Sample Final test

1. (11)

   (a) Solve: $2^{2x+2} = 3^{x-7}$

   (b) Let $f(x) = \ln\left(x^2 - 1\right)$ and $g(x) = 1 - x$, calculate $(f \circ g)(x)$ and determine its domain.

   (c) Calculate the inverse function $f^{-1}(x)$ for $f(x) = \ln(1 - 2x)$, and determine the range of both f and f^{-1}.

2. (7) Calculate the limit if it exists:

   (a) $\lim\limits_{x \to 2} \dfrac{|x-2|(x+3)}{x^2 + x - 6}$

   (b) $\lim\limits_{x \to 1} \dfrac{x-1}{3 - \sqrt{x^2+8}}$

3. (6) Find all asymptotes for $y = \dfrac{\sqrt{9x^2+1}}{x^2 - 25} \dfrac{x^2+1}{x+5}$

4. (12) Calculate the derivatives of:

   (a) $f(x) = x^e e^x + e^2$

   (b) $f(x) = \dfrac{1 + \ln x^2}{1 + x^2}$

   (c) $f(x) = \arctan\left(\sin\left(e^{x^2 \cos x}\right)\right)$

   (d) $f(x) = \sqrt{x}\left(x^{3/2} - x^{-1/2}\right)(x+1)$

   (e) $f(x) = \left(1 + x^2\right)^{\tan x}$.

4. (12) Consider $y = \sqrt{25 + x}$

   (a) Use the definition of derivative to determine $\dfrac{dy}{dx}$

   (b) Calculate the linearization $L(x)$ of $\sqrt{25+x}$ at $a = 0$

   (c) Use $L(x)$ to approximate $\sqrt{30}$.

5. (7) Let $f(x) = x^3 - 2x + 3$.

   (a) Calculate the slope m of the secant line joining the points $\mathbf{A}(-2, f(-2))$ and $\mathbf{B}(0, f(0))$.

   (b) Locate the value $x = c$ (if any) on the interval $(-2, 0)$ such that $f'(c) = m$.

6. (17)

(a) Verify that the point $\mathbf{A}(2,1)$ lies on the curve $C : x^2+2y^2+2 = x^3y^3$ and write an equation of the tangent line to C at $\mathbf{A}$.

(b) A spherical snowball is melting in such way that its diameter D is decreasing at the rate $\frac{dD}{dt} = -.01cm/\min$. At what rate is the volume $V = \frac{4\pi r^3}{3}$ (r is the radius of the snowball) of the snowball decreasing when $D = 9cm$?

(c) Use the l'Hospital's rule to calculate $\lim_{x\to 0} \frac{\sin^2(3x)}{1-\cos(2x)}$.

7. (11)

   (a) Locate the point $\mathbf{A}(x_0, y_0)$ on the straight line $y = x+6$ that is closest to the origin $\mathbf{O}(0,0)$.

   (b) A rectangle is has its base on the $x-axis$ and its upper corners lie on the parabola $y = 12-x^2$. What is the largest area of such rectangle.

8. (14) Let $f(x) = 2x^3 - 21x^2 + 36x - 9$.

   (a) Evaluate $f'(x)$ to determine intervals where $f(x)$ is increasing, intervals where it is decreasing, and all critical $x-$values to identify the local extrema.

   (b) Evaluate $f''(x)$ to determine intervals where $f(x)$ is concave upwards, intervals where it is is concave downwards, and all critical $x-$values to identify the points of inflection.

   (c) Use all the above information to sketch the graph $y = f(x)$.

Bonus (5) Is it possible to have $f(x)$ such that $f(0) = 0, f(2) = 4$ and $f'(x) < 2$ for all $x \in [0,2]$. Give an example of such a function or prove that it is impossible.

Solutions.

1. Then:

   (a) $2^{2x+2} = 3^{x-7} \to (2x+2)\ln 2 = (x-7)\ln 3 \to x = \dfrac{2\ln 2 + 7\ln 3}{\ln 3 - 2\ln 2}$

   (b) If $f(x) = \ln\left(x^2-1\right)$ and $g(x) = 1-x$, then $(f\circ g)(x) = \ln\left((1-x)^2 - 1\right)$ and its domain is $(1-x)^2 - 1 = x^2 - 2x = x(x-2) > 0 \to (-\infty, 0)\cup(2,\infty)$.

   (c) The inverse function $f^{-1}(x)$ for $f(x) = \ln(1-2x)$, and determine the range of both f and f^{-1}.

2. Calculate the limits if it exists:

   (a) $\lim\limits_{x\to 2}\dfrac{|x-2|(x+3)}{x^2+x-6} = \lim\limits_{x\to 2}\dfrac{|x-2|(x+3)}{(x+3)(x-2)} = \lim\limits_{x\to 2}\dfrac{|x-2|}{x-2}$ does not exist as $\lim\limits_{x\to 2^-}\dfrac{|x-2|}{x-2} = -1 \neq 1 = \lim\limits_{x\to 2^+}\dfrac{|x-2|}{x-2}$

   (b) $\lim\limits_{x\to 1}\dfrac{x-1}{3-\sqrt{x^2+8}} = \lim\limits_{x\to 1}\dfrac{(x-1)\left(3+\sqrt{x^2+8}\right)}{9-(x^2+8)} = \lim\limits_{x\to 1}\dfrac{(x-1)\left(3+\sqrt{x^2+8}\right)}{(1-x)(1+x)} = -\lim\limits_{x\to 1}\dfrac{3+\sqrt{x^2+8}}{1+x} = -3$

3. Find all asymptotes for $y = \dfrac{\sqrt{9x^2+1}}{x^2-25}\dfrac{x^2+1}{x+5}$

   a) $x = \pm 5$ are **vertical** asymptotes as when $x \to \pm 5$ then $|y| \to \infty$

   b) $y = \dfrac{\pm x\sqrt{9+\dfrac{1}{x^2}}}{x^2\left(1-\dfrac{25}{x^2}\right)}\dfrac{x^2\left(1+\dfrac{1}{x^2}\right)}{x\left(1+\dfrac{5}{x}\right)} = \dfrac{\pm\left(1+\dfrac{1}{x^2}\right)\sqrt{9+\dfrac{1}{x^2}}}{\left(1-\dfrac{25}{x^2}\right)\left(1+\dfrac{5}{x}\right)} \to y = \pm 3$ are **horizontal** asymptotes, as when $x \to \pm\infty$ then $y \to \pm 3$

4. Calculate the derivatives:

   (a) $\dfrac{d}{dx}\left(x^e e^x + e^2\right) = x^{e-1}e^x(x+e) = x^e e^x + ex^{e-1}e^x$

   (b) $\dfrac{d}{dx}\left(\dfrac{1+\ln x^2}{1+x^2}\right) = \dfrac{(1+x^2)\dfrac{2}{x} - 2x\left(1+\ln x^2\right)}{\left(x^2+1\right)^2} = \dfrac{2-2x^2\ln x^2}{x^5+2x^3+x}$

   (c) $\dfrac{d}{dx}\left(\arctan\left(\sin\left(e^{x^2\cos x}\right)\right)\right) = \dfrac{\cos\left(e^{x^2\cos x}\right)e^{x^2\cos x}\left(2x\cos x - x^2\sin x\right)}{1+\sin^2\left(e^{x^2\cos x}\right)}$

   (d) $\dfrac{d}{dx}\left(\sqrt{x}\left(x^{3/2}-x^{-1/2}\right)(x+1)\right) = \dfrac{d}{dx}\left(x^3+x^2-x-1\right) = 3x^2+2x-1$

(e) $\frac{d}{dx}\left(1+x^2\right)^{\tan x}$: put $y=\left(1+x^2\right)^{\tan x}\to \ln y=\tan x\ln\left(1+x^2\right)\to$
$\frac{y'}{y}=\sec^2 x\ln\left(1+x^2\right)+\frac{2x\tan x}{1+x^2}\to y'=\frac{d}{dx}\left(1+x^2\right)^{\tan x}$
$=\left(1+x^2\right)^{\tan x}\left(\sec^2 x\ln\left(1+x^2\right)+\frac{2x\tan x}{1+x^2}\right)$

5. For $y=\sqrt{25+x}$

(a) Use the definition of derivative to determine
$\frac{dy}{dx}=\lim_{h\to 0}\frac{\sqrt{25+x+h}-\sqrt{25+x}}{h}=$
$\lim_{h\to 0}\frac{\sqrt{25+x+h}-\sqrt{25+x}}{h}\frac{\sqrt{25+x+h}+\sqrt{25+x}}{\sqrt{25+x+h}+\sqrt{25+x}}=$
$\lim_{h\to 0}\frac{(25+x)+h-(25+x)}{h\left(\sqrt{25+x+h}+\sqrt{25+x}\right)}=$
$\lim_{h\to 0}\frac{1}{\sqrt{25+x+h}+\sqrt{25+x}}=\frac{1}{2\sqrt{x+25}}$

(b) The linearization of $\sqrt{25+x}$ at $a=0$ is $L(x)=f(0)+xf'(0)=5+\frac{x}{2\sqrt{25}}=\frac{x}{10}+5$

(c) Use $L(x)$ to approximate $\sqrt{30}$: as $f(x)=\sqrt{25+x}=\sqrt{30}$ when $x=5\to L(5)=5+\frac{5}{2\sqrt{25}}=\frac{11}{2}=5.5\left(\approx 5.4772...=\sqrt{30}\right)$

6. For $f(x)=x^3-2x+3$

(a) the slope of the secant line joining the points $\mathbf{A}(-2,f(-2))=\mathbf{A}(-2,-1)$ and $\mathbf{B}(0,f(0))=\mathbf{B}(0,3)$ is $m=\frac{f(0)-f(-2)}{0-(-2)}=\frac{3-(-1)}{2}=2$

(b) the value $x=c$ on the interval $(-2,0)$ such that $f'(c)=3c^2-2=2\to\left(\frac{2}{3}\sqrt{3}\notin(-2,0)\right)$ $c=-\frac{2}{3}\sqrt{3}$.

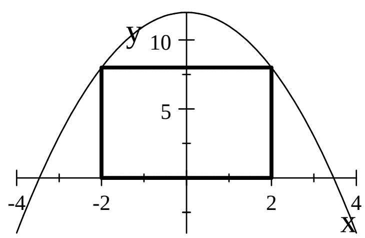
7. (a) The point $\mathbf{A}(2,1)$ lies on the curve $C: x^2+2y^2+2=x^3y^3$ since $2^2+2+2=8=2^3$. Since the differentiation gives: $2x+4yy'=3x^2y^3+3x^3y^2y'\overset{At\ \mathbf{A}}{\to}4+4y'=12+24y'\to y'=-\frac{2}{5}$. The tangent line to C at $\mathbf{A}$ has equation: $y=y'(\mathbf{A})(x-2)+1=-\frac{2}{5}(x-2)+1\to$ $y=\frac{9-2x}{5}$

(b) If a spherical snowball is melting in such way that its diameter $D=2r$ is decreasing at the rate $\frac{dD}{dt}\to\frac{dr}{dt}=\frac{-.01}{2}cm/\min$. Therefore, the rate $V' = 4\pi r^2 r'$ of the volume of the snowball when $D = 9cm \to r = 4.5cm$ is then $V' = 4\pi\left(\frac{81}{4}\right)\frac{-.01}{2}cm/\min = -0.405\,\pi cm/\min$.

(c) Use the l'Hospital's rule to calculate $\lim\limits_{x\to 0}\frac{\sin^2(3x)}{1-\cos(2x)} \overset{"\frac{0}{0}"}{=} \lim\limits_{x\to 0}\frac{6\sin(3x)\cos(3x)}{2\sin(2x)} \overset{"\frac{0}{0}"}{=} \lim\limits_{x\to 0}\frac{18\left(\cos^2(3x)-\sin^2(3x)\right)}{4\cos(2x)} = \frac{9}{2}$.

8. (a) The distance of a point $\mathbf{A}(x,y)$ from the origin is $\sqrt{x^2+y^2} = \sqrt{x^2+(x+6)^2} = \sqrt{2x^2+12x+36} = d(x)$, as $\mathbf{A}$ is on the line $y = x+6$. Therefore, $d'(x) = \frac{2x+6}{\sqrt{2x^2+12x+36}} = 0$ when $x_0 = -3 \to y_0 = 3 \to \mathbf{A}(-3,3)$

(b) A rectangle is has its base $\overline{\mathbf{A}(-x,0)\,\mathbf{B}(x,0)}$ on the $x-axis$ and its upper corners lie on the parabola $y = 12-x^2$. Then its area $A(x)$ is: $A(x) = 2xy = 2x\left(12-x^2\right) = 24x - 2x^3 \to A'(x) = 24 - 6x^2 = 0 \to x = 2$ giving the largest area $A_{\max} = 16$

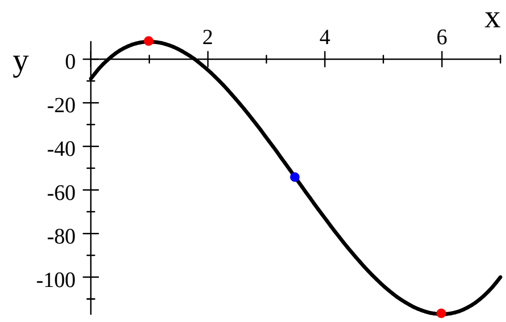


9. For $f(x) = 2x^3 - 21x^2 + 36x - 9$.

(a) Gives $f'(x) = 6x^2 - 42x + 36 = 6(x-1)(x-6) \to f(x)$ is increasing in $(-\infty,1)\cup(6,\infty)$, decreasing in $(1,6)$, and all critical $x-$values are $x_{\max} = 1 \to (1,8)$, $x_{\min} = 6 \to (6,-117)$.

(b) Evaluate $f''(x) = 12x - 42 = 6(2x-7) \to f(x)$ is concave upwards in $(3.5,\infty)$, it is concave downwards in $(-\infty,3.5)$, and the critical $x-$value is $x = \frac{7}{2} \to$ point of inflection is $\left(\frac{7}{2},-\frac{109}{2}\right)$.

(c) The graph $y = f(x)$ is therefore:



Bonus (5) If $f(x)$ is such that $f(0) = 0, f(2) = 4$ and $f'(x) < 2$ for all $x \in [0, 2]$ $\rightarrow$ it satisfies the MVT $\rightarrow$ there must exist $x \in (0, 2) : f'(x) = \dfrac{f(2) - f(0)}{2} = 2 \geq 2 \rightarrow$ therefore such a function does not exist.