Course	Number	Section(s)
Mathematics	203	All
Examination	**Date**	**Pages**
Final	December 2008	3
Instructors		**Course Examiner**
A. Boyarsky, J. Brody, H. Greenspan, H. Proppe, N. Rossokhata		H. Proppe

Special Instructions

▷ **Only approved calculators are allowed.**

Solutions

1. (9 Marks)

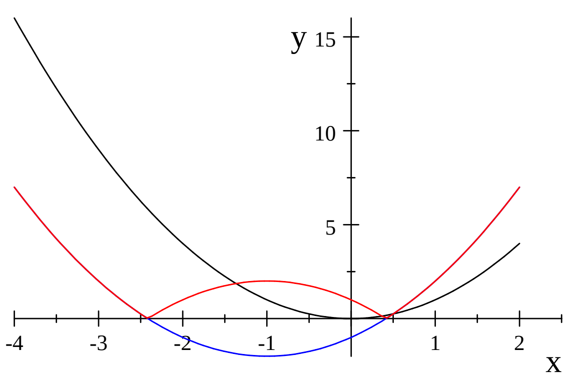
(a) Sketch the graph of the function $f(x) = |(x+1)^2 - 2|$ starting from the graph of the standard parabola $y = x^2$ and using appropriate transformations.

(b) Suppose $f(x) = x^2 - 2x + 1; x \geq 1$. Find $f^{-1}(x)$ and its domain and range.

(c) Solve for x:

$$5^{\log_5(x^2)} = 2 \cdot 4^{\log_4(x)} + e^{\ln 3}$$

Solutions (a)



(b) $f(x) = y = x^2 - 2x + 1 = (x-1)^2$ and since $x \geq 1$ we have $x = 1 + \sqrt{y}$ so the inverse function is $y = f^{-1}(x) = 1 + \sqrt{x}$. The domain is, of course, $[0, \infty)$ and the range $[1, \infty)$. We note, incidentally, that $f \circ f^{-1}(x) = (1 + \sqrt{x})^2 - 2(1 + \sqrt{x}) + 1 = x = f^{-1} \circ f(x)$.

(c)

$$\begin{aligned} 5^{\log_5(x^2)} &= 2 \cdot 4^{\log_4(x)} + e^{\ln 3} \Leftrightarrow \\ x^2 &= 2x + 3 \Leftrightarrow \\ x^2 - 2x - 3 &= (x-3)(x+1) = 0 \end{aligned}$$

by using the fact that the logarithm and exponential functions are inverse to each other. Now x can't be negative, otherwise $\log_4(x)$ in $4^{\log_4(x)}$ is not defined, so the solution must be $x = 3$.

2. (12 Marks) Evaluate the limits:

(a) $\lim_{t \to 1} \frac{t^2 + t - 2}{t^2 + 2t - 3}$ (b) $\lim_{x \to 4} \frac{x^2 - 4x}{2 - \sqrt{x}}$ (c) $\lim_{x \to -\infty} \frac{2x^3 - 7x + 4\sin x}{\sqrt{x^6 + 5x^2 + 1000}}$

Do not use l'Hopital's rule.

Solution (a)

$$\begin{aligned} \lim_{t \to 1} \frac{t^2 + t - 2}{t^2 + 2t - 3} &= \lim_{t \to 1} \frac{(t-1)(t+2)}{(t-1)(t+3)} \\ &= \lim_{t \to 1} \frac{(t+2)}{(t+3)} = \frac{3}{4} \end{aligned}$$

(b)

$$\begin{aligned} \lim_{x \to 4} \frac{x^2 - 4x}{2 - \sqrt{x}} &= \lim_{x \to 4} \frac{x(x-4)}{2 - \sqrt{x}} \cdot \frac{2 + \sqrt{x}}{2 + \sqrt{x}} \\ &= \lim_{x \to 4} \frac{x(x-4)}{4 - x} \cdot \left(2 + \sqrt{x}\right) \\ &= -\lim_{x \to 4} x \cdot \left(2 + \sqrt{x}\right) = -16 \end{aligned}$$

(c)

$$\lim_{x\to-\infty}\frac{2x^3-7x+4\sin x}{\sqrt{x^6+5x^2+1000}} = \lim_{x\to-\infty}\frac{x^3\left(2-7/x^2+4\sin x/x^3\right)}{|x|^3\sqrt{1+5/x+1000/x^3}}$$
$$= \lim_{x\to-\infty}\frac{2x^3}{|x|^3} = -2$$

The reason for the $-$ sign is that when the x^6 comes out from under the square root sign it must have a $+$ sign, even though $x\to-\infty$ because x^6 was a positive number. So we have to write $|x|^3$ (or $-x^3$), not x^3.

3. (10 Marks)

(a) Consider the function $f(x)=\dfrac{x^2+2x-8}{|x-2|}$. Calculate both one-sided limits at the point where the function is undefined.

(b) Find the numbers a and b that make the function

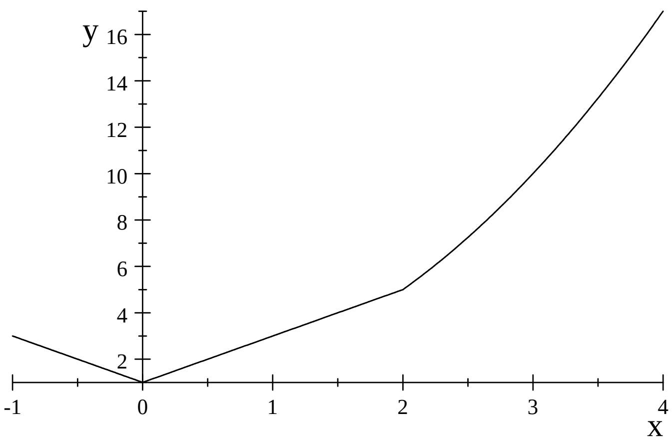
$$f(x)=\begin{cases} 1-2x & \text{if} \quad x\le 0 \\ ax+b & \text{if} \quad 0<x\le 2 \\ x^2+1 & \text{if} \quad x>2 \end{cases}$$

continuous at every point. Sketch the graph of this function.

Solution (a) $f(x)=\dfrac{x^2+2x-8}{|x-2|}=\dfrac{(x-2)(x+4)}{|x-2|}$ and so the point $x=2$ is where f is undefined.

$$\lim_{x\to2^-}\frac{(x-2)(x+4)}{|x-2|} = \lim_{x\to2^-}\frac{(x-2)}{|x-2|}\lim_{x\to2^-}(x+4)$$
$$= (-1)\cdot 6=-6$$
$$\lim_{x\to2^+}\frac{(x-2)(x+4)}{|x-2|} = \lim_{x\to2^+}\frac{(x-2)}{|x-2|}\lim_{x\to2^+}(x+4)$$
$$= (+1)\cdot 6=6$$

(b) Since $f(0)=1$ and $\lim_{x\to0^+}f(x)=b$ the two have to be equal so $b=1$. Also, $f(2)=2a+b=2a+1$ and $\lim_{x\to2^+}f(x)=5$ so $2a+1=5\Longrightarrow a=2$. Here is the graph



4. (15 Marks) Find derivatives of the functions (do not simplify the answer):

(a) $f(x) = (x + x^2)^{-1} \sin 3x$

Solution $f'(x) = -(\sin 3x)\left(x^2 + x\right)^{-2} \cdot (1 + 2x) + 3\cos 3x(x + x^2)^{-1}$

(b) $f(x) = \dfrac{\arctan(x^2)}{\sqrt{x^2 + 1}}$

Solution $f'(x) = \dfrac{\sqrt{x^2+1}\left(\frac{2x}{(1+x^4)}\right) - \arctan x^2 \left(\frac{x}{\sqrt{x^2+1}}\right)}{x^2 + 1}$

(c) $f(x) = (x^3)^2 + 2^x$

Solution $f'(x) = 6x^5 + 2^x \ln 2$

(d) $f(x) = 2x\sqrt{x^2 + \sqrt{1 + x^2}}$

Solution $f'(x) = 2\sqrt{\sqrt{x^2+1} + x^2} + x\dfrac{2x + \frac{x}{\sqrt{x^2+1}}}{\sqrt{\sqrt{x^2+1} + x^2}}$

(e) $f(x) = (\ln x)^{\arctan(x)}$ (use logarithmic differentiation).

Solution $\ln f(x) = \arctan x \cdot \ln(\ln x)$ so $\dfrac{f'(x)}{f(x)} = \dfrac{1}{1+x^2}\ln(\ln x) + \arctan x \dfrac{1}{\ln x}\dfrac{1}{x}$ and

$$\begin{aligned} f'(x) &= f(x)\left(\frac{1}{1+x^2}\ln(\ln x) + \arctan x\frac{1}{\ln x}\frac{1}{x}\right) \\ &= (\ln x)^{\arctan(x)}\left(\frac{1}{1+x^2}\ln(\ln x) + \arctan x\frac{1}{\ln x}\frac{1}{x}\right) \end{aligned}$$

5. (12 Marks)

(a) If $f(x) = \sqrt[3]{x}$, find the linearization $L(x)$ of $f(x)$ at $a = 8$ and use $L(x)$ to estimate $\sqrt[3]{8.5}$ (**Do not** use a calculator to answer this question).

(b) Answer part (a) using differentials, that is, identify dx and calculate df . (**Do not** use a calculator to answer this question).

(c) If $g(x) = x + x^{-1}$, use the definition of the derivative to find $g'(1)$.

(d) Use the appropriate differentiation rule(s) to verify your answer to part (c).

Solutions (a) $L(x) = f(a) + f'(a)(x - a)$ so substituting we have, since $f'(x) = 1/3x^{-2/3}$ $L(x) = f(8) + f'(8)(x - 8) = 2 + \frac{1}{12}(x - 8)$. We have $L(8.5) = 2 + \frac{1}{12}(8.5 - 8) = 2 + \frac{1}{24} = 2.0417$ (note the exact value is $\sqrt[3]{8.5} = 2.0408$)

(b) Using differentials we write

$$\begin{aligned} df &= f'(8)\,dx \\ &= \frac{1}{12}dx \\ &= \frac{1}{12} \cdot \frac{1}{2} = \frac{1}{24} \end{aligned}$$

where $dx = 8.5 - 8 = .5$ and $f'(8) = \frac{1}{12}$. So f changes by $\frac{1}{24}$ from its value at 8 (which is 2) to $2 + \frac{1}{24}$ which is 2.0417 as before.

(c)

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} = \lim_{h\to 0} \frac{\left(x + h + \frac{1}{x+h}\right) - \left(x + \frac{1}{x}\right)}{h} \\ &= \lim_{h\to 0} \frac{h + \frac{1}{x+h} - \frac{1}{x}}{h} \\ &= \lim_{h\to 0} \frac{h}{h} + \lim_{h\to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h} \\ &= 1 + \lim_{h\to 0} \frac{x - (x+h)}{h \cdot x \cdot (x+h)} \\ &= 1 - \lim_{h\to 0} \frac{1}{x \cdot (x+h)} = 1 - \frac{1}{x^2};\ \text{so } g'(1) = 0 \end{aligned}$$

(d) $g'(x) = 1 + (-1)x^{-2} = 1 - \frac{1}{x^2}$ using the power rule and $g'(1) = 0$ as before.

6. (18 Marks)

(a) The equation of a curve defined implicitly is $x^2e^y + x + \ln(1+y) = 2$. Verify that the point $(1, 0)$ belongs to the curve. Find an equation of the tangent line to the curve at this point.

(b) Let $f(x) = x^{1/3}$. Find a number c that satisfies the Mean Value Theorem for the function $f(x)$ on $[0, 1]$.

(c) Use l'Hopital's rule to evaluate

$$\lim_{x\to 0} \frac{\sin x - x}{x^3}$$

Solutions (a) Substitute: $1 \cdot e^0 + 1 + \ln(1+0) = 2$ so the equation is satisfied and the point is on the curve. Next, differentiate:

$$\begin{aligned} x^2e^y + x + \ln(1+y) &= 2 \\ 2xe^y + x^2e^yy' + 1 + \frac{y'}{y+1} &= 0 \text{ now substitute} \\ 2 + y' + 1 + y' &= 0 \\ 2y' &= -3 \\ y' &= -\frac{3}{2} \end{aligned}$$

So an equation of the tangent line is

$$y = -\frac{3}{2}(x-1)$$

(b) $f'(x) = 1/3x^{-2/3}$ which is defined on $(0, 1)$ but not on $[0, 1]$ and we want a number c so that

$$\begin{aligned} \frac{f(1) - f(0)}{1-0} &= f'(c), \text{ i.e.} \\ 1 &= \frac{1}{3}c^{-2/3} \Longrightarrow c^{2/3} = \frac{1}{3} \text{ so} \\ c &= \left(\frac{1}{3}\right)^{3/2} \approx 0.192\,45 \end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 0}\frac{\sin x - x}{x^3} &= \lim_{x\to 0}\frac{\cos x - 1}{3x^2} \text{ l'Hopital's rule still applies:}\\ &= \lim_{x\to 0}\frac{-\sin x}{6x} \text{ l'Hopital's rule still applies:}\\ &= \lim_{x\to 0}\frac{-\cos x}{6} = -\frac{1}{6}\end{aligned}$$

7. (10 Marks)

(a) The equation of a curve in the plane is $e^{xy} = x^2 + y^2$. A particle is moving along this curve at a certain velocity. At the instant that it moves through the point $(0, 1)$ the $y-$coordinate is decreasing at the rate of 3 cm/sec. How fast is the $x-$coordinate changing at this instant and is it increasing or decreasing?

(b) A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

Solutions (a) Differentiate with respect to time and then substitute:

$$\begin{aligned}e^{xy}\left(x\frac{dy}{dt} + y\frac{dx}{dt}\right) &= 2x\frac{dx}{dt} + 2y\frac{dy}{dt}\\ 1\left(0\cdot\frac{dy}{dt} + 1\cdot\frac{dx}{dt}\right) &= 0\cdot\frac{dx}{dt} + 2\cdot\frac{dy}{dt}\\ \frac{dx}{dt} &= 2\frac{dy}{dt} = 2\cdot 3 = 6 \text{ cm/sec}\end{aligned}$$

and since the sign is + it is increasing.

(b) The semicircle (assume it is the upper one) has equation

$$y = \sqrt{4 - x^2}$$

and a rectangle would have vertices $(\pm x, 0)$ and $\left(\pm x, \sqrt{4 - x^2}\right)$ so the area is

$$A = 2x\sqrt{4 - x^2}; x \geq 0$$

differentiate to find local or global extrema:

$$\begin{aligned} A' &= 2\sqrt{4-x^2} - \frac{2x^2}{\sqrt{4-x^2}} = 4 \cdot \frac{2-x^2}{\sqrt{4-x^2}} \\ &= 0 \Leftrightarrow x = \sqrt{2} \text{ and } y = \sqrt{2} \text{ so} \\ A &= 4 \end{aligned}$$

To check that this is a maximum we can either calculate the second derivative which is

$$A'' = 4x\frac{x^2-6}{\left(\sqrt{4-x^2}\right)^3} < 0 \text{ at } x = \sqrt{2}$$

or (easier) note that A' changes sign from positive to negative as x passes through the critical point from left to right. So it is a global maximum since there is only one and $A = 0$ at the endpoints.

8. (14 Marks) Given the function $f(x) = xe^{-x}$,

(a) Find the domain and check for symmetry. Find asymptotes (if any).

(b) Calculate$f'(x)$ and use it to determine interval(s) where the function is increasing, interval(s) where the function is decreasing, and local extrema (if any).

(c) Calculate$f''(x)$ and use it to determine interval(s) where the function is concave upward, interval(s) where the function is concave downward and inflection point(s) (if any).

(d) Sketch the graph of the function.

Solutions (a) The equation shows that the domain is all of $\mathbb{R}$. Replacing x by $-x$ gives a different function - i.e. $f(-x) = -xe^x \neq f(x)$ and also $\neq -f(x)$ so there is no even or odd symmetry. Since the domain is $\mathbb{R}$ there are no vertical asymptotes but we see (using L'Hopital's rule, if necessary) that

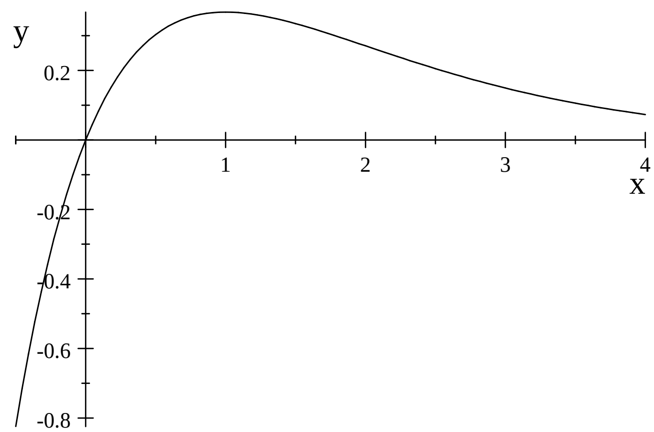
$$\lim_{x\to\infty} xe^{-x} = 0$$

so $y = 0$ is a horizontal asymptote.

(b) $f'(x) = e^{-x} - xe^{-x} = (1-x)\,e^{-x}$ so $x = 1$ is a critical point. Since the derivative is positive for $x < 1$ and negative for $x > 1$ it means the function is increasing on $(-\infty, 1)$, decreasing on $(1, \infty)$ and has a maximum (which must be global) at $x = 1$, $y = e^{-1}$.

(c) $f''(x) = xe^{-x} - 2e^{-x} = (x-2)\,e^{-x}$. Since it changes sign at $x = 2$ this must be an inflection point. The second derivative is positive for $x > 2$ and negative for $x < 2$ so the function is concave down on $(-\infty, 2)$, and concave up on $(2, \infty)$.

(d)



Bonus Question (5 Marks) Let $f(x) = x\sin(|x|)$.

Use the definition of the derivative to show that f is differentiable at $x = 0$ and

find $f'(0)$.

Solution We can't use differentiation rules here to calculate $f'(0)$ because $|x|$ is not differentiable. Also, if we calculate $f'(x)$ for $x \neq 0$ we get

$$\begin{aligned} f'(x) &= \sin x + x\cos x \text{ if } x > 0 \\ f'(x) &= -\sin x - x\cos x \text{ if } x < 0 \end{aligned}$$

and so substituting $x = 0$ in either equation gives the answer 0. But this is incorrect, because neither one is valid when $x = 0$. We have to use the the definition of the derivative:

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{(0+h)\sin(|0+h|) - 0}{h} \\ &= \lim_{h\to 0} \frac{h\sin(|h|)}{h} = \lim_{h\to 0} \sin(|h|) = 0 \end{aligned}$$

and **that** is why the answer is zero!.