ENGR 233 MIDTERM OCT. 20, 2016

INSTRUCTIONS: Answer all six questions. The six questions are equally valued. Formula sheet is attached. Legal calculators are permitted.

1. **Consider the 3-dimensional vector field $\bar{F} =< y^2, 2xy + e^{3z}, 3ye^{3z} >$**
   (a) **Show that this field is conservative.**
   (b) **Find a potential function.**
   (c) **Evaluate $\int_C \bar{F} \cdot d\bar{r}$ where C is the space-curve given by**
   $$\bar{r}(t) =< t^2 + t^3, e^{5t}, 7 >, \quad 0 \leq t \leq 3$$
2. **A mountain is described by the equation $z = f(x, y) = 7 - x^2 - 2y^2$ where z represents altitude, and a skier is standing at the point $(1, 1, 4)$.**
   (a) **Find the directional derivative in the unit direction $< \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} >$.**
   (b) **In which unit direction should the skier face if he or she wants to go downhill the fastest? What is the directional derivative in that direction?**
   (c) **Sketch the level curve $f(x, y) = 4$ and draw the gradient of f at the point $(1, 1)$**
3. **A cannon is on the edge of a 100 ft. cliff, and a cannonball is shot at a 45 degree upward angle with a speed of $\sqrt{18}$ ft/sec.**
   (a) **Find the time at which the cannonball hits the ground.**
   (b) **What is the horizontal range of the cannonball?**
4. **Given the space-curve $\bar{r}(t) =< \cos 2t, \sin 2t, 6t >$:**
   (a) **Find the arc-length as t goes from 0 to 1.**
   (b) **Find the curvature at $t = 0$.**
5. **Consider the surface given by $z = 2x^3 - e^{xy}$.**
   (a) **Find the tangent plane at the point $(1, 0, 1)$.**
   (b) **Find the normal line at the point $(1, 0, 1)$.**
6. (a) **Find the area of the triangle with vertices $(1, 1, 1)$, $(3, 4, 5)$ and $(8, 1, 4)$.**
   (b) **Show that the following three vectors are coplanar:**
   $\bar{a} =< 1, 4, -7 >$, $\bar{b} =< 2, -1, 4 >$, $\bar{c} =< 0, -9, 18 >$.

1 a) $\operatorname{curl} \vec{F} = \begin{pmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ y^2 & 2xy + e^{3z} & 3y e^{3z} \end{pmatrix}$

$= \left\langle 3e^{3z} - 3e^{3z},\ 0-0,\ 2y - 2y \right\rangle = \langle 0, 0, 0 \rangle$

b) $\varphi_x = y^2 \Rightarrow \varphi = y^2 x + g(y,z)$

$\Rightarrow \varphi_y = 2xy + g_y(y,z) = 2xy + e^{3z} \Rightarrow g_y(y,z) = e^{3}$

$\Rightarrow g(y,z) = ye^{3z} + h(z)$

$\Rightarrow \varphi = y^2 x + ye^{3z} + h(z)$

$\Rightarrow \varphi_z = 3ye^{3z} + h'(z) = 3ye^{3z}$

$\Rightarrow h'(z) = 0 \Rightarrow h(z)$ constant (say 0)

$\Rightarrow \boxed{\varphi = y^2 x + ye^{3z}}$

c) $\varphi(36, e^{15}, 7) - \varphi(0, 1, 7)$

$= (e^{15})^2 \cdot 36 + e^{15} e^{21} - 0 - 1 \cdot e^{21}$

(2) $z = f(x,y) = 7 - x^2 - 2y^2$

$\nabla f = \langle -2x, -4y \rangle = \langle -2, -4 \rangle$ at $(1,1)$

a) DD in direction $\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle$

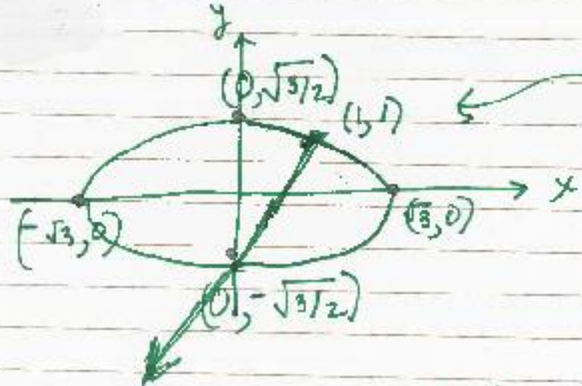
is $\langle -2, -4 \rangle \cdot \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \rangle = -\frac{6}{\sqrt{2}} = -3\sqrt{2}$

b) $\frac{-\nabla f}{\|\nabla f\|} = \frac{\langle 2, 4 \rangle}{\sqrt{20}}$ is direction

DD is

$\nabla f \cdot \frac{\langle 2,4 \rangle}{\sqrt{20}} = \langle -2, -4 \rangle \cdot \frac{\langle 2, 4 \rangle}{\sqrt{20}} = -\sqrt{20}$

c) $f(x,y) = 4$ becomes $x^2 + 2y^2 = 3$



grad. is $\perp$ to curve

not to scale

(3) $\theta = 45^\circ \therefore \cos\theta = \sin\theta = \sqrt{2}/2$

$$x(t) = \left(\sqrt{18}\cos\theta\right)t = 3t$$

$$y(t) = -\tfrac{1}{2}\cdot 32t^2 + \left(\sqrt{18}\sin\theta\right)t + 100$$

$$= -16t^2 + 3t + 100$$

a) $y(t) = 0 \Longrightarrow t \approx 2.6$ sec (by quadratic formula)

b) $x(2.6) \cong 7.8$ ft

④ $\vec{r} = \langle \cos 2t, \sin 2t, 6t \rangle$

$\vec{r}\,' = \langle -2\sin 2t,\ 2\cos 2t,\ 6 \rangle$

$\vec{r}\,'' = \langle -4\cos 2t,\ -4\sin 2t,\ 0 \rangle$

$\|\vec{r}\,'\| = \sqrt{40}$

a) $L = \int_0^1 \sqrt{40}\, dt = \sqrt{40}$

b) $\kappa = \dfrac{\|\vec{r}\,'(0) \times r''(0)\|}{\|r'(0)\|^3}$ $\begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 0 & 2 & 6 \\ -4 & 0 & 0 \end{vmatrix}$

at $t=0$

$= \dfrac{\|\langle 0, -24, 8 \rangle\|}{40^{3/2}}$

$= \dfrac{\sqrt{640}}{40^{3/2}} = .1$

(5) $f(x,y,z) = 2x^3 - e^{xy} - z = 0$

$$\nabla f = \langle 6x^2, -xe^{xy}, -1 \rangle$$

$$= \langle 6, -1, -1 \rangle \quad \text{at} \quad (1,0,1)$$

<u>TP</u> :
$$\begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix} \cdot \begin{pmatrix} x-1 \\ y-0 \\ z-1 \end{pmatrix} = 0$$

$$6x - y - z = 5$$

<u>NL</u>
$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + t\begin{pmatrix} 6 \\ -1 \\ -1 \end{pmatrix}$$

$$x = 1 + 6t$$
$$y = -t$$
$$z = 1 - t$$

⑥

a) $\text{Area} = \frac{1}{2} \left\| \langle 2,3,4 \rangle \times \langle 7,0,3 \rangle \right\|$

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 4 \\ 7 & 0 & 3 \end{vmatrix}$$

$\langle 9, 22, -21 \rangle$

$\therefore \quad \text{Area} = \frac{1}{2}\sqrt{81 + 22^2 + 21^2}$

$= \frac{1}{2}\sqrt{1006} \approx 15.86$

b) Think of volume of parallelopiped:

$|\vec{a} \cdot (\vec{b} \times \vec{c})|$

$$\left| \langle 1, 4, 7 \rangle \cdot \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 2 & -1 & 4 \\ 0 & -9 & 18 \end{vmatrix} \right|$$

$= |\langle 1, 4, 7 \rangle \cdot \langle 18, -36, -18 \rangle|$

$= 0 \quad \therefore$ 3 vectors coplanar (0-volume)