MAT1341 B – TEST 4 – FALL 2016
DGD 1 (FTX–137, 2:30pm)

& DGD 2 (TBT-0019, 2:30 pm)

November 28, 2016.

Instructor: Saeid Molladavoudi.

By entering your name(s) and student number below in your own handwriting, you acknowledge that you have ensured that you are complying with the instruction #2 below.

Family Name:________________

First Name:________________

Student number:________________

Key V1

Your multiple choice answers →

For the marker's use only →

1	C
2	C
subtotal	
3	
4	
5	
[Bonus] 6	
Total	

READ THESE INSTRUCTIONS CAREFULLY.

1. You have 75 minutes to complete this exam.
2. **This is a closed book exam, and no notes of any kind are permitted. The use of calculators, cell phones, or similar devices is not permitted.** All implanted cyber devices not necessary for life-support must be disabled at the beginning of the exam.
3. Read each question carefully, and **answer all questions in the space provided after each question.** For questions 3 to 6, you may use the backs of pages if necessary, but be sure to indicate to the marker that you have done this.
4. Questions 1 and 2 are multiple choice. These questions are worth 1 point each and no part marks will be given. <u>Please record your answers in the space provided above.</u>
5. Questions 3 - 5 are worth 6 points each, and part marks can be earned in each. **The correct answers here require justification written legibly and logically: you must convince the marker that you know why your solution is correct.** Question 6 is a challenging bonus question and is worth 3 points. It is *much* more difficult to obtain marks in the bonus question, so spend your time accordingly. You can earn 100% without attempting Q.6.
6. Where it is possible to check your work, do so.
7. Good luck! Bonne chance!

1. Let A be an 8×6 matrix such that $A\vec{\mathbf{x}} = \vec{0}$ has only the trivial solution. Answer the following questions:

- What is the rank of A?
- Is $A\vec{\mathbf{x}} = \vec{b}$ consistent for all $\vec{b} \in \mathbf{R}^8$?

A. 0, Yes
B. 6, Yes
(C.) 6, No
D. 8, Yes
E. 8, No
F. 2, Yes

$\Leftrightarrow$ Columns of A are L.I.

$\Leftrightarrow$ rank$(A) = n = 6$

$m \times n$

$A\vec{x} = \vec{b}$ is consistent for all $\vec{b} \times \mathbb{R}^m \Leftrightarrow$

Col$(A) = \mathbb{R}^m \Leftrightarrow$ rank$(A) = m$. (Theorem 15.3)

which is not the case here, because

rank$(A) = 6$, but not 8.

2. Let $B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & -1 \\ 1 & 2 & 1 \end{bmatrix}$. The third row of B^{-1} is:

A. $[1 \ \ 2 \ \ 1]$
B. $[-1 \ \ 1 \ \ 1]$
(C.) $[-3 \ \ 1 \ \ 2]$
D. $[0 \ \ 0 \ \ 1]$
E. $[1 \ \ 1 \ \ -1]$
F. B is not invertible.

$$[B \mid I_n] = \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & -1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & -2 & -1 & -1 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|ccc} 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & -2 & -1 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -3 & 1 & 2 \end{array}\right]$$

3. Let

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 2 & -1 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix}.$$

a) Find a basis for the row space of A.

b) Find a basis for the column space of A.

c) Find a basis for $\ker(A) = \{\vec{\mathbf{x}} \in \mathbf{R}^4 \mid A\vec{\mathbf{x}} = \vec{0}\}$.

d) Find the dimension of $\{A\vec{\mathbf{x}} \mid \vec{\mathbf{x}} \in \mathbf{R}^4\}$.

$$A = \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 2 & -1 & 1 \\ 1 & 1 & -1 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & -1 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \tilde{A} \text{ in RREF}$$

a) $\text{Row}(A) = \text{Row}(\tilde{A})$ and the non-zero rows of $\tilde{A}$ would form a basis for $\text{Row}(A)$. So,

$\{(1,1,0,-1), (0,0,1,-3)\}$ is a basis for $\text{Row}(A)$.

b) There are leading ones in the 1st & 3rd columns of $\tilde{A}$. Therefore, $\{(1,2,1), (0,-1,-1)\}$ would form a basis for $\text{Col}(A)$.

c) $[A|\vec{0}] \sim \left[\begin{array}{cccc|c} ① & 1 & 0 & -1 & 0 \\ 0 & 0 & ① & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

$x_2 = s \qquad x_4 = t$

$\rightarrow \left.\begin{array}{l} x_1 = -s+t \\ x_2 = s \\ x_3 = 3t \\ x_4 = t \end{array}\right\} \Rightarrow Ker(A) = \left\{ \begin{pmatrix} -s+t \\ s \\ 3t \\ t \end{pmatrix} \mid s,t \in \mathbb{R} \right\}$

$\rightarrow Ker(A) = \left\{ s\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ 0 \\ 3 \\ 1 \end{pmatrix} \mid s,t \in \mathbb{R} \right\}$

$= span\{(-1,1,0,0),(1,0,3,1)\}$

and since the basic solutions of the kernel would form a basis for Ker(A) (Theorem 15.1 from the text). then $\{(-1,1,0,0),(1,0,3,1)\}$ is a basis for Ker(A).

d)

$\{A\vec{x} \mid \vec{x} \in \mathbb{R}^4\} = Col(A)$

$\Rightarrow dim(Col(A)) = rank(A) = 2.$

4. Let $u_1 = (0, 1, 1, 0)$, $u_2 = (0, 0, 0, 1)$, $u_3 = (0, 1, -1, 0)$, and $U = \text{span}\{u_1, u_2, u_3\}$.

a) Show that $\{u_1, u_2, u_3\}$ is an orthogonal set.

b) Briefly explain why $\{u_1, u_2, u_3\}$ is a basis of U.

c) Find the best approximation to $(1,\ -1,\ 2,\ -1)$ by vectors in U.

d) Extend $\{u_1, u_2, u_3\}$ to a basis of $\mathbf{R}^4$.

a) $\vec{u}_1 \cdot \vec{u}_2 = (0,1,1,0)\cdot(0,0,0,1) = 0 = \vec{u}_2 \cdot \vec{u}_1$

$\vec{u}_1 \cdot \vec{u}_3 = (0,1,1,0)\cdot(0,1,-1,0) = 1-1 = 0 = \vec{u}_3 \cdot \vec{u}_1$

$\vec{u}_2 \cdot \vec{u}_3 = (0,0,0,1)\cdot(0,1,-1,0) = 0 = \vec{u}_3 \cdot \vec{u}_2$

Therefore, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal set.

b) Theorem 19.1 from the textbook:

"vectors in an orthogonal set are L.I." and also $U = \text{span}\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, therefore, $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is a basis for U.

c) $\vec{v} = (1,-1,2,-1)$

$$\text{Proj}_U(\vec{v}) = \frac{\vec{v}\cdot\vec{u}_1}{\|\vec{u}_1\|^2}\vec{u}_1 + \frac{\vec{v}\cdot\vec{u}_2}{\|\vec{u}_2\|^2}\vec{u}_2 + \frac{\vec{v}\cdot\vec{u}_3}{\|\vec{u}_3\|^2}\vec{u}_3$$

$\vec{v}\cdot\vec{u}_1 = (1,-1,2,-1)\cdot(0,1,1,0) = -1+2 = 1$

$\vec{v}\cdot\vec{u}_2 = (1,-1,2,-1)\cdot(0,0,0,1) = -1$

$\vec{v}\cdot\vec{u}_3 = (1,-1,2,-1)\cdot(0,1,-1,0) = -1-2 = -3$

$$\|\vec{u}_1\|^2 = \vec{u}_1 \cdot \vec{u}_1 = (0,1,1,0)\cdot(0,1,1,0) = 1+1 = 2$$

$$\|\vec{u}_2\|^2 = \vec{u}_2 \cdot \vec{u}_2 = (0,0,0,1)\cdot(0,0,0,1) = 1$$

$$\|\vec{u}_3\|^2 = \vec{u}_3 \cdot \vec{u}_3 = (0,1,-1,0)\cdot(0,1,-1,0) = 1+1 = 2$$

$$\Rightarrow \text{Proj}_U(\vec{v}) = \frac{1}{2}\vec{u}_1 - \vec{u}_2 - \frac{3}{2}\vec{u}_3.$$

d)

$$A = \begin{bmatrix} \vec{u}_1 \\ \vec{u}_2 \\ \vec{u}_3 \\ \vec{u}_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ \hline & \vec{u}_4 & & \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ \hline & \vec{u}_4 & & \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ \hline & \vec{u}_4 & & \end{bmatrix}$$

$$\sim \begin{bmatrix} 0 & \textcircled{1} & 1 & 0 \\ 0 & 0 & \textcircled{1} & 0 \\ 0 & 0 & 0 & \textcircled{1} \\ \hline & \vec{u}_4 & & \end{bmatrix} \Rightarrow \text{if } \vec{u}_4 = (1,0,0,0) \in \mathbb{R}^4,$$

then $\{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ would form a basis for $\mathbb{R}^4$.

5(a). State whether the following is true or false, and justify your answer.

- If you say the statement may be false, you must give an explicit example - with numbers, matrices, or functions (as is appropriate), if possible, or an argument using theorems and facts from class.

- If you say the statement is always true, you must give a clear explanation.

(i) Let $\{u, v\}$ be linearly independent in $\mathbf{R}^{2016}$. If A is an invertible 2016×2016 matrix, then $\{Au, Av\}$ is also linearly independent in $\mathbf{R}^{2016}$.

write the dependence relation for these vectors:

$0 = a(A\vec{u}) + b(A\vec{v}) \Rightarrow A(a\vec{u}) + A(b\vec{v}) = 0$

$\Rightarrow$ Since A is invertible: $\underbrace{A^{-1}A}_{I}(a\vec{u}) + \underbrace{A^{-1}A}_{I}(b\vec{v}) = A^{-1}0 = 0$

$\Rightarrow a(I\vec{u}) + b(I\vec{v}) = 0 \Rightarrow a\vec{u} + b\vec{v} = 0$

but $\vec{u}$ & $\vec{v}$ are L.I., so $a = b = 0$

ANSWER: True

(ii) The rows of a 3 by 5 matrix are always linearly dependent.

A counter example:

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix}_{3\times 5}$$

the rows are LI.

ANSWER: False

5(b). Let A be a $n \times n$ matrix with real entries. Give three additional statements **equivalent to**

"There is $\vec{b} \in \mathbf{R}^n$ such that the system $A\vec{\mathbf{x}} = \vec{b}$ is inconsistent,"

one each in terms of:

(i) The general solution of $A\vec{\mathbf{x}} = \vec{0}$:

does not contain the unique solution $\vec{x} = \vec{0}$.

(ii) The column space of A:

$Col(A) \neq \mathbb{R}^n$

(iii) The invertibility (or not) of A:

A is not invertible

6. [Bonus/Challenge] Suppose A is an invertible 15×15 matrix and B is any 15×12 matrix with $\text{rank}(B) = 12$. Prove carefully that $\text{rank}(AB) = 12$.

(You <u>cannot</u> choose A or B: your proof must work for all A and B satisfying the conditions above.)

$A: n \times n,\ n = 15$

$B: n \times p,\ p = 12 \Rightarrow$ in block column format:

$$B = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \cdots & \vec{b}_p \end{bmatrix}_{n \times p}$$

$\text{rank}(B) = p \Leftrightarrow$ Columns of B are L.I,

$AB: n \times p$ and in block column format:

$$AB = \begin{bmatrix} A\vec{b}_1 & A\vec{b}_2 & \cdots & A\vec{b}_p \end{bmatrix}_{n \times p},$$

where $A\vec{b}_1 : n \times 1$, $A\vec{b}_2 : n \times 1, \cdots, A\vec{b}_p : n \times 1$.

Now, let's form the dependence relations for these vectors:

$$a_1(A\vec{b}_1) + a_2(A\vec{b}_2) + \cdots + a_p(A\vec{b}_p) = \vec{0}$$

$$\Rightarrow A(a_1\vec{b}_1) + A(a_2\vec{b}_2) + \cdots + A(a_p\vec{b}_p) = \vec{0}$$

multiply both sides from left by A^{-1} (Since A is invertible)

$$\Rightarrow \underbrace{(A^{-1}A)}_{I_n}(a_1\vec{b}_1) + \underbrace{(A^{-1}A)}_{I_n}(a_2\vec{b}_2) + \cdots + \underbrace{(A^{-1}A)}_{I_n}(a_p\vec{b}_p) = \vec{0}$$

(You may use this page for rough work or solutions that did not fit on previous pages.)

$$\Rightarrow a_1 (\underbrace{I_n \vec{b}_1}_{\vec{b}_1}) + a_2 (\underbrace{I_n \vec{b}_2}_{\vec{b}_2}) + \cdots + a_p (\underbrace{I_n \vec{b}_p}_{\vec{b}_p}) = \vec{0}$$

$$\Rightarrow a_1 \vec{b}_1 + a_2 \vec{b}_2 + \cdots + a_p \vec{b}_p = \vec{0}$$

but we know that the columns of B are LI, i.e. the only solution to the dependence relation is the trivial solution, so,

$$a_1 = a_2 = \cdots = a_p = 0,$$

which is also the only solution to the dependence relation of the columns of AB. Therefor, the columns of AB, i.e. $A\vec{b}_1, \ldots A\vec{b}_p$ are also LI.

Hence,

$$\text{rank}(AB) = \#\text{ columns of } AB = p = 12$$