Final Exam: Calculus notes

General

Domain: The set of elements *x* that can be inputs for a function *f*
Range: The set of elements *y* that are outputs of a function *f*

Increasing Function: A function is increasing over an interval A if for all $x \in A$, the property $x1 < x2 \rightarrow f(x1) < f(x2)$ holds.
Decreasing Function: A function is decreasing over an interval A if for all $x \in A$, the property $x1 < x2 \rightarrow f(x1) > f(x2)$ holds.

Even Function: A function with the property that for all values of *x*: $f(-x) = f(x)$
Odd Function: A function with the property that for all values of *x*: $f(-x) = -f(x)$

A function is neither even nor odd if it does not satisfy either of these properties. When sketching, it is helpful to keep in mind that even functions are symmetric about the y-axis and that odd functions are symmetric about the origin (0, 0).

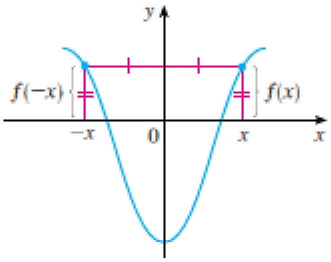


FIGURE 19 An even function

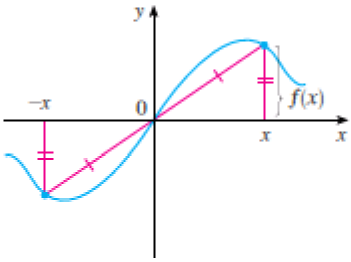
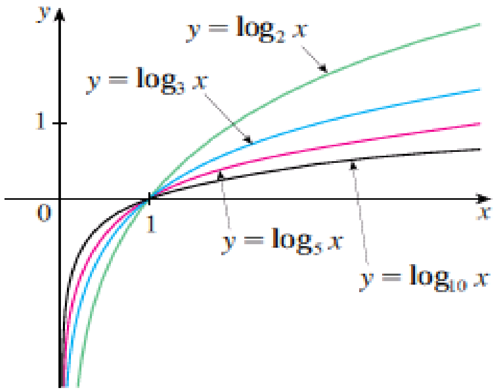


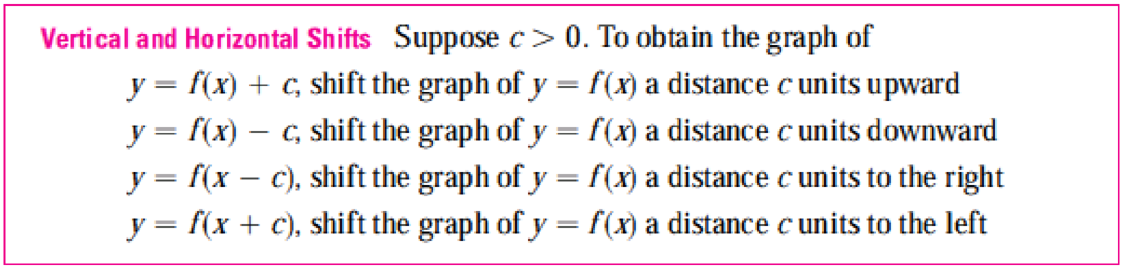
FIGURE 20 An odd function

$$\sin(x + 2\pi) = \sin x \qquad \cos(x + 2\pi) = \cos x$$

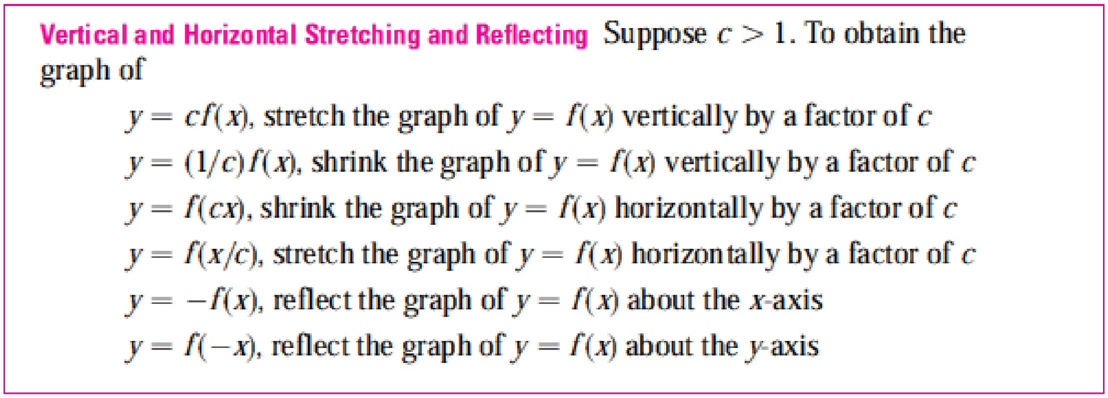
Exponential functions: $a^x = y$
Power functions: $x^a = y$
Logarithmic functions



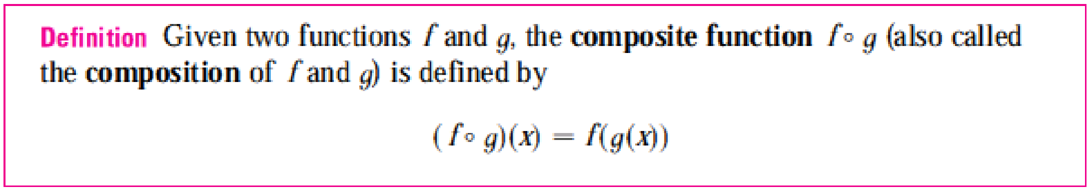
Domain = (0, infinity); **Range** = (-infinity, infinity)


Vertical and Horizontal Shifts Suppose $c > 0$. To obtain the graph of

$y = f(x) + c$, shift the graph of $y = f(x)$ a distance c units upward
$y = f(x) - c$, shift the graph of $y = f(x)$ a distance c units downward
$y = f(x - c)$, shift the graph of $y = f(x)$ a distance c units to the right
$y = f(x + c)$, shift the graph of $y = f(x)$ a distance c units to the left


Vertical and Horizontal Stretching and Reflecting Suppose $c > 1$. To obtain the graph of

$y = cf(x)$, stretch the graph of $y = f(x)$ vertically by a factor of c
$y = (1/c)f(x)$, shrink the graph of $y = f(x)$ vertically by a factor of c
$y = f(cx)$, shrink the graph of $y = f(x)$ horizontally by a factor of c
$y = f(x/c)$, stretch the graph of $y = f(x)$ horizontally by a factor of c
$y = -f(x)$, reflect the graph of $y = f(x)$ about the x-axis
$y = f(-x)$, reflect the graph of $y = f(x)$ about the y-axis


Definition Given two functions f and g, the **composite function** $f \circ g$ (also called the **composition** of f and g) is defined by

$$(f \circ g)(x) = f(g(x))$$

Absolute Value

Definition:

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

Properties and Rules:

- $|x| = a \quad \Rightarrow \quad x = \pm a$
- $|x| < a \quad \Rightarrow \quad x = -a < x < a$
- $|x| > a \quad \Rightarrow \quad x > a \text{ or } x < -a$

$$|xy| = |x||y|$$

$$\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$$

Triangle Inequality:

$$|x + y| \leq |x| + |y|$$

Example. Given that $|x| < 2$, show that $\left|\frac{x-1}{2x^2+5}\right| \leq \frac{3}{5}$

First, we split the fraction and apply the triangle inequality to obtain:

$$\left|\frac{x - 1}{2x^2 + 5}\right| \leq \left|\frac{x}{2x^2 + 5}\right| + \left|\frac{-1}{2x^2 + 5}\right|$$

Note that $2x^2 + 5 \geq 5$ for any value of *x*. Therefore, if we replace the denominator with 5, we are shrinking it and thereby making the entire rational expression larger. Hence:

$$\left|\frac{x}{2x^2 + 5}\right| + \left|\frac{-1}{2x^2 + 5}\right| \leq \left|\frac{x}{5}\right| + \left|\frac{-1}{5}\right|$$

After applying properties of absolute value, we can obtain the following expression:

$$\frac{|x| + |-1|}{|5|}$$

Since |x| < 2, as provided in the question, we can safely substitute 2 for x in the expression.

$$\frac{|x| + |-1|}{|5|} = \frac{|x| + 1}{5} \leq \frac{3}{5}$$

Heavy-side functions

$$H(x) = \begin{cases} 1 & \text{if} x \geq 0 \\ 0 & \text{if} x < 0 \end{cases}$$

Example. Sketch $f(x) = H(x + 1) + |x|$

Start out by looking for key points. Applying the Heaviside definition to H(x + 1), we can see that H(x + 1) = 1 if $x + 1 \geq 0$, or $x \geq -1$. Similarly, H(x + 1) = 0 if x < -1. Next, by the definition of absolute value, we have that the key point for |x| is at x = 0.

Use these points to establish 3 cases:

1. $x < -1$
2. $-1 \leq x < 0$
3. $x \geq 0$

In case 1, we have $f(x) = 0 + (-x) = -x$.
In case 2, we have $f(x) = 1 + (-x) = -x + 1$.
In case 3, we have $f(x) = 1 + (x) = x + 1$
We finish by sketching each of these lines on the appropriate x-intervals.

Example. Sketch the inequality $|x + y| \leq 1$.

We have two cases to consider. Our goal is to get inequalities where we isolate *y*.

- Case 1: $x + y \geq 0$, which implies that $\boldsymbol{y \geq -x}$
  - We have $x + y \leq 1$, or $\boldsymbol{y \leq 1 - \mathrm{x}}$
- Case 2: $x + y < 0$, which implies that $\boldsymbol{y} < -x$
  - We have $-(x + y) \leq 1$, or $\boldsymbol{y \geq -\mathrm{x} - 1}$

To finish, sketch each of the lines defining the y-regions and shade in their intersection.

One-to-One Functions

A function is **one-to-one** if it never takes the same y-value twice, that is, it has the property:

$$f(x1) = f(x2) \rightarrow x1 = x2$$

A function is one-to-one if it passes the **horizontal line test**. If a function is increasing and decreasing on different intervals, it cannot be one-to-one unless it is discontinuous.

Example. Identify which of the following functions are one-to-one.

- $y = \ln(x)$ [YES]
- $y = \cos(x)$ [NO]
- $y = x^n$, where n is an even integer [NO]
- $y = x^n$, where n is an odd integer [YES]

Example. Prove that $f(x) = \sqrt{x^2 + 5}$ is one-to-one on [0, ∞).

To prove this, we need to use the defintion of a one-to-one function. Initialize $f(x1) = f(x2)$. Thus,

$$\sqrt{(x1)^2 + 5} = \sqrt{(x2)^2 + 5}$$

By squaring both sides and subtracting off the constant term, we end up with:

$$(x1)^2 = (x2)^2$$

Normally, by taking the square root of both sides, we would have:

$$\pm x1 = \pm x2$$

However, we are only considering the interval [0, ∞), therefore, *x1* and *x2* are both positive. Therefore, $x1 = x2$. We have proved $f(x1) = f(x2) \rightarrow x1 = x2$.
Hence, $f(x)$ is one-to-one on [0, ∞).

Inverse Functions

A function $f(x)$ that is one-to-one with domain A and range B has an **inverse function** $f^{-1}(x)$ with domain B and range A.
$f^{-1}(x)$ reverses the operations of $f(x)$ in the opposite direction
$f^{-1}(x)$ is a reflection of $f(x)$ about the line *y = x*

$$f^{-1}(a) = b \rightarrow f(b) = a$$

Cancellation Identity: Let $f(x)$ and $g(x)$ be functions that are inverses of each other.
Then: $f\left(g(x)\right) = x \quad and \quad g(f(x) = x$
The cancellation identity can be applied only if x is in the domain of the inside function.

To find the equation of $f^{-1}(x)$, we first switch x and y within the equation of $f(x)$. Then, if possible, we isolate for y, or leave the function implicitly defined.

Inverse Trigonometric Functions

Given that trigonometric functions are periodic, we know that they are not one-to-one. Therefore, to define an inverse trigonometric function, we must restrict the domain of the corresponding trigonometric function to make it one-to-one.

Trigonometric Function	Restriction on the Domain	Inverse Trigonometric Function	Domain and Range of Inverse
$\boldsymbol{y = \sin(x)}$	$\frac{-\pi}{2} \leq x \leq \frac{\pi}{2}$	$y = \arcsin(x)$	Domain $-1 \leq x \leq 1$ Range $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$
$\boldsymbol{y = \cos(x)}$	$0 \leq x \leq \pi$	$y = \arccos(x)$	Domain $-1 \leq x \leq 1$ Range $0 \leq y \leq \pi$
$\boldsymbol{y = \tan(x)}$	$\frac{-\pi}{2} < x < \frac{\pi}{2}$	$y = \arctan(x)$	Domain $x \in \mathbb{R}$ Range $\frac{-\pi}{2} < y < \frac{\pi}{2}$

Example. Solve $\cos(\arccos(x)) = \frac{1}{2}$

We know that x is an element of *arccos(x)* if $-1 \leq x \leq 1$. Therefore, we can apply the cancellation identity, and get $x = \frac{1}{2}$

Example. Evaluate $\arcsin(\sin\frac{5\pi}{4})$.

Note that $\frac{5\pi}{4}$ is not in the domain of the domain-restricted sin (x). Therefore, we cannot apply the cancellation identity.

Start with $\sin\frac{5\pi}{4}$. Since $\frac{5\pi}{4}$ falls in the third quadrant, by CAST rule, we have that $\sin\frac{5\pi}{4}$ is negative. Taking $\frac{\pi}{4}$ (the related acute angle), we can use a special triangle to give us the result $\sin\frac{5\pi}{4} = -\frac{1}{\sqrt{2}}$.

Now, we look at $\arcsin(-\frac{1}{\sqrt{2}})$. Considering it as a function y, we have its range as $\frac{-\pi}{2} \leq y \leq \frac{\pi}{2}$

Thus, we know our solution will fall in this range. We need to find a y such that $\sin(y) = -\frac{1}{\sqrt{2}}$

Within our range, we have quadrants 1 and 4. As sin is negative in quadrant 4, we can further restrict the range to $\frac{-\pi}{2} \leq y \leq 0$

The only angle that works is $\frac{-\pi}{4}$

Limits

1 Definition Suppose $f(x)$ is defined when x is near the number a. (This means that f is defined on some open interval that contains a, except possibly at a itself.) Then we write

$$\lim_{x \to a} f(x) = L$$

and say "the limit of $f(x)$, as x approaches a, equals L"

if we can make the values of $f(x)$ arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

$$\lim_{x \to 0} \frac{\sin x}{x} = 1 \qquad \lim_{x \to 0} \sin \frac{\pi}{x} \text{ does not exist}$$

3 $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = L$ and $\lim_{x \to a^+} f(x) = L$

EXAMPLE 7 Show that $\lim_{x \to 0} |x| = 0$.

SOLUTION Recall that

$$|x| = \begin{cases} x & \text{if } x \geqslant 0 \\ -x & \text{if } x < 0 \end{cases}$$

Since $|x| = x$ for $x > 0$, we have

$$\lim_{x \to 0^+} |x| = \lim_{x \to 0^+} x = 0$$

For $x < 0$ we have $|x| = -x$ and so

$$\lim_{x \to 0^-} |x| = \lim_{x \to 0^-} (-x) = 0$$

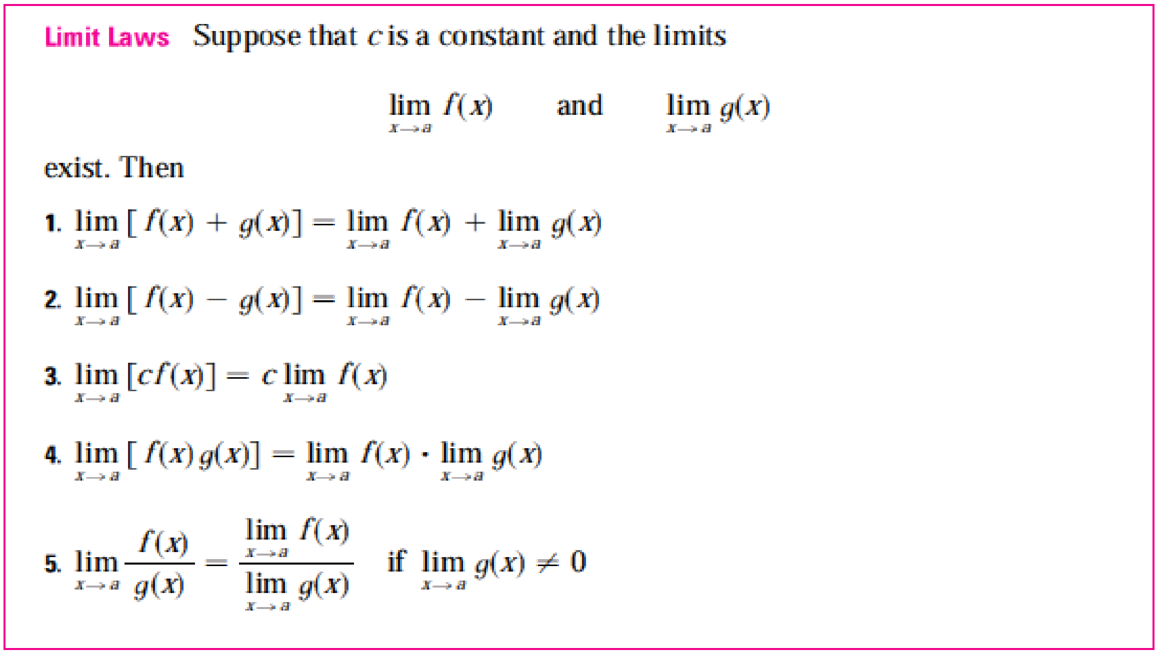
Therefore, by Theorem 1,

$$\lim_{x \to 0} |x| = 0$$

6 Definition The line $x = a$ is called a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following statements is true:

$$\lim_{x \to a} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = \infty$$

$$\lim_{x \to a} f(x) = -\infty \qquad \lim_{x \to a^-} f(x) = -\infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

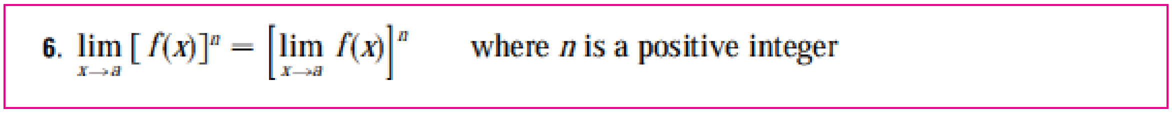


Limit Laws Suppose that c is a constant and the limits

$$\lim_{x \to a} f(x) \quad \text{and} \quad \lim_{x \to a} g(x)$$

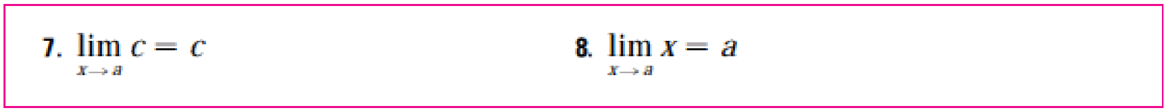
exist. Then

1. $\lim_{x \to a} [f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a} [f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a} [cf(x)] = c \lim_{x \to a} f(x)$
4. $\lim_{x \to a} [f(x)\, g(x)] = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
5. $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ if $\lim_{x \to a} g(x) \neq 0$

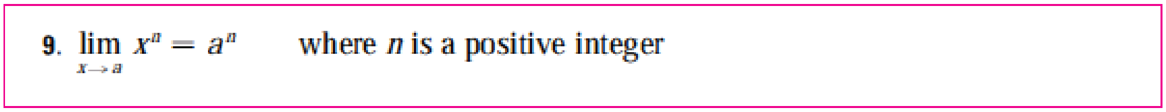


6. $\lim_{x \to a} [f(x)]^n = \left[\lim_{x \to a} f(x)\right]^n$ where n is a positive integer

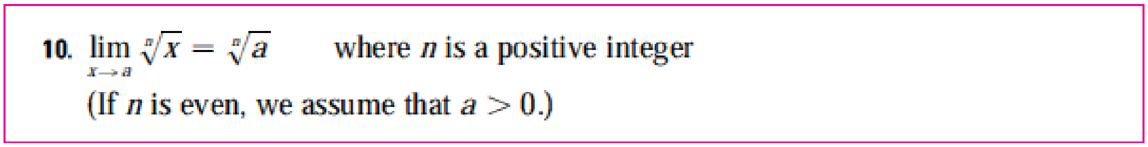
$$\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$$



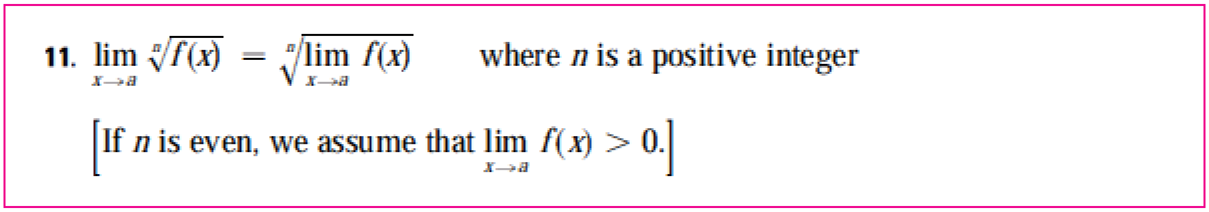
7. $\lim_{x \to a} c = c$
8. $\lim_{x \to a} x = a$



9. $\lim_{x \to a} x^n = a^n$ where n is a positive integer

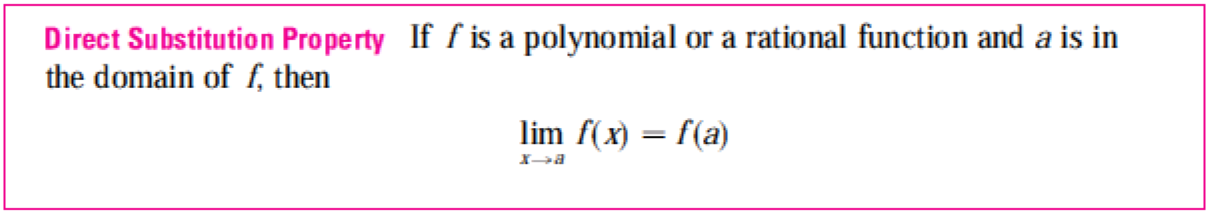


10. $\lim_{x \to a} \sqrt[n]{x} = \sqrt[n]{a}$ where n is a positive integer
    (If n is even, we assume that $a > 0$.)



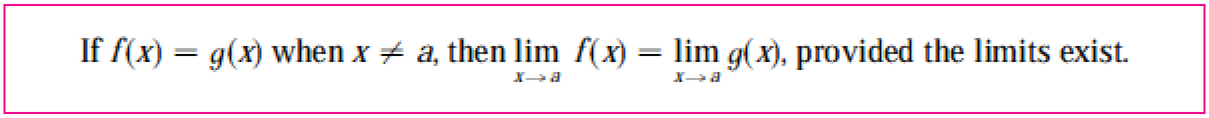
11. $\lim_{x \to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \to a} f(x)}$ where n is a positive integer

    [If n is even, we assume that $\lim_{x \to a} f(x) > 0$.]

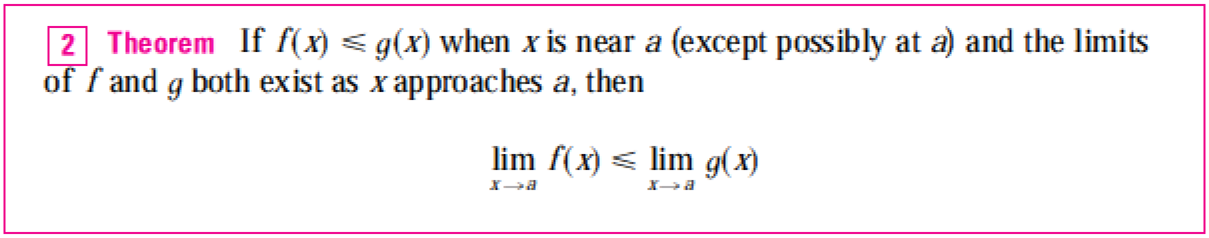


Direct Substitution Property If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x \to a} f(x) = f(a)$$



If $f(x) = g(x)$ when $x \neq a$, then $\lim_{x \to a} f(x) = \lim_{x \to a} g(x)$, provided the limits exist.



2 Theorem If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x)$$

The formal definition of the limit

$\lim_{x \to a} f(x) = L$ if given any $\varepsilon > 0$, we can find a $\delta > 0$ such that:

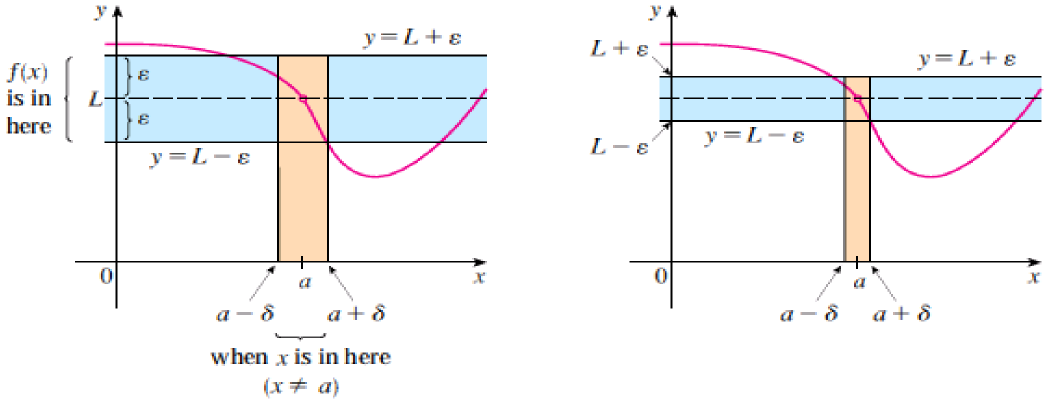
$$0 < |x - a| < \delta \to |f(x) - L| < \varepsilon$$

Since |x – a| is the distance from x to a, and |f(x) – L| is the distance from f (x) to L, and since $\mathcal{E}$ can be arbitrarily small, the definition of a limit can be expressed in words as follows:

Lim x→a f(x) = L means that the distance between f(x) and L can be made arbitrarily small by taking the distance from x to a sufficiently small (but not 0)

Alternatively, lim x→a f(x) = L means that the values of f(x) can be made as close as we please to L by taking x close enough to a (but not equal to a)

Lim f(x)= L as x→a means that for every $\mathcal{E}$>0 , we can find δ>0 such that if x lies in an open interval (a – δ, a + δ) and x ≠ a, then f(x) lies in the open interval (L – $\mathcal{E}$, L + $\mathcal{E}$)



Left hand limit:

$\lim_{x \to a-} f(x) = L$ if given any $\varepsilon > 0$, we can find a $\delta > 0$ such that:

$$a - \delta < x < a \to |f(x) - L| < \varepsilon$$

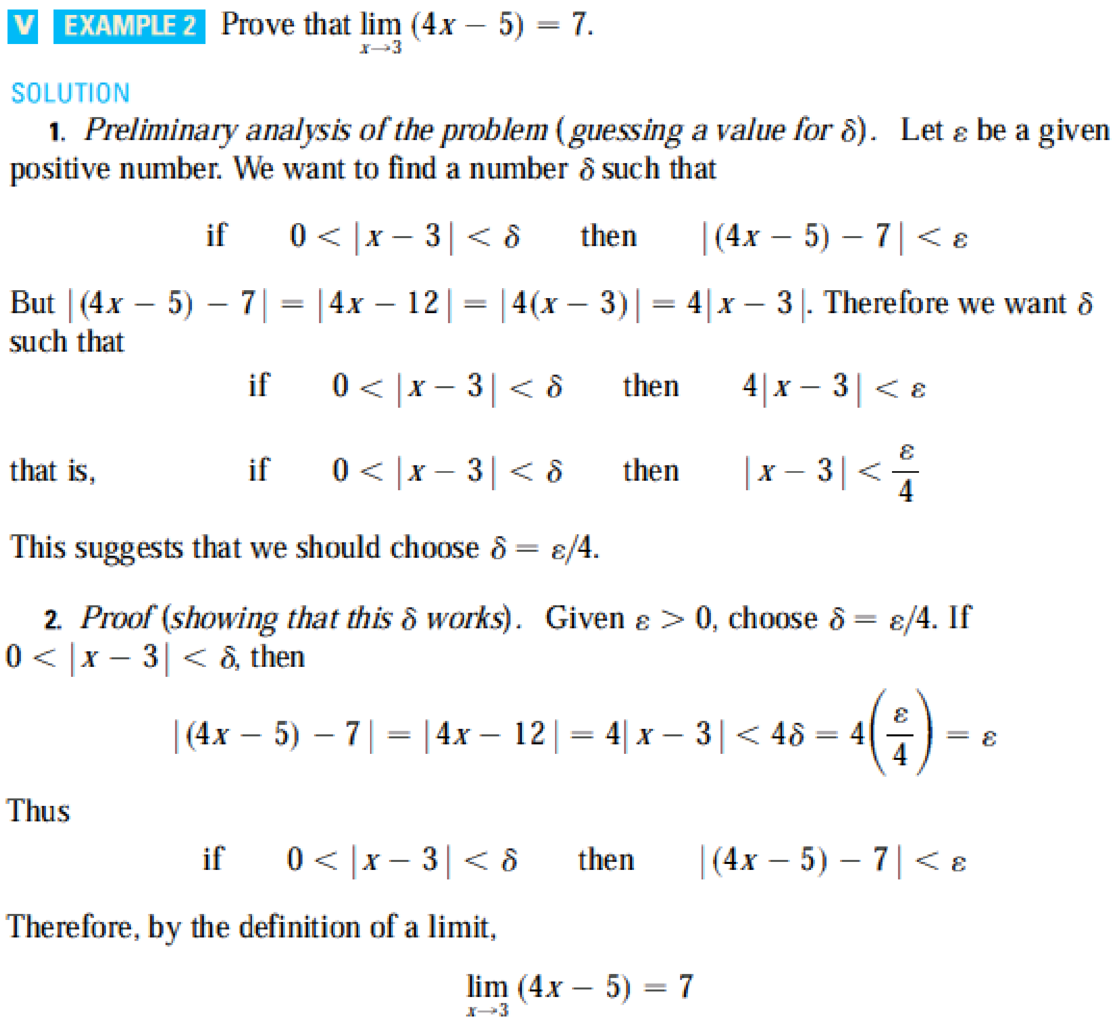
Right hand limit:

$\lim_{x \to a+} f(x) = L$ if given any $\varepsilon > 0$, we can find a $\delta > 0$ such that:

$$a < x < a + \delta \to |f(x) - L| < \varepsilon$$

The Squeeze Theorem

If f(x) ≤ g(x) ≤ h(x) when x is near a (except possible at a) and $\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$, then $\lim_{x \to a} g(x) = L$



V EXAMPLE 2 Prove that $\lim_{x\to 3}(4x-5)=7$.

SOLUTION

1. *Preliminary analysis of the problem (guessing a value for δ).* Let ε be a given positive number. We want to find a number δ such that

$$\text{if} \quad 0<|x-3|<\delta \quad \text{then} \quad |(4x-5)-7|<\varepsilon$$

But $|(4x-5)-7| = |4x-12| = |4(x-3)| = 4|x-3|$. Therefore we want δ such that

$$\text{if} \quad 0<|x-3|<\delta \quad \text{then} \quad 4|x-3|<\varepsilon$$

that is,

$$\text{if} \quad 0<|x-3|<\delta \quad \text{then} \quad |x-3|<\frac{\varepsilon}{4}$$

This suggests that we should choose $\delta = \varepsilon/4$.

2. *Proof (showing that this δ works).* Given $\varepsilon>0$, choose $\delta=\varepsilon/4$. If $0<|x-3|<\delta$, then

$$|(4x-5)-7| = |4x-12| = 4|x-3| < 4\delta = 4\left(\frac{\varepsilon}{4}\right) = \varepsilon$$

Thus

$$\text{if} \quad 0<|x-3|<\delta \quad \text{then} \quad |(4x-5)-7|<\varepsilon$$

Therefore, by the definition of a limit,

$$\lim_{x\to 3}(4x-5)=7$$

Example. Evaluate $\lim_{x\to 0}\frac{\tan(x)}{x}$

$$\tan(x)=\frac{\sin(x)}{\cos(x)}$$

Applying this to our function and dividing, we have:

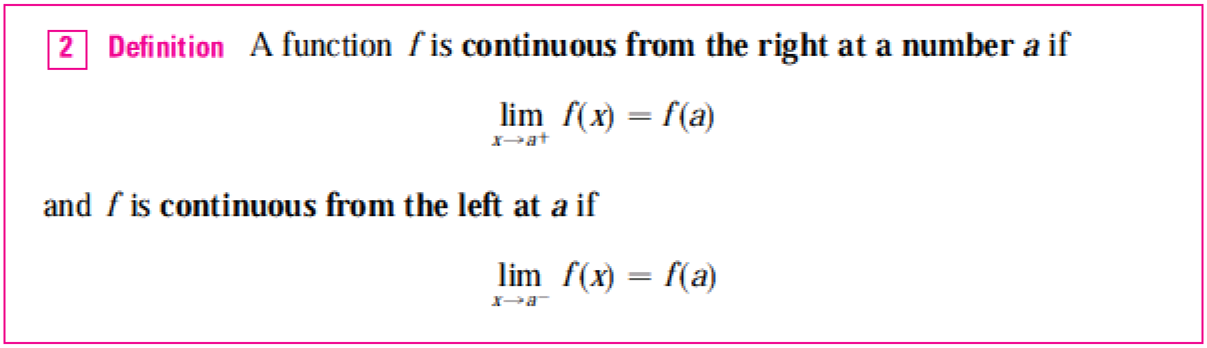
$$\frac{\tan(x)}{x}=\frac{\sin(x)}{x\cdot\cos(x)}=\frac{\sin(x)}{x}\cdot\frac{1}{\cos(x)}$$

Substituting directly, we can evaluate the limit as:

$$\lim_{x\to 0}\frac{\tan(x)}{x}=\lim_{x\to 0}\frac{\sin(x)}{x}\cdot\frac{1}{\cos(x)}=1\cdot 1=1$$

Continuity

A function $f(x)$ is continuous at a point $x = a$ if $\lim_{x \to a} f(x) = f(a)$.
A function is continuous over an interval A if it is continuous on every x in A.



2 Definition A function f is **continuous from the right at a number** a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is **continuous from the left at** a if

$$\lim_{x \to a^-} f(x) = f(a)$$

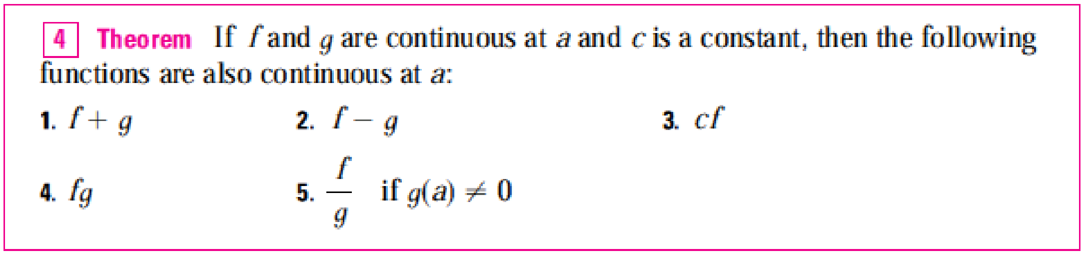
Example. Find the values for *b* and *c* such that the function is continuous at x = 2.

$$f(x) = \begin{cases} x^2 + 1 & x > 2 \\ c & x = 2 \\ x + b & x > 2 \end{cases}$$

Observe the one-sided limits around *x* = 2. We need $\lim_{x \to 2+} f(x) = \lim_{x \to 2-} f(x)$. Setting them to equal each other, we have:

$$\lim_{x \to 2+} x + b = \lim_{x \to 2-} x^2 + 1$$
$$2 + b = 5$$
$$b = 3$$

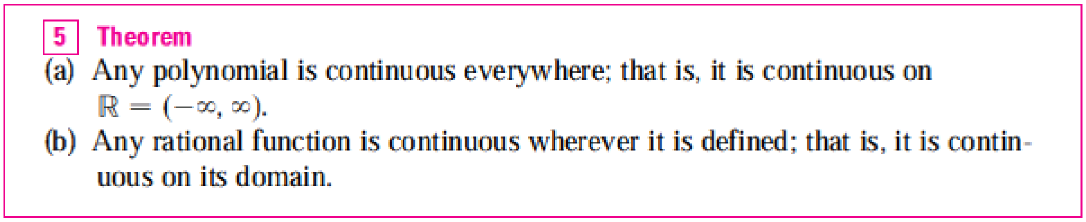
With this value of b, we have $\lim_{x \to 2+} f(x) = \lim_{x \to 2-} f(x) = \lim_{x \to 2} f(x) = 5$



4 Theorem If f and g are continuous at a and c is a constant, then the following functions are also continuous at a:

1. $f + g$ **2.** $f - g$ **3.** cf

4. fg **5.** $\dfrac{f}{g}$ if $g(a) \neq 0$



5 Theorem
(a) Any polynomial is continuous everywhere; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$.
(b) Any rational function is continuous wherever it is defined; that is, it is continuous on its domain.

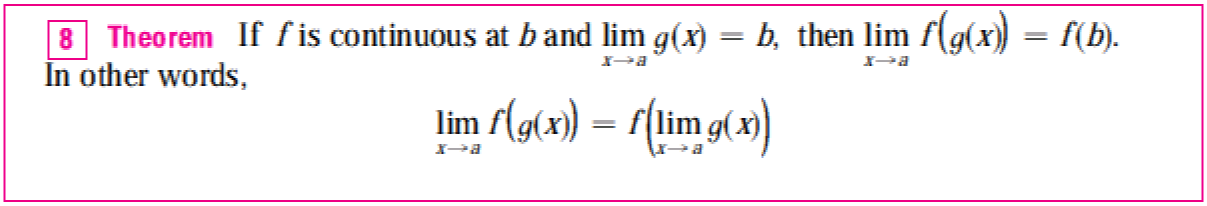
6 $$\lim_{\theta \to 0} \cos \theta = 1 \qquad \lim_{\theta \to 0} \sin \theta = 0$$

Note that all polynomial, exponential, rational, root, logarithmic, trigonometric, and reciprocal functions are continuous **on their domains**.

Example. Identify where $f(x) = e^{x^2+\sin(x)}$ is continuous.
$x^2$and $\sin(x)$ are both continuous on all real numbers *x*. Thus, by continuity theorems, their sum is also continous. $e^x$is continuous on all real numbers *x*. Therefore, by continuity theorems, its composition with $x^2 + \sin(x)$ is also continuous on all real numbers.

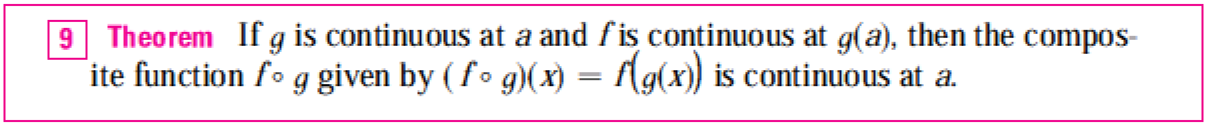
Types of Discontinuities

- Infinite: When a function has a vertical asymptote
- Jump: When the one-sided limits do not equal one another
- Removable: When the limit does not equal the function value at a point
- Infinite Oscillations: there are an infinite number of oscillations in a neighborhood of a point
    - EX: $f(x) = \sin\left(\frac{1}{x}\right)$

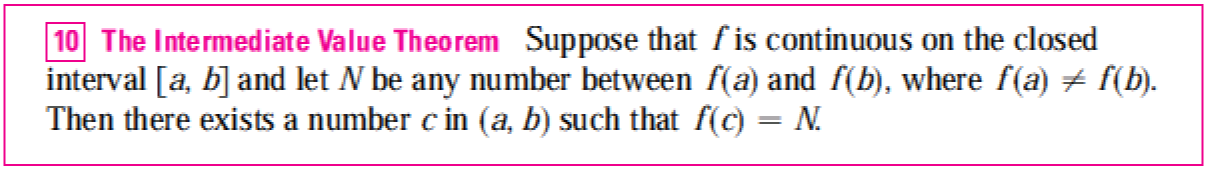


8 Theorem If f is continuous at b and $\lim_{x\to a} g(x) = b$, then $\lim_{x\to a} f(g(x)) = f(b)$. In other words,

$$\lim_{x\to a} f(g(x)) = f\left(\lim_{x\to a} g(x)\right)$$



9 Theorem If g is continuous at a and f is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at a.



10 The Intermediate Value Theorem Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number c in (a, b) such that $f(c) = N$.

The **Method of Bisection** is an algorithm that allows us to approximate the location of the root.

Example. Approximate the root of $f(x) = x^4 - 4x^3 + 1$ between 0 and 1 to $\frac{1}{8}$ accuracy.
We start by taking the midpoint of the interval containing the root, in this case $x = 0.5$. Now, evaluating the function at this point, we get $f(0.5) = 0.5625$.
Since the function is decreasing on the interval, that means at $x = 0.5$, it has not reached its root yet. Thus, we can narrow our interval to $(0.5, 1)$.

Now we repeat our algorithm by taking the midpoint of $(0.5, 1)$, which is 0.75. Note that this point approximates the root with an accuracy of $\frac{1}{4}$ or 0.25.
$f(0.75) = -0.37$, meaning we have gone past the root. Thus, we can narrow our interval again to $(0.5, 0.75)$.
The midpoint of the interval is 0.625, which gives us an accuracy of $\frac{1}{8}$, as required.

Tangents

There are two limit-based derivative definitions we can use, sometimes referred two as **difference quotients** or **first principles**:

$f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$ $\qquad$ $f'(a) = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$

The 2nd definition is **Newton's quotient**. Both give the derivative of a function at a point, which geometrically refers to the slope of the tangent line to the function at that point.

In single-variable calculus, the **differentiability** of a function at a point refers to the existence of the derivative at that point.

Example. Determine if $f(x) = \begin{cases} x^5 \sin\left(\frac{3}{x}\right) & x \neq 0 \\ 0 & x = 0 \end{cases}$ is differentiable at x = 0.

Because this function is piecewise, we cannot use our derivative rules. We must use the derivative definition. Differentiability depends on the existence of the limit. Newton's quotient:

$$f'(0) = \lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}$$

$$f'(0) = \lim_{x \to 0} \frac{f(x)}{x}$$

Note that we are interested in the limit as x **approaches** 0 – that is to say, it does not **equal** 0.

$$f'(0) = \lim_{x \to 0} \frac{x^5 \sin(\frac{3}{x})}{x} = \lim_{x \to 0} x^4 \sin(\frac{3}{x})$$

Now, by applying squeeze theorem, we can easily see that the limit is 0. Therefore, the function is differentiable at x = 0.

Theorem. If a function is differentiable at a point, it is also continuous at that point. Sometimes it is easier to show continuity at a point by proving differentiability at that point. Note that by the Contrapositive Law, we also have the statement: **"If a function is NOT continuous at a point, then it is NOT differentiable at the point**."

Power Rule $If\ f(x) = ax^n, then\ f'(x) = nax^{n-1}$

Product Rule $[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$

Quotient Rule $\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{f'(x)g(x) - f(x)g\prime(x)}{[g(x)]^2}$

Chain Rule $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$ Type equation here.

Exponential Derivatives $\frac{d}{dx} e^x = e^x$ $\qquad$ $\frac{d}{dx} a^x = a^x \cdot \ln(a)$

Note: If *e* or *a* are raised to a function (ie. not just *x* on its own), then we must apply Chain Rule

$\frac{d}{dx}\sin(x) = \cos(x)$	$\frac{d}{dx}\cos(x) = -\sin(x)$
$\frac{d}{dx}\tan(x) = \sec^2(x)$	$\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$
$\frac{d}{dx}\sec(x) = \tan(x)\sec(x)$	$\frac{d}{dx}\cot(x) = -\csc^2(x)$
$\frac{d}{dx}(\arcsin(x)) = \frac{1}{\sqrt{1-x^2}}$	$\frac{d}{dx}(\arctan(x)) = \frac{1}{x^2+1}$
$\frac{d}{dx}(\ln(x)) = \frac{1}{x}$	$\frac{d}{dx}(\log_a(x)) = \frac{1}{x\ln(a)}$

Implicit Differentiation

We use implicit differentiation on a function when the function is not written explicitly in terms of y. Differentiate every term, on both sides of the equation. Recall that we treat y as a function, so when we differentiate a term involving y, we need to apply Chain Rule.

Example. Find the derivative of $e^y + \sin(y) = x^2 + \frac{1}{x}$

$$e^y\left(\frac{dy}{dx}\right) + \cos(y)\left(\frac{dy}{dx}\right) = 2x + (-\frac{1}{x^2})$$

Collecting like terms, and solving explicitly for the derivative:

$$\left(\frac{dy}{dx}\right)(e^y + \cos(y)) = 2x - \frac{1}{x^2}$$

$$\frac{dy}{dx} = \frac{2x - \frac{1}{x^2}}{e^y + \cos(y)}$$

Logarithmic Differentiation

When we have a large function that would require several iterations of Product and/or Quotient Rule, applying logarithmic differentiation may help to simplify it.

To use logarithmic differentiation, take the ln of both sides of the equation. Then, simplify using log laws. Once this is done, implicitly differentiate.

Example. Find the derivative of $y = x^x$

Method 1: Logarithmic Differentiation

$$\ln(y) = \ln(x^x)$$
$$\ln(y) = x \cdot \ln(x)$$

Applying implicit differentiation:

$$\frac{1}{y}\cdot\frac{dy}{dx} = [1](\ln(x)) + (x)\frac{1}{x}$$
$$\frac{dy}{dx} = y \cdot (\ln(x) + 1)$$
$$\frac{dy}{dx} = x^x(\ln(x) + 1)$$

Method 2

Note that $e^{\ln(blah)} = blah$, where blah can be anything.

$$y = x^x = e^{\ln(x^x)}$$
$$y = e^{xln(x)}$$

Differentiating both sides:

$$\frac{dy}{dx} = \frac{d}{dx}(e^{xln(x)})$$
$$\frac{dy}{dx} = e^{xln(x)}([1](\ln(x)) + (x)\frac{1}{x})$$
$$\frac{dy}{dx} = x^x(\ln(x) + 1)$$

Applications of differentiation

Sometimes, it can be easier to take a derivative than a limit. If we get a limit expression that mimics the form of the first principles definition of the derivative, then instead of taking that limit, we can just evaluate the derivative.

Example. Evaluate $\lim_{x\to 0}\frac{\ln(1+x)}{x}$

Since $\ln(1) = 0$:

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{\ln(1+x) - \ln(1)}{x}$$

Replace *x* with *h,* where *a* = 1 and $f(x) = \ln(x)$:

$$\lim_{h\to 0}\frac{f(a+h) - f(a)}{h} = \lim_{x\to 0}\frac{\ln(1+x) - \ln(1)}{x} = f'(1)$$

$$\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$$

Through this method, we can derive the following two definitions of e in the form of limits:

$$\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e$$
$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Related Rate Problems

A related rate problem is a question involving applied differential calculus. We interpret the derivative as a rate of change. Usually, there is more than 1 equation and more than 1 unknown variable and/or rate to deal with. We use the relation between rates and variables.

Example. Two runners are running perpendicular to each other. Runner A runs at a rate of 5 m/s, while Runner B runs at a rate of 4 m/s. After 10 seconds, how quickly is the distance between the runners increasing?

Step 1. Define Variables

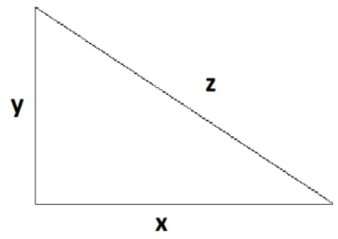
Let x be the distance travelled by Runner A.

Let y be the distance travelled by Runner B.

Let z be the distance between the runners.

Let t be the time in seconds since the runners started running.

*IMPORTANT: y is NOT a function of x!



Step 2. Information Given

$$\frac{dx}{dt} = 5 \ \ and \ \ \frac{dy}{dt} = 4$$
$$z^2 = x^2 + y^2$$

Step 3. Information Needed

$$\frac{dz}{dt} \ \ at \ t = 10$$

Step 4. Solve

Applying implicit differentiation:

$$2z\left(\frac{dz}{dt}\right) = 2x\left(\frac{dx}{dt}\right) + 2y\left(\frac{dy}{dt}\right)$$
$$2z\left(\frac{dz}{dt}\right) = 2x(5) + 2y(4) = 10x + 8y$$

At t = 10, $x = 5 \cdot 10 = 50 \ \ and \ \ y = 4 \cdot 10 = 40.$

$z = \sqrt{(50)^2 + (40)^2} = \sqrt{4100} = 10\sqrt{41}$

$$\frac{dz}{dt} = \frac{10x + 8y}{2z} = \frac{10(50) + 8(40)}{2(10\sqrt{41})} = \frac{41}{\sqrt{41}} = \sqrt{41}$$

Thus, our answer is $\sqrt{41}$ m/s.

Extreme Values

A function f on domain D has an **absolute maximum** at c if $f(c) \geq f(x) \ \forall \ x \in D$ or an **absolute minimum** at c if $f(c) \leq f(x) \ \forall \ x \in D$.

A function f has a **local maximum or minimum** at c if $f(c) \geq f(x) \ or \ f(c) \leq f(x)$ in a small open interval around c.

A function f has a **critical number** at c if $f'(c) = 0$ or $f'(c)$ does not exist.

Fermat's Theorem If f has a local maximum or minimum at c and $f'(c)$ exists, then $f'(c) = 0$

Logical Equivalent

If f has a local maximum or minimum at c, then $f'(c) = 0$ OR $f'(c)$ does not exist.

- Note that the converse is not true

Extreme Value Theorem

If f is continuous on [a, b] then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ for some $c, d \in [a, b]$.

If f satisfies the conditions of the EVT, we use the **Closed Interval Method** (aka **Max/Min Algorithm**) to find the maximum and minimum:

1. Find all critical values in (a, b)
2. Evaluate f at all critical values and at the endpoints
3. The largest of these is the absolute maximum, the smallest is the absolute minimum

Example. Find the maximum and minimum of $f(x) = \frac{\ln(x)}{x}$ on [1, 3]

f is continuous on [1, 3]

$$f'(x) = \frac{\left[\frac{1}{x}\right](x) - \ln(x)\,[1]}{x^2} = \frac{1 - \ln(x)}{x^2}$$

$f'(x)$ exists for all $x \in [1, 3]$

$$f'(x) = 0$$
$$1 - \ln(x) = 0$$
$$\ln(x) = 1$$
$$x = e$$
$$f(1) = \ln(1) = 0$$
$$f(3) = \frac{\ln(3)}{3}$$
$$f(e) = \frac{\ln(e)}{e} = \frac{1}{e}$$

Therefore, the maximum is $\frac{1}{e}$ and the minimum is 0.

Mean Value Theorem If f is a function such that:

1. f is continuous on [a, b]
2. f is differentiable on (a, b),

then $\exists\, c \in (a, b)$ such that $f'(c) = \frac{f(b) - f(a)}{b - a}$

- there is at least one tangent line that is parallel to the secant line

Rolle's Theorem If f satisfies the third condition that $f(a) = f(b)$ (in addition to the MVT conditions), then $\exists\ c \in (a,b)$ such that $f'(c) = 0$

Example. Given that $f(2) = 10$ and that $3 \leq f'(x) \leq 5\ \forall\ x$, find the range of possible values for $f(4)$.

f is differentiable for all $x \rightarrow f$ is continuous for all x

Therefore, we can apply the MVT for any interval. Since we know $f(2)$ and want to know $f(4)$, let's choose (2, 4).

By the MVT, $\exists\ c \in (2,4)$ such that $f'(c) = \frac{f(4)-f(2)}{4-2}$

$$2f'(c) = f(4) - 10$$
$$f(4) = 2f'(c) + 10$$
$$2(3) + 10 \leq f(4) \leq 2(5) + 10$$
$$16 \leq f(4) \leq 20$$

Constant Function Theorem If $f'(x) = 0\ \forall\ x \in (a,b)$ then $f(x)$ is constant on (a,b).

Corollary. If $f'(x) = g'(x)\ \forall\ x \in (a,b)$, then $(f-g)(x)$ is constant on (a,b).

Increasing Function Theorem If $f'(x) > 0$ for all x in an interval A, then $f(x)$ is increasing on A. If $f'(x) < 0$ for all x in an interval A, then $f(x)$ is decreasing on A.

Example. Show that $f(x) = 2x - \sin(x) - 1$ has exactly one root.

f is continuous for all x, $f(\pi) = 2\pi - 1 > 0$, and $f(0) = -1 < 0$

Thus, by the IVT, f has at least one root.

$f'(x) = 2 - \cos(x) > 0$ for all x

Therefore, by the IFT, f is increasing for all x. Thus, f has exactly one root.

Curve Sketching

1. **Domain:** Look for restrictions
2. **Symmetry:** Check for even or odd symmetry
3. **Asymptotes**
   - Vertical asymptotes for rational functions
   - Horizontal asymptotes by examining $\lim_{x\to\infty} f(x)$
4. **Intervals of Increase/Decrease**
   - **Increasing/Decreasing Test:** Use the Increasing Function Theorem to determine the increasing and decreasing intervals.
5. **Local Maximum/Minimum**

- **First Derivative Test**: If $f'(x)$ changes from negative to positive at a critical number c, then there is a local minimum at c. If $f'(x)$ changes from positive to negative at a critical number c, then there is a local maximum at c. If $f'(x)$ does not change sign at c, then there is no local extrema at c.

6. **Concavity and Points of Inflection**
   - If $f''(x) > 0$ for all x in an interval, then f is concave up on the interval.
   - If $f''(x) < 0$ for all x in an interval, then f is concave down on the interval.
   - **Second Derivative Test:** If f is continuous at a point P and it changes from concave up to concave down or vice versa, then P is a point of inflection.
7. **Sketch**

Example. Sketch $f(x) = x \cdot ln(x)$

1. **Domain:** We can't take the ln of a number less than or equal to 0, $D = \{x > 0\}$
2. **Symmetry:** Neither even nor odd symmetry
3. **Asymptotes:** As $x \to \infty,\ f(x) \to \infty$

Since $D = \{x > 0\}$, we need to examine $\lim_{x\to 0} x \cdot \ln(x) = 0$ (by L'Hôpital's Rule)

4. **Intervals of Increase/Decrease**

$$f'(x) = [1]\ln(x) + (x)\left[\frac{1}{x}\right]$$
$$f'(x) = \ln(x) + 1$$

Set $f'(x) = 0$.

$$\ln(x) = -1$$
$$e^{\ln(x)} = e^{-1}$$
$$x = e^{-1} \text{ (critical number)}$$

Examine the sign of $f'(x)$ for x-values less than and greater than e^{-1}

$$f'(1) = \ln(1) + 1 = 1 > 0$$
$$f'\left(\frac{1}{10}\right) = \ln\left(\frac{1}{10}\right) + 1 \approx -1.302 < 0$$

Therefore, $f(x)$ is decreasing on $(0, e^{-1})$ and increasing on (e^{-1}, ∞).

5. **Local Maximum/Minimum**

$x = e^{-1}$ is the only critical number of $f(x)$.

From part 4, we know that a local minimum occurs here.

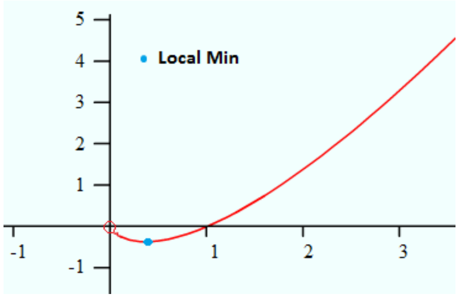
$$f(e^{-1}) = -\frac{1}{e} \approx -0.37$$

6. **Concavity and Points of Inflection**

$f''(x) = \dfrac{1}{x}$

For all $x > 0$, $f''(x) > 0$ → function is concave up for all x → no inflection points

7. Sketch



L'Hôpital's Rule

Recall the **indeterminate forms** for limits. These are expressions that are obtained through substituting, and which do not allow you to evaluate the limit:

$$0^0 \quad 1^\infty \quad (\infty - \infty) \quad \frac{\pm\infty}{\pm\infty} \quad (0 \cdot \infty) \quad \infty^0 \quad \frac{0}{0}$$

For $\frac{\pm\infty}{\pm\infty}$ and $\frac{0}{0}$, we can use **L'Hôpital's Rule** help us evaluate these limits.

Consider $\lim_{x \to a} \frac{f(x)}{g(x)}$ where by substituting *x* = *a*, we end up with one of our two specific indeterminate forms. Then, L'Hôpital's Rule states that:

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example. Evaluate $\lim_{x \to 0} x \cdot \ln(x)$

Note that the indeterminate form of this limit is $0 \cdot -\infty$. We want to get the limit into one of the two special indeterminate forms so we can apply L'Hôpital's Rule.

$$\lim_{x \to 0} x \cdot \ln(x) = \lim_{x \to 0} \frac{\ln(x)}{\frac{1}{x}} \qquad \left[\frac{-\infty}{\pm\infty}\right]$$

Applying L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{\ln(x)}{\frac{1}{x}} = \lim_{x \to 0} \frac{\frac{1}{x}}{\frac{-1}{x^2}} = \lim_{x \to 0} -\frac{x^2}{x} = \lim_{x \to 0} -x = 0$$

Example. Evaluate $\lim_{x \to 0} \frac{1}{x} - \frac{1}{\sin(x)}$

$$\lim_{x \to 0} \frac{1}{x} - \frac{1}{\sin(x)} = \lim_{x \to 0} \frac{\sin(x) - x}{x \cdot sin(x)} \qquad \left[\frac{0}{0}\right]$$

Applying L'Hôpital's Rule:

$$\lim_{x\to 0}\frac{\sin(x)-x}{x\cdot sin(x)}=\lim_{x\to 0}\frac{\cos(x)-1}{[1]\sin(x)+x[\cos(x)]}\qquad\left[\frac{0}{0}\right]$$

Applying L'Hôpital's Rule again:

$$\lim_{x\to 0}\frac{\cos(x)-1}{[1]\sin(x)+x[\cos(x)]}=\lim_{x\to 0}-\frac{\sin(x)}{\cos(x)+([1]\cos(x)+(x)\sin(x))}=-\frac{0}{2}=0$$

Newton's Method

Recall that the Intermediate Value Theorem allows us to determine if a function has a root on a particular interval. **Newton's Method** lets us narrow in on the root using derivatives.

1. Start by guessing x_0, an approximation of the function's root
2. Use the formula $x_{n+1}=x_n-\frac{f(x_n)}{f'(x_n)}$ in order to get an improved approximation
3. Continue iterating Newton's Method until the approximation is good enough

Example. Approximate the root of $f(x)=x^3+x-1$

Note that $f'(x)=3x^2+1$

The function is continuous for all x.

$f(0)=-1$ and $f(1)=1$

Thus, by the Intermediate Value Theorem, there is a root in the interval $x\in(0,1)$.

For our initial value, choose the midpoint, $x_0=0.5$

By applying Newton's Method iteratively:

$$x_1=x_0-\frac{f(x_0)}{f'(x_0)}=0.5-\frac{-0.375}{1.75}\approx 0.7142$$

$$x_2=x_1-\frac{f(x_1)}{f'(x_1)}\approx 0.6831$$

$$x_3=x_2-\frac{f(x_2)}{f'(x_2)}\approx 0.6823$$

Anti-Differentiation

Anti-differentiation is the opposite operation of differentiation.

Reverse Power Rule. $If\ f'(x)=x^p, then\ f(x)=\frac{x^{p+1}}{p+1}+C$

Note that since the derivative of a constant is 0, we **always** need to add " + C" to our antiderivatives so that our answer is general!

Table of Anti-Derivatives

$f'(x)$	$f(x)$

$\frac{1}{x}$	$\ln\lvert x\rvert + C$
e^x	$e^x + C$
$\sin(x)$	$-\cos(x) + C$
$\cos(x)$	$\sin(x) + C$
$\sec^2(x)$	$\tan(x) + C$
$\sec(x)\tan(x)$	$\sec(x) + C$
$\frac{1}{\sqrt{1-x^2}}$	$\arcsin(x) + C$
$\frac{1}{1+\mathrm{x}^2}$	$\arctan(x) + C$

Application to Motion

Let $s(t)$ be the position of an object with respect to time.
Let $v(t)$ be the velocity of an object with respect to time.
Let $a(t)$ be the acceleration of an object with respect to time.
Note that $s'(t) = v(t)$ and $v'(t) = a(t)$.

Example. A stone is dropped from 450 m above ground. When dropped, it is accelerated by the force of gravity, $-9.8\ m/s^2$. Determine the height of the stone above the ground at time t. How long does it take for the stone to hit the ground?

We know that $a(t) = -9.8$. Anti-differentiating:

$$v(t) = -9.8t + C$$

Note that $v(0) = C = 0$, since the stone had no initial velocity at t = 0.
Therefore, $v(t) = -9.8t$. Anti-differentiating:

$$s(t) = -\frac{9.8t^2}{2} + K$$

Note that $s(0) = K = 450$, since that was the position of the stone at t = 0.
Therefore,

$$s(t) = -\frac{9.8t^2}{2} + 450 = -4.9t^2 + 450$$

The stone hits the ground when $s(t) = 0$. Then,

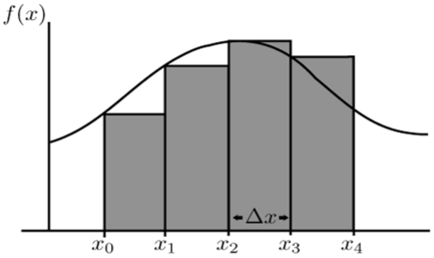
$$4.9t^2 = 450$$

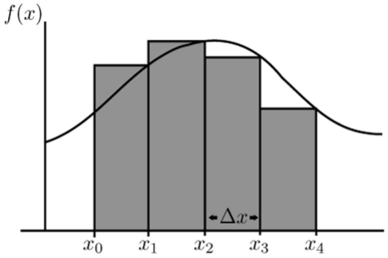
$$t^2 = \frac{450}{4.9}$$

$$t = \sqrt{\frac{450}{4.9}} \approx 9.58\ s$$

Riemann Sums

A Riemann sum involves splitting up part of a function's domain into uniform intervals. For each interval, we draw rectangles either from the left **(L_n, left Riemann sum)** or the right **(R_n, right Riemann sum)** of the function value for each interval. $\sum_{i=1}^{n} f(x_{i)}\,\Delta x$





We then add up the areas of all the rectangles.

Example. For $f(x) = \ln(x)$ over the interval [1, 3], determine L_4 and R_4

Start by determining the length of each interval:

$$\Delta x = \frac{b-a}{n} = \frac{3-1}{4} = \frac{1}{2}$$

For a left sum, we start our intervals with x = a = 1:

$$L_4 = f(1)\left(\frac{1}{2}\right) + f\left(\frac{3}{2}\right)\left(\frac{1}{2}\right) + f(2)\left(\frac{1}{2}\right) + f\left(\frac{5}{2}\right)\left(\frac{1}{2}\right)$$

$$L_4 = \frac{1}{2}\left[f(1) + f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right)\right]$$

$$L_4 = \frac{1}{2}\left[\ln(1) + \ln\left(\frac{3}{2}\right) + \ln(2) + \ln\left(\frac{5}{2}\right)\right]$$

$$L_4 \approx 1.007451510$$

For a right sum, we start our intervals with x = a + Δx = $\frac{3}{2}$:

$$R_4 = \frac{1}{2}\left[f\left(\frac{3}{2}\right) + f(2) + f\left(\frac{5}{2}\right) + f(3)\right]$$

$$R_4 = \frac{1}{2}\left[\ln\left(\frac{3}{2}\right) + \ln(2) + \ln\left(\frac{5}{2}\right) + \ln(3)\right]$$

$$R_4 \approx 1.556757654$$

Definite Integrals

By increasing the number of rectangles in a Riemann sum, our approximation of the area under the curve gets better. Hence, we define the **definite integral** from $x = a$ to $x = b$ as follows:

$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_{i)}\,\Delta x$ where $\Delta x = \frac{b-a}{n}$

This gives us the area under the curve over the interval [a, b].

Fundamental Theorem of Calculus

Part 1. If f is continuous on the interval [a, b], the integral function defined by:

$$g(x) = \int_a^x f(t)\,dt \qquad a \le x \le b$$

is continuous on [a, b] and differentiable on (a, b) and $g'(x) = f(x)$.

Part 2. If f is continuous on the interval [a, b] and $F'(x) = f(x)$, then:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Thus, the definite integral is the same as the anti-derivative function taken over a specific interval.

Note that the **indefinite integral** of a function is its general anti-derivative. For example, if $F'(x) = f(x)$:

$$\int f(x) = F(x) + C$$

Example. Evaluate $\int_0^{\ln(8)} 2e^{-2t}\,dt$

We know the derivative of an exponential function is the same as the original function. Remembering to account for Chain Rule, we see:

$$\int 2e^{-2t}\;dt = \frac{2e^{-2t}}{-2} + C = -e^{-2t}$$

We can easily verify this answer by differentiating.

Now, by FTC 2:

$$\int_0^{\ln(8)} 2e^{-2t}\,dt = [-e^{-2t}]\begin{matrix}\ln(8)\\ 0\end{matrix}$$

$$\int_0^{\ln(8)} 2e^{-2t}\,dt = \left[-e^{-2\ln(8)}\right] - [-e^{0}]$$

By properties of logarithms:

$$\int_0^{\ln(8)} 2e^{-2t}\,dt = \left[-e^{\ln(8)^{-2}}\right] - [-1]$$

$$\int_0^{\ln(8)} 2e^{-2t}\,dt = [-8^{-2}] + 1$$

$$\int_0^{\ln(8)} 2e^{-2t}\,dt = 1 - \frac{1}{64} = \frac{63}{64}$$

Area Problems

The anti-derivative of a function can be used to compute the area under the function's curve.

Example. Evaluate $\int_{-5}^{5}\sqrt{25-x^2}\ dx$ without using anti-derivatives.

Note that if we let $y=\sqrt{25-x^2}$, we see that the integral represents the area under the top half of a circle with radius 5, since the integral only includes the positive square root.

Thus, $\int_{-5}^{5}\sqrt{25-x^2}\ dx=\frac{\pi(5)^2}{2}=\frac{25\pi}{2}$

In order to find the **area between curves** $f(x)$ and $g(x)$, we need to find:

$$\int_a^b (f(x)-g(x))\,dx$$

where a and b are the x-axis bounds on the area and $f(x)$ is the "upper" function.

Example. Find the area bounded by $y=2\cos(x)+1, y=1-2\sin(x)\,, x=0, x=\frac{3\pi}{4}$

$$A=\int_0^{\frac{3\pi}{4}}[2\cos(x)+1-(1-2\sin(x))]\ dx$$

$$A=\int_0^{\frac{3\pi}{4}}[2\cos(x)+2\sin(x)]\ dx$$

$$A=2\int_0^{\frac{3\pi}{4}}[\cos(x)+\sin(x)]\ dx$$

$$A=2[\sin(x)-\cos(x)]\begin{matrix}3\pi/4\\0\end{matrix}$$

$$A=2\left[\left(\sin\left(\frac{3\pi}{4}\right)-\cos\left(\frac{3\pi}{4}\right)\right)-(\sin(0)-\cos(0))\right]$$

$$A=2\left[\left(\frac{1}{\sqrt{2}}-\left(-\frac{1}{\sqrt{2}}\right)\right)-(-1)\right]$$

$$A=2\sqrt{2}+2$$

Sometimes there may be 2 area functions to integrate separately and then add together.

Exercise. Find the area enclosed by the curve $f(x)=x^3+x^2$ and the line $g(x)=2x$.

We are not given our bounds of integration, so we determine them using the intersection.

$$2x=x^3+x^2$$

$$x^3+x^2-2x=0$$

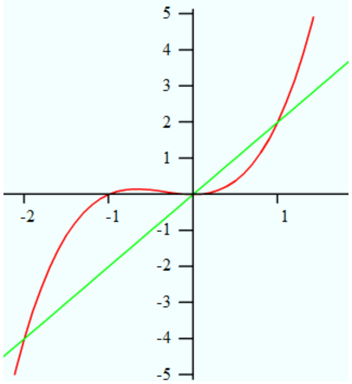
$$x(x+2)(x-1)=0$$

Points of intersection: $x = -2, 0, 1$
By examining the graph of the function, we can see that there is an area where f > g and another where g > x. Therefore, we calculate these areas separately and add them up at the end.

$$A = A1 + A2$$

$$A = \int_{-2}^{0} (x^3 + x^2 - 2x)\, dx + \int_{0}^{1} \left(2x - (x^3 + x^2)\right) dx$$

The answer should be $A = \frac{37}{12}$



Integration by Substitution

The method of substitution allows us to turn a complex integral into an easily solvable one.

$$\int f\left(g(x)\right) g'(x)\, dx$$

Let $u = g(x)$.
Then $\frac{du}{dx} = g'(x) \rightarrow du = g'(x)\, dx$
So:

$$\int f\left(g(x)\right) g'(x)\, dx = \int f(u)\, du$$

In other words, we usually want an integral where a function and its derivative (or a multiple of it) appear.

Example. Evaluate $\int_0^1 \sqrt{x^2 + x + 3}\,(2x + 1)\, dx$
Let $u = x^2 + x + 3$. Then, $du = 2x + 1\, dx$.
Note that $x = 0 => u = 3$ and $x = 1 => 5$

Therefore: $\int_0^1 \sqrt{x^2 + x + 3}\,(2x + 1)\, dx = \int_3^5 \sqrt{u}\ du$

$$\int_3^5 \sqrt{u}\ du = \left[\frac{u^{\frac{3}{2}}}{\frac{3}{2}}\right]_3^5 = \left[\frac{2u^{\frac{3}{2}}}{3}\right]_3^5$$

$$\int_3^5 \sqrt{u}\ du = \frac{2}{3}[5^{\frac{3}{2}} - 3^{\frac{3}{2}}]$$

Example. Evaluate $\int \frac{x}{\sqrt{9x^2-4}}\ dx$
Let $u = 9x^2 - 4$. Then $du = 18x\, dx => \frac{1}{18} du = x\, dx$

$$\int \frac{x}{\sqrt{9x^2 - 4}}\ dx = \int \frac{1}{\sqrt{u}} \cdot \frac{1}{18}\ du$$

By properties of integrals:

$$\int \frac{1}{\sqrt{u}} \cdot \frac{1}{18}\, du = \frac{1}{18} \int \frac{1}{\sqrt{u}}\, du = \frac{1}{18} \int u^{-\frac{1}{2}}\, du$$

Thus, by the reverse power rule:

$$\frac{1}{18} \int u^{-\frac{1}{2}}\, du = \frac{1}{18}\left(2u^{\frac{1}{2}} + C\right) = \frac{1}{9}\sqrt{u} + C$$

Finally, we substitute our value for u back in:

$$\int \frac{x}{\sqrt{9x^2 - 4}}\, dx = \frac{1}{9}\sqrt{9x^2 - 4} + C$$

Example. Evaluate $\int \frac{e^x}{1+e^{2x}}\, dx$

Let $u = e^x$. Then $du = e^x dx$.

$$\int \frac{e^x}{1 + e^{2x}}\, dx = \int \frac{1}{1 + u^2} du$$

Recalling the derivative of $\arctan(x)$:

$$\int \frac{1}{1 + u^2} du = \arctan(u) + C$$

$$\int \frac{e^x}{1 + e^{2x}}\, dx = \arctan(e^x) + C$$

Proofs

Limit Sum Law

If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a}[f(x) + g(x)] = L + M$

Let $\varepsilon > 0$ be given.

If $0 < |x - a| < \delta$, then $|f(x) + g(x) - (L + M)| < \varepsilon$

By Triangle Inequality:

$$|f(x) + g(x) - (L + M)| \leq |f(x) - L| + |g(x) - M|$$

$|f(x) + g(x) - (L + M)| < \varepsilon$ if $|f(x) - L| < \frac{\varepsilon}{2}$ and $|g(x) - M| < \frac{\varepsilon}{2}$

Then, there exist $\delta 1\ and\ \delta 2$ such that:

$$\text{If } 0 < |x - a| < \delta 1, \text{ then } |f(x) - L| < \frac{\varepsilon}{2}$$

$$\text{If } 0 < |x - a| < \delta 2, \text{ then } |g(x) - M| < \frac{\varepsilon}{2}$$

Let $\delta = \min(\delta 1, \delta 2)$
Thus, if $0 < |x - a| < \delta$, then $0 < |x - a| < \delta 1$ and $0 < |x - a| < \delta 2$

Therefore, $|f(x) + g(x) - (L + M)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

Hence, $\lim_{x \to a}[f(x) + g(x)] = L + M$

■

Differentiability implies Continuity

If a function is differentiable at a point, it is also continuous at that point.

For *x* close to a point *a*, we have:

$$f(x) = \frac{f(x) - f(a)}{x - a}(x - a) + f(a)$$

Taking limits, we have:

$$\lim_{z \to a} f(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a}(x - a) + \lim_{x \to a} f(a)$$

$$\lim_{z \to a} f(x) = f'(a) \cdot (0) + f(a)$$

$$\lim_{z \to a} f(x) = f(a)$$

Therefore, $f(x)$ is continuous at $x = a$

■

Product Rule

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + f'(x)g(x)$$

Using first principles:

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x)g(x)}{h}$$

Adding and subtracting $f(x + h)g(x)$ in the numerator:

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h \to 0} \frac{f(x + h)g(x + h) - f(x + h)g(x) - f(x)g(x) + f(x + h)g(x)}{h}$$

$$\frac{d}{dx}[f(x)g(x)] = \lim_{h \to 0} f(x + h) \cdot \frac{[g(x + h) - g(x)]}{h} + g(x)\frac{f(x + h) - f(x)}{h}$$

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

Increasing Function Theorem

If $f'(x) > 0$ on (a, b), then f is increasing on (a, b)

Let $x1, x2$ be any two numbers in (a, b), with $x1 < x2$
Since $f'(x) > 0$, f must be differentiable on (a,b)
And therefore on $(x1, x2)$, and f is continuous on $[x1, x2]$.
Applying MVT on $(x1, x2)$:

There exists a number c such that $f'(c) = \frac{f(x2)-f(x1)}{x2-x1}$

Since f'(x)>0 in (a,b), then f'(c)>0

⇨ $\frac{f(x2)-f(x1)}{x2-x1} > 0$
⇨ $f(x2) - f(x1) > 0$
⇨ $f(x2) > f(x1)$

So for any $x2 > x1$ in (a, b), we have $f(x2) > f(x1)$

I.E. f is increasing on (a, b).

The Fundamental Theorem of Calculus – Part 2

If f is continuous on $[a, b]$, then:

$$\int_a^b f(x)dx = F(b) - F(a)$$

where F is any antiderivative of f, that is, a function such that $F' = f$

Let $g(x) = \int_a^x f(t)dt$

From FTC1, $g'(x) = f(x)$ (g is an antiderivative of f)

Let F be another antiderivative of f on $[a, b]$.

Then notice $\frac{d}{dx}\big(F(x) - g(x)\big) = F'(x) - g'(x) = f(x) - f(x) = 0$

By CVT, $F(x) - g(x) = C$, where C is constant on $[a, b]$

$F(b) - g(b) = C$ (1)

$F(a) - g(a) = C$ (2)

Notice that $g(a) = \int_a^a f(t)dt = 0$

Then $F(a) = C$

Sub into (1):

$F(b) - g(b) = F(a)$

$g(b) = F(b) - F(a)$

- ⇨ $\int_a^b f(t)dt = F(b) - F(a)$
- ⇨ $\int_a^b f(x)dx = F(b) - F(a)$

Fermat's Theorem (For Local Maximum)

If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$

If f has a local max at c, then $f(c) \geq f(x)$ for all x near c. Then for some h close to 0, $f(c) \geq f(c+h)$

$f(c+h) - f(c) \leq 0$ (1)

For $h > 0$, we divide by h, $\frac{f(c+h)-f(c)}{h} \leq 0$

And taking the limit as $h \to 0^+$:

$$\lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq \lim_{h\to 0^+} 0 = 0$$

Since $f'(c)$ exists, then:

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^+} \frac{f(c+h)-f(c)}{h} \leq 0$$

$f'(c) \leq 0$

For h<0, we divide (1) by h, $\frac{f(c+h)-f(c)}{h} \geq 0$

And taking the limit as $h \to 0^-$:

$$\lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq \lim_{h\to 0^-} 0 = 0$$

Since $f'(c)$ exists, then:

$$f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} = \lim_{h\to 0^-} \frac{f(c+h)-f(c)}{h} \geq 0$$

$f'(c) \geq 0$

So we've shown that $f'(c) \leq 0$ and $f'(c) \geq 0$

Thus $f'(c) = 0$