1 Chapter 14 - Suggested 'Core' Problems

Problem # 1:

Given that the total demand is $P = 10 - Q$, where Q is total industry output (i.e. $Q = Q_1 + Q_2$). Assume that firm 1's cost structure is $C_1(Q_1) = Q_1$, and firm 2's cost structure is $C_2(Q_2) = Q_2$.

a) Show that if firm 1 were the only firm to produce then, $P_1^* = 5.5$ and $Q_1^* = 4.5$

b) Show that if the two firm collude (cartel) industry output will be 4.5 and $P^* = 5.5$

b) Show that if the two firms carry out Cournot competition then $Q_1^* = 3$, $Q_2^* = 3$ and $P* = 4$

Answer:

a)

$$
\begin{aligned}
&\text{Max}_{\ Q_1} \quad \text{Profits of firm 1} \\
&\text{Max}_{\ Q_1} \quad P(Q)Q_1 - C_1(Q_1) \\
&\text{Max}_{\ Q_1} \quad (10 - Q_1 - 0)Q_1 - Q_1 \\
&\text{Max}_{\ Q_1} \quad 9Q_1 - Q_1^2 \\
&9 - 2Q_1 = 0 \\
&Q_1^* = 4.5 \\
&P_1^* = 10 - 4.5 = 5.5 \\
\text{Profits of firm } 1 &= 5.5 \cdot 4.5 - 4.5 = 20.25
\end{aligned}
$$

Since firm 2 does not produce, it has no profit.

b)

$$
\begin{aligned}
&\text{Max}_{\ Q_1,Q_2} \quad \text{Profits of firm 1} + \text{Profits of firm 2} \\
&\text{Max}_{\ Q_1,Q_2} \quad [P(Q)Q_1 - C_1(Q_1)] + [P(Q)Q_2 - C_2(Q_2)] \\
&\text{Max}_{\ Q_1,Q_2} \quad [(10 - Q_1 - Q_2)Q_1 - Q_1] + [(10 - Q_1 - Q_2)Q_2 - Q_2] \\
&\text{Max}_{\ Q_1} \quad 9Q_1 - Q_1^2 - 2Q_1Q_2 + 9Q_2 - Q_2^2 \\
&9 - 2Q_1 - 2Q_2 = 0 \qquad (1) \\
&9 - 2Q_1 - 2Q_2 = 0 \qquad (2)
\end{aligned}
$$

Note: There are two equation, and two unknowns. Yet, the two equations are the same so one cannot solve for Q_1 and Q_2 individually. One can say, however, that

$$
\begin{aligned}
9 - 2Q_1 - 2Q_2 &= 0 \\
9 - 2(Q_1 + Q_2) &= 0 \\
9 - 2Q &= 0 \\
Q^* &= 4.5
\end{aligned}
$$

and thus $P^* = 5.5$. The two firms have both a marginal cost of 1 and no fixed costs. As such, if one is maximizing total profits it does not matter which firm produces the output because the cost is identical and constant per unit of output. One can assume that they split the output (i.e. $Q_1^* = Q_2^* = 2.25$), and thus profits will be \$ 10.125 each.

Note: If one had a quadratic cost structure, the two first order conditions would be different, and one could solve for the optimal Q_1^* and Q_2^* (Check for yourself).

c)

$$
\begin{aligned}
&\text{Max}_{\,Q_1} \quad \text{Profits of firm 1} \\
&\text{Max}_{\,Q_1} \quad PQ_1 - C_1(Q_1) \\
&\text{Max}_{\,Q_1} \quad (10 - Q_1 - Q_2)Q_1 - Q_1 \\
&\text{Max}_{\,Q_1} \quad 9Q_1 - Q_1^2 - Q_1Q_2 \\
&9 - 2Q_1 - Q_2 = 0 \qquad (3)
\end{aligned}
$$

$$
\begin{aligned}
&\text{Max}_{\,Q_2} \quad \text{Profits of firm 2} \\
&\text{Max}_{\,Q_2} \quad PQ_2 - C_2(Q_2) \\
&\text{Max}_{\,Q_2} \quad (10 - Q_1 - Q_2)Q_2 - Q_2 \\
&\text{Max}_{\,Q_2} \quad 9Q_2 - Q_1Q_2 - Q_2^2 \\
&9 - Q_1 - 2Q_2 = 0 \qquad (4)
\end{aligned}
$$

There are two equations (equations (3) and (4)), and two unknowns (Q_1 and Q_2). Multiply equation (3) by -2 and add to equation (4)

$$
\begin{aligned}
&-18 + 4Q_1 + 2Q_2 = 0 \\
&+ \\
&9 - Q_1 - 2Q_2 = 0
\end{aligned}
$$

$$
\begin{aligned}
-9 + 3Q_1 &= 0 \\
Q_1^* &= 3
\end{aligned}
$$

sub back optimal solution into equation (3)

$$
\begin{aligned}
9 - 2Q_1 - Q_2 &= 0 \\
9 - 2(3) - Q_2 &= 0 \\
Q_2^* &= 3
\end{aligned}
$$

Given the demand function one can calculate the equilibrium price

$$P^* = 10 - 3 - 3$$
$$P^* = 4$$

$$\text{Profits of firm } 1 = 4 \cdot 3 - 3 = 9$$
$$\text{Profits of firm } 2 = 4 \cdot 3 - 3 = 9$$

Note that the profits of each firm will be less than if they colluded.

Problem # 2: - Cournot model with homogeneous product

Given that the total demand is $P = 10 - Q$, and that firm 1's cost are $C_1(Q_1) = Q_1$, and firm 2's cost are $C_2(Q_2) = 2Q_2$, calculate the Cournot outcome.

Answer:

$$\begin{aligned}
&\text{Max}_{\ Q_1} \quad \text{Profits of firm 1} \\
&\text{Max}_{\ Q_1} \quad PQ_1 - C_1(Q_1) \\
&\text{Max}_{\ Q_1} \quad (10 - Q_1 - Q_2)Q_1 - Q_1 \\
&\text{Max}_{\ Q_1} \quad 9Q_1 - Q_1^2 - Q_1Q_2 \\
&9 - 2Q_1 - Q_2 = 0 \qquad (5)
\end{aligned}$$

$$\begin{aligned}
&\text{Max}_{\ Q_2} \quad \text{Profits of firm 2} \\
&\text{Max}_{\ Q_2} \quad PQ_2 - C_2(Q_2) \\
&\text{Max}_{\ Q_2} \quad (10 - Q_1 - Q_2)Q_2 - 2Q_2 \\
&\text{Max}_{\ Q_2} \quad 8Q_2 - Q_1Q_2 - Q_2^2 \\
&8 - Q_1 - 2Q_2 = 0 \qquad (6)
\end{aligned}$$

There are two equations (equations (1) and (2)), and two unknowns (Q_1 and Q_2). Solve

$$Q_1^* = 10/3$$
$$Q_2^* = 7/3$$

Given the demand function one can calculate the equilibrium price

$$P = 10 - 10/3 - 7/3 = 13/3$$

$$\text{Profits of firm } 1 = 13/3 \cdot 10/3 - 10/3 = 11.11$$
$$\text{Profits of firm } 2 = 13/3 \cdot 7/3 - 2(7/3) = 5.44$$

Problem # 3: - Cournot model with differentiated products

The demand faced by firm 1 is $P_1 = 10 - 2Q_1 - Q_2$, and the demand faced by firm 2 is $P_2 = 10 - Q_1 - 3Q_2$. In addition firm 1's cost are $C_1(Q_1) = Q_1$, and firm 2's cost are $C_1(Q_2) = 2Q_2$. Set up the problem faced by each firm.

Answer: Answer:

$$\begin{aligned} &\text{Max}_{Q_1} \quad \text{Profits of firm 1} \\ &\text{Max}_{Q_1} \quad P_1Q_1 - C_1(Q_1) \\ &\text{Max}_{Q_1} \quad (10 - 2Q_1 - Q_2)Q_1 - Q_1 \end{aligned}$$

$$\begin{aligned} &\text{Max}_{Q_2} \quad \text{Profits of firm 2} \\ &\text{Max}_{Q_2} \quad P_2Q_2 - C_2(Q_2) \\ &\text{Max}_{Q_2} \quad (10 - Q_1 - 3Q_2)Q_2 - 2Q_2 \end{aligned}$$

Problem # 4: - Cournot model with homogeneous products

Given that the total demand is $P = 10 - Q$, and that firm 1's cost are $C_1(Q_1) = Q_1$, and firm 2's cost are $C_2(Q_2) = aQ_2$. Set up the problem faced by each firm. If a were to increase, would you expect it to affect firm 1's optimal output decision?

Answer:

$$\begin{aligned} &\text{Max}_{Q_1} \quad \text{Profits of firm 1} \\ &\text{Max}_{Q_1} \quad PQ_1 - C_1(Q_1) \\ &\text{Max}_{Q_1} \quad (10 - Q_1 - Q_2)Q_1 - Q_1 \end{aligned}$$

$$\begin{aligned} &\text{Max}_{Q_2} \quad \text{Profits of firm 2} \\ &\text{Max}_{Q_2} \quad PQ_2 - C_2(Q_2) \\ &\text{Max}_{Q_2} \quad (10 - Q_1 - Q_2)Q_2 - aQ_2 \end{aligned}$$

It will affect the output decision of firm 1. Why? First, one can see from firm 2's optimization problem that a higher cost will affect its optimal choice. We know that a higher marginal cost will make it produce less. Second, the output decision of firm 2 affects the price firm 1 faces. Since firm 2 produces, it gives more room for firm 1 to produce, so it will increase its output.

Problem # 5: - Bertrand model with homogeneous product

Given that the total demand is $P = 10 - Q$, and that firm 1's cost are $C_1(Q_1) = Q_1$, and firm 2's cost are $C_2(Q_2) = 2Q_2$, calculate the Bertrand outcome.

Answer:

$$\begin{aligned}
P_1^* &= 1.99 \\
P_2^* &= 2 \\
Q_1^* &= 10 - 1.99 = 8.01 \\
Q_2^* &= 0 \\
\text{Profits of firm 1} &= 1.99 \cdot 8.01 - 8.01 \\
\text{Profits of firm 2} &= 2 \cdot 0 - 2(0)
\end{aligned}$$

Problem # 6: - Stackelberg model with homogeneous product

Given that the total demand is $P = 10 - Q$, and that firm 1's cost are $C(Q_1) = Q_1$, and firm 2's cost are $C_2(Q_2) = 2Q_2$, calculate the Stackelberg outcome. Assume that firm 1 chooses output first.

Answer: Start in Period 2

$$\begin{aligned}
&\text{Max}_{\ Q_2} \quad \text{Profits of firm 2} \\
&\text{Max}_{\ Q_2} \quad PQ_2 - C_2(Q_2) \\
&\text{Max}_{\ Q_2} \quad (10 - Q_1 - Q_2)Q_2 - 2Q_2 \\
&\text{Max}_{\ Q_2} \quad 8Q_2 - Q_1Q_2 - Q_2^2 \\
&8 - Q_1 - 2Q_2 = 0 \\
Q_2 &= \frac{8 - Q_1}{2} \qquad (7)
\end{aligned}$$

Now Back to Period 1

$$\begin{aligned}
&\text{Max}_{\ Q_1} \quad \text{Profits of firm 1} \\
&\text{Max}_{\ Q_1} \quad PQ_1 - C_1(Q_1) \\
&\text{Max}_{\ Q_1} \quad (10 - Q_1 - Q_2)Q_1 - Q_1 \\
&\text{Max}_{\ Q_1} \quad 9Q_1 - Q_1Q_2 - Q_1^2 \\
&\text{Max}_{\ Q_1} \quad 9Q_1 - Q_1(\frac{8 - Q_1}{2}) - Q_1^2 \\
&\text{Max}_{\ Q_1} \quad 5Q_1 - Q_1^2/2 \\
&5 - Q_1 = 0 \\
&Q_1^* = 5 \qquad (8)
\end{aligned}$$

Substitute (4) into (3)

$$
\begin{aligned}
Q_2^* &= (8-5)/2 = 3/2 \\
P^* &= 10 - 5 - 3/2 = 7/2 \\
\text{Profits of firm 1} &= 7/2 \cdot 5 - 5 = 12.5 \\
\text{Profits of firm 2} &= 7/2 \cdot 3/2 - 2(3/2) = 2.25
\end{aligned}
$$

Problem # 7: - Stackelberg model with homogeneous product

Given that the total demand is $P = 10 - Q$, and that firm 1's cost are $C_1(Q_1) = Q_1$, and firm 2's cost are $C_2(Q_2) = 2Q_2$, calculate the Stackelberg outcome. Assume that firm 2 chooses output first.

Answer: Start in Period 2

$$
\begin{aligned}
&\text{Max}_{\ Q_1} \quad \text{Profits of firm 1} \\
&\text{Max}_{\ Q_1} \quad PQ_1 - C_1(Q_1) \\
&\text{Max}_{\ Q_1} \quad (10 - Q_1 - Q_2)Q_1 - Q_1 \\
&\text{Max}_{\ Q_1} \quad 9Q_1 - Q_1Q_2 - Q_1^2 \\
&9 - Q_2 - 2Q_1 = 0 \\
Q_1 = &\frac{9 - Q_2}{2} \qquad (9)
\end{aligned}
$$

Now Back to Period 1

$$
\begin{aligned}
&\text{Max}_{\ Q_2} \quad \text{Profits of firm 2} \\
&\text{Max}_{\ Q_2} \quad PQ_2 - C_(Q_2) \\
&\text{Max}_{\ Q_2} \quad (10 - Q_1 - Q_2)Q_2 - 2Q_2 \\
&\text{Max}_{\ Q_2} \quad 8Q_2 - Q_1Q_2 - Q_2^2 \\
&\text{Max}_{\ Q_2} \quad 8Q_2 - (\frac{9 - Q_2}{2})Q_2 - Q_2^2 \\
&\text{Max}_{\ Q_2} \quad (7/2)Q_2 - Q_2^2/2 \\
&7/2 - Q_2 = 0 \\
&Q_2^* = 7/2 \qquad (10)
\end{aligned}
$$

Substitute (6) in to (5)

$$
\begin{aligned}
Q_1^* &= (9 - (7/2))/2 = 11/4 \\
P^* &= 10 - (7/2) - (11/4) = 15/4 \\
\text{Profits of firm 1} &= 15/4 \cdot 11/4 - (11/4) = 7.56 \\
\text{Profits of firm 2} &= 15/4 \cdot 7/2 - 2(7/2) = 6.13
\end{aligned}
$$