = \frac{3/8 - 1/2}{2} \\ &= -\frac{1}{16} \end{aligned}$$

(b) $f'(x) = -\dfrac{1}{(3x-1)^2}$ so if $1 \le c \le 3$ and $f'(c) = m = -\dfrac{1}{16}$ we have

$$\begin{aligned} -\frac{1}{(3c-1)^2} &= -\frac{1}{16} \\ (3c-1)^2 &= 16 \\ 3c - 1 &= 4 \text{ (since } c > 0) \\ c &= \frac{5}{3} \end{aligned}$$

9

7 (a)

$$\begin{aligned} V(r) &= \frac{4}{3}\pi r^3 \\ \frac{dV}{dr} &= \lim_{h \to 0} \frac{V(r+h) - V(r)}{h} = \frac{4}{3}\pi \lim_{h \to 0} \frac{(r+h)^3 - r^3}{h} \\ &= \frac{4}{3}\pi \lim_{h \to 0} \frac{r^3 + 3r^2h + 3rh^2 + h^3 - r^3}{h} \\ &= \frac{4}{3}\pi \lim_{h \to 0} \frac{h\ \ 3r^2 + 3rh + h^2)}{h} = \frac{4}{3}\pi \lim_{h \to 0}\ \ 3r^2 + 3rh + h^2) \\ &= \frac{4}{3}\pi \cdot 3r^2 = 4\pi r^2 \end{aligned}$$

(b) The formula is

$$\begin{aligned} L(r) &= f(a) + f'(a)(r-a) \\ &= \frac{4}{3}\pi a^3 + 4\pi a^2 (r-a) \end{aligned}$$

(c) We use the estimate $dV \approx \Delta V = L(r) - \dfrac{4}{3}\pi a^3$ and $\Delta r = r - a$ to get

$$\begin{aligned} L(r) - \frac{4}{3}\pi a^3 &= 4\pi a^2 (r-a) \\ dV &= 4\pi a^2 \Delta r \\ \Delta r &= \frac{dV}{4\pi a^2} = \frac{340}{4\pi \cdot (52)^2} \\ &= .01\text{cm} \end{aligned}$$

12

8 (a) $f(x) = xe^{-x^2}$ on $\left[-\frac{1}{2}, 1\right]$.

$$\begin{aligned} f'(x) &= e^{-x^2} + xe^{-x^2}(-2x) \\ &= e^{-x^2}\left(1-2x^2\right) \\ f'(x) &= 0 \Leftrightarrow 1-2x^2 = 0 \\ &\Leftrightarrow 2x^2 = 1 \end{aligned}$$

and so the critical numbers are $x = 1/\sqrt{2}$ and $x = -1/\sqrt{2}$ (but this one is not in the given interval). So we check: ; $f(-1/2) = -\frac{1}{2}e^{-\frac{1}{4}} \simeq -0.3894$; $f\left(1/\sqrt{2}\right) = \frac{1}{\sqrt{2}}e^{-\frac{1}{2}} \simeq 0.42888$; and $f(1) = e^{-1} \simeq 0.36788$. So the absolute minimum is $-\frac{1}{2}e^{-\frac{1}{4}} \simeq -0.3894$ at $-1/2$ and the absolute maximum is $\frac{1}{\sqrt{2}}e^{-\frac{1}{2}} \simeq 0.42888$ at $1/\sqrt{2}$.

(b) Let A be the area of the can. Given the radius is r and the height is h we see that the total area is the sum of the area of the base $\left(\pi r^2\right)$ plus the area of the cylindrical side $(2\pi rh)$

$$A = \pi r^2 + 2\pi rh; r > 0$$

Also the volume

$$\begin{aligned} V &= \pi r^2 h = 1000 \text{ so} \\ h &= \frac{1000}{\pi r^2} \text{ and} \\ A &= \pi r^2 + 2\pi r \cdot \frac{1000}{\pi r^2} = \pi r^2 + \frac{2000}{r} \\ A' &= 2\pi r - \frac{2000}{r^2} = 0 \Leftrightarrow r^3 = \frac{1000}{\pi} \text{ so} \\ r &= \frac{10}{\sqrt[3]{\pi}} = h \end{aligned}$$

Since

$$A'' = 2\pi + 4000r^{-3} > 0$$

we have a local minimum (and hence an absolute minimum).

16

9 (a) $f(x) = 2x^2 - x^4$ is defined for all x because it is a polynomial, so the domain is $\mathbb{R} = (-\infty, \infty)$. It is an even function since $f(-x) = f(x)$.

There are no vertical asymptotes because the domain is $\mathbb{R}$ (so there are no points a where the function can have limit ∞ or $-\infty$). There are no horizontal asymptotes because $\lim_{x\to\pm\infty} f(x) = -\infty$. So there are no asymptotes.

(b)

$$\begin{aligned} f'(x) &= 4x - 4x^3 = 4x\ \ 1 - x^2\big) \\ &= 4x(1-x)(1+x) \end{aligned}$$

The critical numbers are $0, -1$ and 1. Constructing the table to see where it is increasing/decreasing we have

	$x < -1$	$x = -1$	$-1 < x < 0$	$x = 0$	$0 < x < 1$	$x = 1$	$x > 1$
$4x$	$-$	$-$	$-$	0	$+$	$+$	$+$
$1 - x$	$+$	$+$	$+$	$+$	$+$	0	$-$
$1 + x$	$-$	0	$+$	$+$	$+$	$+$	$+$
$f'(x)$	$+$	0	$-$	0	$+$	0	$-$
	$\nearrow$	Max.	$\searrow$	min.	$\nearrow$	Max.	$\searrow$

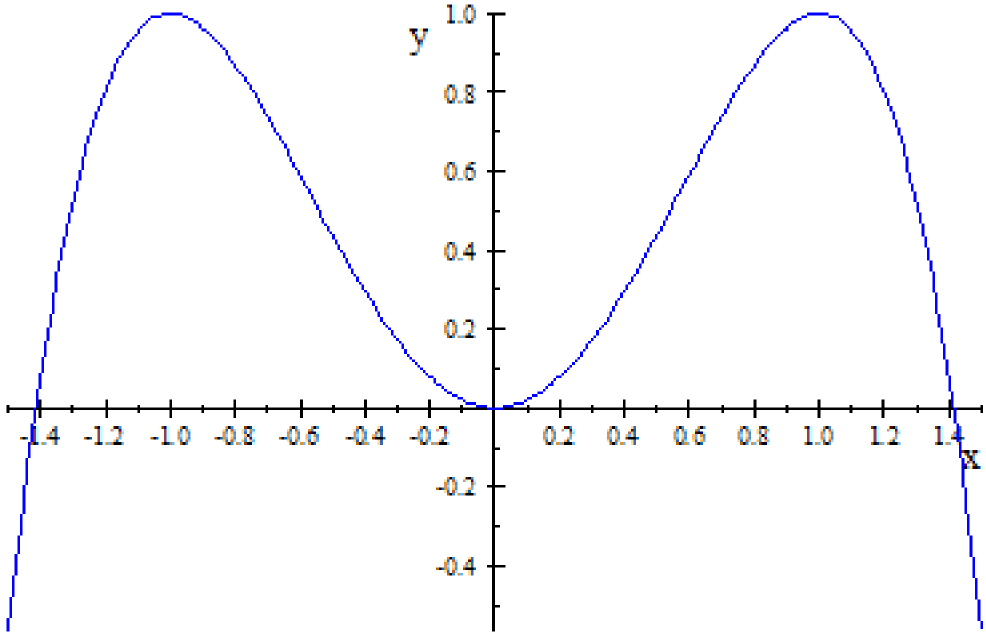
We can read off the intervals and the local extrema: local maxima at $(-1, 1)$ and $(1, 1)$ and a local minimum at $(0, 0)$.

(c)

$$\begin{aligned} f''(x) &= 4 - 12x^2 = 4\ \ 1 - 3x^2\big) \\ &= 0 \Longleftrightarrow x = \pm 1/\sqrt{3} \end{aligned}$$

Here we will use the even symmetry and note that on $\ 0, 1/\sqrt{3}\big)$ $f''(x) > 0$ so the curve is concave up. By symmetry the curve is concave up on $\ -1/\sqrt{3}, 1/\sqrt{3}\big)$. For $x > 1/\sqrt{3}$ (and $x < -1/\sqrt{3}$) it is concave down, so these points: $\ \pm 1/\sqrt{3}, 1/3\big) = (\pm 0.577, 0.333)$ are inflection points.

(d) The graph:



5

Bonus

$$f(x) = \frac{\sin(ax)}{x-a}$$

$$\lim_{x\to a} f(x) = \frac{a\cos(ax)}{1} = a\cos\ a^2) \text{ by L'Hôpital'srule}$$

If $a = 1$

$$\lim_{x\to 1} \frac{\sin(x)}{x-1} = \frac{\cos(x)}{1} = \cos(1) \text{ by L'Hôpital'srule}$$

What is wrong? L'Hôpital's rule doesn't apply if $\sin\ a^2) \neq 0$ since the original expression $f(x) = \frac{\sin(ax)}{x-a}$ is not of the form $\frac{0}{0}$ near $x = a$: the numerator is near $\sin\ a^2)$ while the denominator is near 0. If $a = 0$ we get $\frac{0}{x} = 0$ which has limit 0 anyway, without L'Hôpital's rule. If not, we can only use L'Hôpital's rule when $\sin\ a^2) = 0$, i.e. when $a^2 = \pi n$ or $a = \sqrt{n\pi}$; $n = 1, 2, 3, \cdots$.

Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	203	All
Examination	**Date**	**Pages**
Final	April 2011	3

Instructors	**Course Examiners**
A. Fenwick, R. Lan, I. Pendev, N. Rossokhata	A. Atoyan & H. Proppe

Special Instructions

▷ **Only Sharp EL 531 or Casio FX 300 MS calculators are allowed.**

MARKS

[11] **1.** (a) Sketch the graph of the function $f(x) = |x^2 - 2x|$ starting from the graph of the standard parabola and using appropriate transformations.

(b) Suppose $f(x) = x^3 - 1$ and and $g(x) = \sqrt[3]{1+x}$. Find $f \circ g$, $g \circ f$, and $f \circ f$.

(c) Find the inverse of the function $f(x) = 2 - e^x$. Determine the domain and range of f and f^{-1}.

[8] **2.** Evaluate the limits:

(a) $\displaystyle \lim_{x\to -4} \frac{\sqrt{x^2+9}-5}{x+4}$ (b) $\displaystyle \lim_{x\to -\infty} \frac{\sqrt{9x^6-x}}{x^3+1}$

Do not use l'Hôpital's rule.

[11] **3.** (a) Consider the function $f(x) = \dfrac{|x-1|(x+3)}{x^2-1}$.

Calculate both one-sided limits at the point(s) where the function is undefined.

(b) Find parameters a and b such that the function

$$f(x) = \begin{cases} \dfrac{|x+1|}{x+1}, & \text{if} \quad x < -1 \\ ax, & \text{if} \quad -1 \le x < 1 \\ bx+2, & \text{if} \quad x \ge 1 \end{cases}$$

will be continuous at every point.

[16] **4.** Find derivatives of the functions (you don't have to simplify the answers):

(a) $f(x) = \dfrac{(\sqrt{x} - 2x)^2 - 3x^2}{x}$;

(b) $f(x) = e^2 + e^{x^2} \ln(1 - 5x)$;

(c) $f(x) = \dfrac{\sin^3 x}{\cos(2x) + \tan(x)}$;

(d) $f(x) = \arctan\left(\sqrt{1 + \ln(3x - 1)}\right)$;

(e) $f(x) = (1 + \sin x)^{\tan(x)}$ (use logarithmic differentiation).

[12] **5.** Given the function $f(x) = \sqrt{36 - x}$,

(a) Use appropriate differentiation rules to find the derivative of the function.

(b) Derive the same result for $f'(x)$ using the definition of the derivative as the limit of the difference quotient.

(c) Find the linear approximation of $f(x)$ at $a = 0$.

(d) Use this linear approximation to approximate $\sqrt{32}$.

[17] **6.** (a) Verify that the point (2,1) belongs to the curve defined implicitly by the equation $x^2 + 2y^2 + 2 = x^3y^3$, and find an equation of the tangent line to the curve at this point.

(b) Two cars are moving away from the intersection of two orthogonal streets at the speeds $v_1 = 15$ m/s going east, and $v_2 = 20$ m/s going north. How fast is the distance between the cars increasing at the instant when the first car is at the distance $x = 30$ m and the second car is at the distance $y = 40$ m from the intersection?

(c) Use l'Hôpital's rule to evaluate $\lim\limits_{x \to 0} \dfrac{\sin^2(5x)}{1 - \cos(x)}$.

[6] **7.** Let $f(x) = \sqrt{1+4x}$.

(a) Find the slope m of the secant line joining the points $(0, f(0))$ and $(2, f(2))$.

(b) Find all point(s) $x = c$ (if any) on the interval $[0, 2]$ such that $f'(c) = m$.

[5] **8.** The top and bottom margins of a poster are each 6 cm and the side margins are each 4 cm. If the printed material on the poster is a rectangle with an area of 384 cm^2, find the dimensions of the poster with the the smallest area.

[14] **9.** Given the function $f(x) = \dfrac{x^2}{x^2+9}$,

(a) Find the domain and check for symmetry. Find all horizontal and vertical asymptotes (if any).

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function.

[5] **Bonus Question**

Suppose f and g are both concave upward on $(-\infty, \infty)$. What condition on f is sufficient to ensure that the composite function $h(x) = f(g(x))$ is also concave upward?

Department of Mathematics & Statistics

Course	**Number**	**Section(s)**
Mathematics	203	All
Examination	**Date**	**Pages**
Final	December 2006	3
Instructors		**Course Examiner**
A. Boyarsky, J. Brody, E. Duma, J. Hayes, T. Hughes, Y. Khidirov, H. Proppe, M. Mei		H. Proppe

Special Instructions

▷ Calculators are **not** allowed.

MARKS

[10] **1.** (a) Suppose $f(x) = \sqrt{1-x}$ and $g(x) = \sin^2 x$. Find $f \circ g \circ f$ and $g \circ f \circ g$. Simplify

(b) Find the inverse of the function $f(x) = e^{x^3} - 1$. Determine the domain and range of f and f^{-1}.

[10] **2.** Evaluate the limits:

(a) $\displaystyle \lim_{x \to 4} \frac{\sqrt{5x-4}-4}{2x^3 - 32x}$ (b) $\displaystyle \lim_{x \to \infty} \frac{3x^2(\sqrt{x}+1)^3}{(2x+1)^3(\sqrt{x}-1)}$

Do not use l'Hopital's rule.

[12] **3.** (a) Consider the function $f(x) = \dfrac{|x-3|}{x^2 - 5x + 6}$.

Calculate both one-sided limits at the point(s) where the function is undefined.

(b) Find parameters a and b such that the function

$$f(x) = \begin{cases} \dfrac{2}{x^2}, & \text{if } x \le -1 \\ ax + b, & \text{if } -1 < x \le 0 \\ 4 - x^2, & \text{if } x > 0 \end{cases}$$

will be continuous at every point. Sketch the graph of this function.

[12] **4.** Find derivatives of the functions (do not simplify the answer):

(a) $f(x) = (x^2 + \frac{1}{x})^2 \cos 4x$;

(b) $f(x) = \frac{\arcsin^2 x}{\sqrt{1-x^2}}$;

(c) $f(x) = \tan^3(x + e^{5x})$;

(d) $f(x) = (\arctan 2x)^{\sqrt{x}}$ (use logarithmic differentiation).

[12] **5.** Given the function $f(x) = \frac{2x}{x+2}$,

(a) Use appropriate differentiation rules to find the derivative of the function.

(b) Use the definition of derivative to verify (a).

(c) Find the linear approximation of the function at $x_0 = 2$.

(d) Use the linear approximation above to approximate $f(2.2)$.

[18] **6.** (a) The equation of a curve defined implicitly is $y^3 \ln(e + x^2) = x^3y - 3x + 1$. Verify that the point $(0, 1)$ belongs to the curve. Find an equation of the tangent line to the curve at this point.

(b) Let $f(x) = \ln^2 \sin x$. Find $f''(x)$.

(c) Use l'Hopital's rule to evaluate $\lim_{x\to 0} \frac{x \sin 2x}{e^{x^2} - 1}$.

[10] **7.** (a) A particle is moving along the plane curve $x^2y = 8$. At the moment when $x = 1$ the x-coordinate is increasing at a rate of 3 cm/sec. Is the y-coordinate increasing or decreasing at this moment? How fast?

(b) A rectangle $ABCD$ has sides parallel to the coordinate axes and point A is located at the origin. Point C lies on the graph of the function $y = -\ln x$ and has a positive y coordinate. Find the coordinates of the point C so that the area of the rectangle is maximized.

[16] **8.** Given the function $f(x) = x^2 e^x$,

(a) Find the domain and check for symmetry. Find asymptotes (if any).

(b) Calculate $f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function.

[5] **Bonus Question**

Given the equation $x^5 + x = 1$,

(a) Show that there is a root between $\frac{1}{2}$ and 1.

(b) Show that the equation has exactly one root.

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number	Sections
Mathematics	203	All
Examination	**Date**	**Pages**
Final	April 2015	3
Instructors:	H. Greenspan, J. Nam, B. Rhodes, S. Vikram, Y. Zhao	**Course Examiner** A. Atoyan
Special Instructions:	Only approved calculators are allowed. **Show all your work for full marks.**	

MARKS

[11] **1.** (a) Let $f(x) = x^2 - 4$ and $g(x) = \sqrt{4 - x}$. Find $g \circ f$ and $f \circ g$ and determine the domain of each of these composite functions.

(b) Find the range of the function $f = e^{2x} + 2e^x$, the inverse function f^{-1}, and the range of f^{-1}. (HINT: assume $e^x = u$ to see how to find f^{-1})

[10] **2.** Evaluate the limits.

(a) $\displaystyle\lim_{x \to -2} \frac{x^2 - x - 6}{4 - x^2}$ (b) $\displaystyle\lim_{x \to 0} \frac{\sqrt{x + 9} - 3}{x}$

Do not use l'Hôpital rule.

[6] **3.** Find all horizontal and vertical asymptotes of the function

$$f(x) = \frac{|x|\sqrt{4x^2 + 1} - 2x^2}{x^2 - 3}$$

[15] **4.** Find the derivatives of the following functions:

(a) $f(x) = \dfrac{2\sqrt{x^5} - x^{3/2}}{x^2}$

(b) $f(x) = \ln \dfrac{x^4}{x - 3}$

(c) $f(x) = e^3 + \arctan(e^x - e^{-x})$

(d) $f(x) = \dfrac{e^x}{1 + \cos(x^2)}$

(e) $f(x) = (1 + x^2)^{2x}$ (use logarithmic differentiation)

[15] 5. (a) Verify that the point (2 0) belongs to the curve defined by the equation $y + x\sqrt{1+y^2} + 2 = x^2$, and find the equation of the tangent line to the curve at this point.

(b) A particle is moving along a circle with radius $r = 5\,\mathrm{m}$ described by the equation $x^2 + y^2 = 25$ in the (x, y) plane. At the point $(-4, 3)$ the x-coordinate changes at the rate $\frac{dx}{dt} = 15\,\frac{\mathrm{m}}{\mathrm{sec}}$. How fast is the y coordinate changing at that instant?

(c) Use the l'Hôpital's rule to evaluate the $\lim\limits_{x\to 0} \dfrac{e^x - x - 1}{x^2 + x^3}$.

[6] 6. Let $f(x) = \dfrac{x}{3x-1}$.

(a) Find the slope m of the secant line joining the points $(1, f(1))$ and $(3, f(3))$.

(b) Find all points $x = c$ (if any) on the interval [1,3] such that $f'(c) = m$.

[9] 7. The volume of a sphere with radius r is given by the formula $V(r) = \frac{4\pi}{3}r^3$.

(a) Use the **definition of the derivative** to show that $\frac{dV}{dr} = 4\pi r^2$.

(b) If a is a given fixed value for r, write the formula for the linearization of the volume function $V(r)$ at a.

(c) Use this linearization to calculate the thickness Δr (in centimeters) of a layer of paint on the surface of a spherical ball with radius $r = 52\,\mathrm{cm}$ if the total volume of paint used is $340\,\mathrm{cm}^3$.

[12] 8. **(a)** Find the absolute extrema of $f(x) = x\,e^{-x^2}$ on the interval $[-\frac{1}{2}, 1]$.

(b) Find the radius r and the height h of the a cylindrical can that is open at the top and has a volume $1000\,\mathrm{cm}^3$, but has the smallest possible surface area.

[16] 9. Given the function $f(x) = 2x^2 - x^4$.

(a) Find the domain of f and check for symmetry. Find asymptotes of f (if any).

(b) Calculate $f'(x)$ and use it to determine intervals where the function is increasing, intervals where it is decreasing, and the local extrema (if any).

(c) Calculate $f''(x)$ and use it to determine intervals where the function is concave upward, intervals where the function is concave downward, and the inflection points (if any).

(d) Sketch the graph of the function $f(x)$ using the information obtained above.

[5] **Bonus Question**

We know that a function f is differentiable on the interval [0,2] and has values $f(0) = 0$, $f(1) = 1$ and $f(2) = -1$. Is this information sufficient to claim, using the Mean Value theorem, that the tangent line to the graph of $f(x)$ must be horizontal at least at one point x in the interval (0,2)? Explain why yes or why not.

CONCORDIA UNIVERSITY
Department of Mathematics & Statistics

Course	Number	Sections
Mathematics	203	All
Examination	**Date**	**Pages**
Alternate	December 2014	3
Instructors:	Z. Ben Salah, A. Boyarsky, J. Brody, I. Gorelyshev, T. Hughes, P. Moore	**Course Examiner** A. Atoyan
Special Instructions:	Only approved calculators are allowed **Show all your work for full marks.**	

MARKS

[11] **1.** (a) Solve for x: $2\log_2(x) - \log_2(x+3) = 5^{\log_5(2)}$

(b) Let $f(x) = \dfrac{1}{x^2-1}$ and $g(x) = \sqrt{1+x}$. Find $f \circ g$ and determine its domain.

(c) Given the function $f(x) = (8+2^x)^{1/3}$, find the inverse function f^{-1}, the range of f and the range of f^{-1}.

[7] **2.** Find the limit if it exists (**do not** use l'Hôpital's rule.):

(a) $\displaystyle\lim_{x\to 1} \frac{x^2+2x-3}{|x-1|}$ (b) $\displaystyle\lim_{x\to -2} \frac{3-\sqrt{x^2-3x-1}}{x^2-4}$

[6] **3.** Find all horizontal and vertical asymptotes of the function

$$f(x) = \frac{\sqrt{9x^4+2x^2+1}}{x^2+4x}$$

[15] **4.** Find the derivatives of the following functions (you don't need to simplify your final answer, but you must show how you calculate it):

(a) $f(x) = (x^{3/2}+1)(x^{3/2}-1)\tan x$

(b) $f(x) = (x+e^{-x})\ln^2(x)$

(c) $f(x) = \sqrt{x\sin(x^3)+\sin(x^3-x)}$

(d) $f(x) = \dfrac{3^x}{3^x+3^{-x}} + 3^2$

(e) $f(x) = (\cos x + x^2)^{5x}$ (use logarithmic differentiation)

[12] 5. Consider the function $y = 3x + x^{-1}$.

(a) Use the **definition of derivative** to find the formula for dy/dx.

(b) Find the linearization $L(x)$ of the function $y(x)$ at $a = 2$.

(c) Find the differential dy and evaluate it for the values $x = 2$ and $dx = 0.1$.

[7] 6. Let $f(x) = \dfrac{x+1}{x+3}$.

(a) Find the slope m of the secant line joining the points $(-1, f(-1))$ and $(2, f(2))$.

(b) Find all points $x = c$ (if any) on the interval [-1,2] such that $f'(c) = m$.

[17] 7. (a) Verify that the point (1,2) belongs to the curve defined by the equation $y^3 - x\,y - 2\sqrt{3+x^2} = 2$, and find an equation of the tangent line to the curve at this point.

(b) At 1 PM, ship A is 5 kilometers strictly to the west of ship B. Ship A is sailing west at speed 20 km/hour and ship B is sailing north at 30 km/hour. How fast (in km/hour) is the distance between ships changing at 3 PM?

(c) Use l'Hôpital's rule to evaluate the $\lim\limits_{x\to 0} \dfrac{\sin(x) - x}{e^{x^3} - 1}$.

[11] 8. (a) Find the point (x_0, y_0) on the curve $y = 2\sqrt{x}$ that is closest to the point (3,0).

(b) A box with a square base is to be constructed with a volume of 50 m^3. The material for the bottom and the sides of the box costs \$2/m^2, and the material for the top costs \$5/m^2. Find the dimensions that minimize the cost of the box.

[14] 9. The function $f(x)$ and its derivatives are given:

$$f(x) = \frac{(1+x)}{x^2}, \qquad f'(x) = \frac{-(x+2)}{x^3}, \qquad f''(x) = \frac{2(x+3)}{x^4}.$$

(a) Find the domain of $f(x)$, check for symmetry, and also find asymptotes (if any).

(b) Find intervals where the function $f(x)$ is increasing, intervals where it is decreasing, and the local maxima and local minima (if any) of $f(x)$.

(c) Determine intervals where the function $f(x)$ is concave upward, intervals where the function is concave downward, and the inflection points of $f(x)$.

(d) Sketch the graph of the function $f(x)$ using the information obtained above.

[5] **Bonus Question.** Let $p(x) = x^4 + a^2x^2 - 2a^2x$, where a is any real number. Prove that the graph $y = p(x)$ has at least one point of local minimum on the interval $(-1, 1)$.