Solution to Assignment 4

MAT1322D, Fall 2016

1. (3 marks) Consider series $\sum_{n=1}^{\infty} \frac{1}{2^n \sqrt{n^2+1}} (x+1)^n$. For which values of *x* is this series absolutely convergent, conditionally convergent, or divergent?

Solution. The center of the series is $x = -1$. The radius of convergence is

$$\lim_{n\to\infty} \left| \frac{c_n}{c_{n+1}} \right| = \lim_{n\to\infty} \left| \left(\frac{1}{2^n \sqrt{n^2+1}} \right) \left(2^{n+1} \sqrt{(n+1)^2+1} \right) \right| = 2 \lim_{n\to\infty} \sqrt{\frac{n^2+2n+2}{n^2+1}} = 2.$$

Hence, this series is absolutely convergent in interval (–1 – 2, –1 + 2) = (–3, 1), and it is divergent in (–∞, –3) and (1, ∞).

When $x = 1$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{2^n \sqrt{n^2+1}} 2^n = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$. Since $\frac{1}{\sqrt{n^2+1}} \geq \frac{1}{\sqrt{n^2+3n^2}} = \frac{1}{2n}$, and the series $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, this series diverges. (You can also use the limit comparison test to show that this series is divergent).

When $x = -3$, the series becomes $\sum_{n=1}^{\infty} \frac{1}{2^n \sqrt{n^2+1}} (-2)^n = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n^2+1}}$. By alternating series test, this series converges. Since series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$ diverges, this series is convergent but not absolutely convergent at $x = -3$.

Summarizing:

This series is absolutely convergent in (–3, 1); it is convergent but not absolutely convergent at $x = -3$; it is divergent in (–∞, –3) and [1, ∞).

2. (not marked) Consider the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{7^n n} (2x+3)^n$. For which values of *x* is this series absolutely convergent, conditionally convergent, or divergent?

Solution. Center is $x = -\frac{3}{2}$. This series is absolutely convergent when

$\lim_{n\to\infty} \left| \frac{(2x+3)^{n+1}}{7^{n+1}(n+1)} \frac{7^n n}{(2x+3)^n} \right| = \frac{1}{7} \lim_{n\to\infty} \left| \frac{n}{n+1} \right| |2x+3| = \frac{1}{7} |2x+3| < 1$. When $| 2x + 1 | < 7$, i.e.,

$-7 < 2x + 3 < 7$, or $-10 < 2x < 4$, $-5 < x < 2$, this series is absolutely convergent. Radius of convergence $R = \frac{1}{2}(2 + 5) = \frac{7}{2}$. When $x < -5$ or $x > 2$, this series is divergent.

When $x = -5$, the series is $\sum_{n=1}^{\infty} \frac{1}{n}$. It is the harmonic series, which diverges.

When $x = 2$, the series is $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. It converges by the alternating series test.

This power series is absolutely convergent when $-5 < x < 2$, it is conditionally convergent when $x = 2$, and it is divergent when $x \le -5$ or $x > 2$.

Interval of convergence is $(-5, 2]$.

3. (not marked) Find the Maclaurin series of $y = \frac{x}{(1+x^2)^2}$.

Solution. Take the derivative on both sides of $\frac{1}{1-t} = 1 + t + t^2 + ... = \sum_{n=0}^{\infty} t^n$, we have

$$\frac{1}{(1-t)^2} = 1 + 2t + 3t^2 + ... = \sum_{n=0}^{\infty} (n+1)t^n .$$

Let $t = -x^2$. $\frac{1}{(1+x^2)^2} = 1 - 2x^2 + 3x^4 - ... = \sum_{n=0}^{\infty} (-1)^n (n+1) x^{2n}$.

Finally, $\frac{x}{(1+x^2)^2} = x - 2x^3 + 3x^5 - ... = \sum_{n=0}^{\infty} (-1)^n (n+1) x^{2n+1}$.

4. (3 marks) The Maclaurin series of he function $y = \tan x$ is

$$\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \frac{17}{315}x^7 + ... \ .$$

Recall that $\frac{d}{dx}\tan x = \frac{1}{\cos^2 x}$. Use this result to find the first four non-zero terms of the Maclaurin series of the function $y(x) = \int_0^x \frac{1}{\cos^2(t^2)} dx$.

Solution. Differentiate the Maclaurin series of the function $\tan x$.

$$\frac{1}{\cos^2 x} = 1 + x^2 + \frac{2}{3}x^4 + \frac{17}{45}x^6 + \ldots \ .$$

Substitute x^2 for x.

$$\frac{1}{\cos^2(x^2)} = 1 + (x^2)^2 + \frac{2}{3}(x^2)^4 + \frac{17}{45}(x^2)^6 + \ldots = 1 + x^4 + \frac{2}{3}x^8 + \frac{17}{45}x^{12} + \ldots \ .$$

$$y(x) = \int_0^x \frac{1}{\cos^2(t^2)}\,dt = \left[t + \frac{1}{5}t^5 + \frac{2}{27}t^9 + \frac{17}{13\times 45}t^{13} + \ldots \right]_{t=0}^{x} = x + \frac{1}{5}x^5 + \frac{2}{27}x^9 + \frac{17}{13\times 45}x^{13} + \ldots .$$

5. (a) (2 marks) Use the binomial series to find the first three non-zero terms of the Maclaurin series of the function $y = \dfrac{1}{\sqrt{4+t^2}}$.

(b) (not marked) Recall that $\dfrac{d}{dx}\ln(x + \sqrt{4+x^2}) = \dfrac{1}{\sqrt{4+x^2}}$. Therefore,

$$\int_0^x \frac{1}{\sqrt{4+t^2}}\,dt = \left[\ln\left(t + \sqrt{4+t^2}\right) \right]_{t=0}^{x} = \ln\left(x + \sqrt{4+x^2}\right) - \ln 2 .$$

Find the first four non-zero terms of the Maclaurin series of the function $y = \ln(x + \sqrt{4+x^2})$.

Solution. (a) By the binomial series with $k = -\dfrac{1}{2}$,

$$\frac{1}{\sqrt{1+t}} = 1 - \frac{1}{2}t + \frac{3}{8}t^2 - ..$$

Since $\dfrac{1}{\sqrt{4+x^2}} = \dfrac{1}{2\sqrt{1+\left(\frac{x}{2}\right)^2}}$, let $t = \left(\dfrac{x}{2}\right)^2$.

$$\frac{1}{\sqrt{4+x^2}} = \frac{1}{2\sqrt{1+(x/2)^2}} = \frac{1}{2}\left(1 - \frac{1}{2}\left(\frac{x}{2}\right)^2 + \frac{3}{8}\left(\frac{x}{2}\right)^4 - \ldots\right) = \frac{1}{2} - \frac{1}{16}x^2 + \frac{3}{256}x^4 - \ldots .$$

(b) $\ln(x + \sqrt{4+x^2}) = \ln 2 + \displaystyle\int_0^x \left(\frac{1}{2} - \frac{1}{16}t^2 + \frac{3}{256}t^4 + \ldots\right)dx = \ln 2 + \frac{1}{2}x - \frac{1}{48}x^3 + \frac{3}{1280}x^5 - \ldots .$

6. (2 marks) If the Taylor series of a function at the center $x = -1$ is

$$f(x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{2^n n^2} (x+1)^{2n-1} = \frac{1}{2}(x+1) - \frac{1}{2^2 2^2}(x+1)^3 + \frac{1}{2^3 3^2}(x+1)^5 - \frac{1}{2^4 4^2}(x+1)^7 + \dots,$$

find the fourth and the seventh derivative of this function at $x = -1$.

Solution. In Taylor series, the coefficient of x^k is $c_k = \dfrac{f^{(k)}(-1)}{k!}$. When $n = 4$, since $c_4 = 0$, $f^{(4)}(-1) = 0$. When $k = 7$, $\dfrac{f^{(4)}(-1)}{4!} = -\dfrac{1}{2^4 4^2}$. Hence, $f^{(4)}(-1) = -\dfrac{4!}{2^4 4^2} = \dfrac{3}{32}$.