MATH 138 Midterm

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7.1 - Integration by Parts

Derived from the relationship in the
Product Rule for differentiation, integration by parts relies on the following:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Or alternatively

$$\int u\,dv = uv - \int v\,du$$

The most useful application is when there is a product of functions, the key is to let one function equal *u* and the other to equal *dv* – not v. You then integrate *dv* by itself to get *v*.

In some cases, especially when there are trigonometric functions involved, integration by parts is needed again on $\int v\,du$, and part of the result will be the original integral. In this case, it can be isolated moved to the other side, and continue solving as usual.

For definite integrals

$$\int_a^b u\,dv = uv\Big]_a^b - \int_a^b v\,du$$

7.2 - Trigonometric Integration

Here, a combination of trigonometric identities and substitution rule techniques are used to solve integrals with trigonometric functions.

Note usually one of the other functions is kept as a term so when the substitution is made, it can be cancelled when the variable of integration is changed. Ie. If u = sinx, du = cos x dx, so 1 cos x should be kept so that du can be substituted for dx without introducing another experession.

For $\sin x$ and $\cos x$

If *(The power of)*	**Then**	**Substitute**
sin x = odd	Keep one sin x term, and convert the rest into (1-$\cos^2$x).	u = cos x
cos x = odd	Keep one sin x term, and convert the rest into (1-$\sin^2$x).	u = sin x
sin x and cos x = even (or 0)	Use the half angle identities to convert each $\sin^2$ x and $\cos^2$x into cos (2x) or sin (2x)	Solve as needed

For sec x and tan x

If *(Power of)*	**And there is**	**Then**	**Substitute**
tan x = odd	any sec x	Keep one tan x term and convert the rest into (1 -$\sec^2$ x).	u = sec x
sec x = even	At least 2 sec x	Keep one $\sec^2$ x term, and convert the rest into (1-$\tan^2$x).	u = tan x
tan x = even	sec x = odd	Convert $\tan^2$x into (1 -$\sec^2$ x)	Integrate by parts

For all other cases, the following can be used, in conjunction with other techniques

$$\int \sec x \, dx = \ln|\sec x + \tan x| + c \qquad \int \tan x \, dx = \ln|\sec x| + c$$

$$\int \csc x \, dx = \ln|\csc x - \cot x| + c \qquad \int \cot x \, dx = \ln|\sin x| + c$$

7.3 - Trig Substitution

If	Substitute	Square Root Becomes
$\sqrt{x^2+a^2}$	$x = a \tan\theta$	$a \sec\theta$
$\sqrt{x^2-a^2}$	$x = a \sec\theta$	$a \tan\theta$
$\sqrt{a^2-x^2}$	$x = a \sin\theta$	$a \cos\theta$

These substitutions allow the expression under the square root to be of only one term, and so the square root can be gotten rid of that way. Then solve the trigonometric expression as usual.

When using these substitutions, at the substitution step, remember to restrict the domain of the function so it is one-to-one. This step is a technicality and your values should not affect the answer.

7.4 – Integration by Partial Fractions

To integrate rational functions, split it up into a sum of partial fractions. Make sure the degree of the numerator is less than the degree of the denominator. If not, perform long division and integrate the quotient normally and apply the below to the remainder.

First, factor the denominator so all the terms are either in the form (ax + b) or (ax^2 + bx + c). Then split the fraction, with the denominator being a factor and the numerator being an arbitrary variable.

$$\frac{mx+n}{rx^2+sx+t} = \frac{A}{ax+b} + \frac{B}{cx+d}$$

Then, cross multiply all the fractions together and rearrange in terms of x

$$\frac{mx+n}{rx^2+sx+t} = \frac{A(cx+d)+B(ax+b)}{rx^2+sx+t} = \frac{(cA+aB)x+(dA+bB)}{rx^2+sx+t}$$

This gives a system of equations for A and B to solve for, in this case, m = cA + aB and n = dA + bB, from comparing each term of the original numerator to the equivalent term of the new fraction. Then, plug A and B back into the separated equation and integrate.

Notes: If there is more than one occurrence of the denominator $(ax+b)^n$, the partial fraction of that part is

$$\frac{mx+n}{(ax+b)^n} = \frac{A}{ax+b} + \frac{B}{(ax+b)^2} + \cdots + \frac{Z}{(ax+b)^n}$$

If there is a quadratic factor that cannot be further factored, the same approach can be used

$$\frac{mx+n}{rx^3+sx^2+x+t} = \frac{A}{ax+b} + \frac{Bx+C}{cx^2+dx+e}$$

And solve for A, B, C etc. However, at the end, there will still be a term in which there is a quadratic in the denominator to integrate, and that is best done by completing the square in the denominator, and then integrate that term by parts.

Some useful formulas to know when using integration by parts:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

$$\int \frac{1}{(ax+b)^n}\,dx = \frac{1}{a(1-n)}\frac{1}{(ax+b)^{n-1}} + c$$

$$\int \frac{1}{ax^2+b}\,dx = \frac{1}{2a}\ln|ax^2+b| + c$$

$$\int \frac{1}{(ax^2+b)^n}\,dx = \frac{1}{2a(1-n)}\frac{1}{(ax^2+b)^{n-1}} + c$$

$$\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + c$$

6.2 – Volumes by Disks

If given an area on the x-y plane, integration in 2-D can be done by dividing the area into infinitely many rectangles, and taking the sum of the areas of those rectangles, where each rectangle has an area of x(h(x)), where h(x) is the height of the area at point x (equal to f(x) – g(x) when the area is enclosed by two functions) and x approaches zero. Essentially, vertical lines of length h(x). Therefore, the area of the whole area is $\int_a^b h(x)dx$

In 3D, for some solid object, the volume can be thought of in the same way, except instead of lines, it is a sum of infinitely many cross-sections. (For example, the volume of a cylinder is the sum of infinitely many circular slices.) Therefore, the volume of a solid is

$$\int_a^b A(x)dx$$

Where A(x) is a function that defines the area of a cross sectional piece of the solid.

Note that this area does not have to be a continuous area and can be defined in terms of different variables. When you have questions that say “Given an area bounded by f(x) and g(x), what is the volume of the solid when it is rotated about some line”, the cross-section piece is often donut/washer shaped.

6.3 – Volumes by Cylindrical Shells

Similar to the above, except this time, imagine dividing a shape into infinitely many, different sized cylindrical shells. The volume is therefore the sum of infinitely many cylindrical shells. (Refer to textbook for diagrams). The volume of one shell is circumference X height X width, but since there are infinitely many shells, each one has a width of dx. Therefore, the volume of the entire solid is

$$\int_a^b (circumference\ of\ shell)(height\ of\ shell)dx$$

The circumference will be twice the distance from the center (radius). A shape rotated around the y axis will have a radius x for each shell, so the circumference will be $2\pi x$. The height can be found in a similar way as shown above. (Ie. If the solid was formed by rotating the region bounded by $y = -x^2 + 2$ and $y = 1$ around the y-axis, circumference would be $2\pi x$ and the height would be $(-x^2 + 2) – 1$.) With this technique, there's no need to think about whether or not the solid has a hol or not, what shape it is, etc, although it still helps to visualize it.

7.7 – Approximate Integration

Midpoint Rule

$$M_n = \left(\frac{b-a}{n}\right)[f(x_1) + f(x_2) + \cdots + f(x_n)]$$

Where x_n is the midpoint of $[x_{i-1}, x_i]$.

Trapezoid Rule

$$T_n = \left(\frac{b-a}{2n}\right)[f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)]$$

Where $x_i = a + i\,[(b-a)/n]$

Simpson's Rule

$$S_n = \left(\frac{b-a}{3n}\right)[f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)]$$

Where n is even

The error bound E is the difference between the integral and the approximation, and is expressed as an upper bound. (For example, I = $T_n - E_T$, where I is the definite integral)

For midpoint and trapezoid rules, f’’(x) <= K for [a,b], K is the upper bound of the second derivative. For Simpson’s rule, f’’’’(x) <= K on [a,b], K is the upper bound of the fourth derivative.

$$|E_T| \leq \frac{K(b-a)^3}{12n^2}$$

$$|E_M| \leq \frac{K(b-a)^3}{24n^2}$$

$$|E_S| \leq \frac{K(b-a)^5}{180n^4}$$

Note these are upper bounds, and even though in general one rule is better than another in terms of error, this is not necessarily true in all situations.

7.8 – Improper Integrals

Sometimes integrals may not be on a finite interval or may not be continuous on an interval. To deal with them, we can use limits.

Type 1 - For integrals evaluated at an endpoint at infinity, if the interval exists throughout the interval, it can be evaluated as

$$\int_a^{\infty} f(x)dx = \lim_{t\to\infty}\int_a^t f(x)dx \quad or \quad \int_{-\infty}^{b} f(x)dx = \lim_{t\to-\infty}\int_t^b f(x)dx$$

If the limit exists, the integral is convergent, otherwise, it is divergent.

Type 2 – For integrals that are not defined at an endpoint,

Continuous on [a, b)

$$\int_a^b f(x)dx = \lim_{t\to b-}\int_a^t f(x)dx$$

Continuous on (a, b]

$$\int_a^b f(x)dx = \lim_{t \to a+} \int_t^b f(x)dx$$

Discontinuous at c (where both integrals are convergent)

$$\int_a^b f(x)dx = \lim_{t \to c-} \int_a^t f(x)dx + \lim_{t \to c+} \int_t^b f(x)dx$$

Note that we have to look at an integral before solving it to see if it is improper or not, and choose the appropriate technique to use. Using regular techniques on an improper integral will not yield the correct answer.

Some Useful Formulas to Know

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C (n \neq 1)$$

$$\int \frac{1}{n} \, dx = \ln|x| + C$$

$$\int e^x \, dx = e^x + C$$

$$\int a^x \, dx = \frac{a^x}{\ln a} + C$$

$$\int \sin x \, dx = -\cos x + C$$

$$\int \cos x \, dx = \sin x + C$$

$$\int \sec^2 x \, dx = \tan x + C$$

$$\int \csc^2 x \, dx = -\cot x + C$$

$$\int \sec x \tan x \, dx = \sec x + C$$

$$\int \csc x \cot x \, dx = -\csc x + C$$

$$\int \frac{1}{1+x^2} \, dx = \arctan x + C$$

$$\int \frac{1}{\sqrt{1-x^2}} \, dx = \arcsin x + C$$

$$\int \sinh x \, dx = \cosh x + C$$

$$\int \cosh x \, dx = \sinh x + C$$

EXAMPLE QUESTIONS AND SOLUTIONS

Originally prepared by Vincent Chan

EXAMPLE **1.** Calculate

$$\int \frac{x^2}{(1+x^2)(\arctan x - x)}\,dx.$$

SOLUTION**.** The strategy is not immediately obvious. There is no clear simplification to be made, no trigonometric functions to substitute, no radicals, and no rational functions. However, notice that the derivative of $\arctan x$ will yield a $\frac{1}{1+x^2}$ term, so we try for a direct antiderivative. To get $(\arctan x - x)$ in the denominator, we use a logarithm:

$$\ln(\arctan x - x).$$

Taking the derivative, we see by chain rule that

$$\begin{aligned}\frac{d}{dx}\ln(\arctan x - x) &= \frac{1}{\arctan x - x}\left(\frac{1}{1+x^2} - 1\right)\\ &= \frac{1-1-x^2}{(1+x^2)(\arctan x - x)} = \frac{-x^2}{(1+x^2)(\arctan x - x)}.\end{aligned}$$

Thus, to get to our integrand, we require a negative sign:

$$\int \frac{x^2}{(1+x^2)(\arctan x - x)}\,dx = -\ln(\arctan x - x) + C.$$

EXAMPLE **2.** Calculate

$$\int e^{a\theta}\cos b\theta\,d\theta.$$

SOLUTION**.** We have a product of functions, so we try integration by parts:

$$u = e^{a\theta} \qquad dv = \cos b\theta\,d\theta$$
$$du = ae^{a\theta}\,d\theta \qquad v = \frac{1}{b}\sin b\theta$$

Then

$$\int e^{a\theta}\cos b\theta\,d\theta = \frac{1}{b}e^{a\theta}\sin b\theta - \frac{a}{b}\int e^{a\theta}\sin b\theta\,d\theta$$

The integral is no easier than what we started with, but we can use integration by parts a second time, to get a recurring integral:

$$u = e^{a\theta} \qquad dv = \sin b\theta\,d\theta$$
$$du = ae^{a\theta}\,d\theta \qquad v = -\frac{1}{b}\cos b\theta$$

Then

$$\int e^{a\theta}\cos b\theta\,d\theta = \frac{1}{b}e^{a\theta}\sin b\theta - \frac{a}{b}\left[-\frac{1}{b}e^{a\theta}\cos b\theta + \frac{a}{b}\int e^{a\theta}\cos b\theta\right]$$

We solve for our original integral using simple algebra now.

$$\frac{a^2+b^2}{b^2}\int e^{a\theta}\cos b\theta\,d\theta = \frac{1}{b}e^{a\theta}\sin b\theta + \frac{a}{b^2}e^{a\theta}\cos b\theta + C$$
$$\implies \qquad \int e^{a\theta}\cos b\theta\,d\theta = \frac{e^{a\theta}}{a^2+b^2}(b\sin b\theta + a\cos b\theta) + C.$$

EXAMPLE **3.** Calculate

$$\int e^x \sin(x+e^x)\sin(x-e^x)\,dx.$$

SOLUTION. We begin by simplifying the integrand. The similarity of the sine factors leads us to use the identity

$$\sin(A)\sin(B) = \frac{1}{2}[\cos(A-B)-\cos(A+B)].$$

Then

$$\begin{aligned}\int e^x \sin(x+e^x)\sin(x-e^x)\,dx &= \frac{1}{2}\int e^x[\cos(2e^x)-\cos(2x)]\,dx\\ &= \frac{1}{2}\int e^x\cos(2e^x)\,dx - \frac{1}{2}\int e^x\cos(2x)\,dx.\end{aligned}$$

The first integral can be completed by inspection or by using the substitution $u = e^x$:

$$\int e^x\cos(2e^x)\,dx = \frac{1}{2}\sin(2e^x) + C.$$

The second integral is a special case of Example 2, with $a = 1$ and $b = 2$:

$$\int e^x\cos(2x)\,dx = \frac{e^x}{1+4}(2\sin(2x)+\cos(2x)) + C.$$

Together,

$$\begin{aligned}\int e^x \sin(x+e^x)\sin(x-e^x)\,dx &= \frac{1}{2}\frac{1}{2}\sin(2e^x) - \frac{1}{2}\frac{e^x}{1+4}(2\sin(2x)+\cos(2x)) + C\\ &= \frac{1}{4}\sin(2e^x) - \frac{e^x}{10}(2\sin(2x)+\cos(2x)) + C.\end{aligned}$$

EXAMPLE **4.** Calculate

$$\int_{25}^{64}\frac{\sqrt{\sqrt{x}-4}-1}{x(\sqrt{x}-4)^2}\,dx.$$

SOLUTION. We have a radical, we we try the substitution $u = \sqrt{\sqrt{x}-4}$. Then $x = (u^2+4)^2$, so that $dx = 2(u^2+4)(2u)\,du$, and $(\sqrt{x}-4)^2 = u^4$. To alter the bounds, when $x = 25$, then $u = \sqrt{\sqrt{25}-4} = 1$ and when $x = 64$, then $u = \sqrt{\sqrt{64}-4} = 2$. We thus have

$$\begin{aligned}\int_{25}^{64}\frac{\sqrt{\sqrt{x}-4}-1}{x(\sqrt{x}-4)^2}\,dx &= \int_1^2 \frac{u-1}{(u^2+4)^2u^4}4(u^2+4)u\,du\\ &= 4\int_1^2\frac{u-1}{u^3(u^2+4)}\,du.\end{aligned}$$

Since we have a rational function, we can try to apply partial fractions. Write

$$\frac{u-1}{u^3(u^2+4)} = \frac{A}{u} + \frac{B}{u^2} + \frac{C}{u^3} + \frac{Du+E}{u^2+4},$$

which, after multiplying by $u^3(u^2+4)$, yields

$$\begin{aligned} u-1 &= Au^2(u^2+4) + Bu(u^2+4) + C(u^2+4) + (Du+E)u^3 \\ &= (A+D)u^4 + (B+E)u^3 + (4A+C)u^2 + (4B)u + (4C). \end{aligned}$$

Comparing coefficients gives

$$A+D=0, \qquad B+E=0, \qquad 4A+C=0, \qquad 4B=1, \qquad 4C=-1.$$

Solving,

$$A=\frac{1}{16}, \qquad B=\frac{1}{4}, \qquad C=-\frac{1}{4}, \qquad D=-\frac{1}{16}, \qquad E=-\frac{1}{4}.$$

Then,

$$\begin{aligned} \int_{25}^{64} \frac{\sqrt{\sqrt{x}-4}-1}{x(\sqrt{x}-4)^2}\,dx &= 4\int_1^2 \frac{\frac{1}{16}}{u} + \frac{\frac{1}{4}}{u^2} + \frac{-\frac{1}{4}}{u^3} + \frac{-\frac{1}{16}u-\frac{1}{4}}{u^2+4}\,du \\ &= \frac{1}{4}\int_1^2 \frac{du}{u} + \int_1^2 \frac{du}{u^2} - \int_1^2 \frac{du}{u^3} - \frac{1}{4}\int_1^2 \frac{u}{u^2+4}\,du - \int_1^2 \frac{du}{u^2+4} \\ &= \frac{1}{4}\ln(u)\Big|_1^2 - \frac{1}{u}\Big|_1^2 + \frac{1}{2u^2}\Big|_1^2 - \frac{1}{8}\ln(u^2+4)\Big|_1^2 - \frac{1}{2}\arctan\left(\frac{u}{2}\right)\Big|_1^2 \\ &= \frac{1}{4}\ln(2) - \frac{1}{2} + 1 + \frac{1}{8} - \frac{1}{2} - \frac{1}{8}\ln(8) + \frac{1}{8}\ln(5) - \frac{1}{2}\arctan(1) + \frac{1}{2}\arctan\left(\frac{1}{2}\right) \\ &= \frac{1}{4}\ln(2) + \frac{1}{8} - \frac{3}{8}\ln(2) + \frac{1}{8}\ln(5) - \frac{\pi}{8} + \frac{1}{2}\arctan\left(\frac{1}{2}\right) \\ &= \frac{1-\pi}{8} + \frac{1}{8}\ln\left(\frac{5}{2}\right) + \frac{1}{2}\arctan\left(\frac{1}{2}\right). \end{aligned}$$

EXAMPLE **5.** Calculate $\displaystyle\int_a^1 \arctan(x^{-1})\,dx$, for $0 < a < 1$.

SOLUTION. The strategy is not immediately obvious. There is no clear simplification to be made, no trigonometric functions, no radicals, and no rational functions. We might be able to use integration by parts, using the "1" trick. (Notice that the integrand is continuous for $0 < a < 1$). We have

$$u = \arctan(x^{-1}) \qquad\qquad dv = dx$$

$$du = \frac{1}{1+x^{-2}}\left(-\frac{1}{x^2}\right)dx = -\frac{1}{x^2+1}\,dx \qquad\qquad v = x$$

Then

$$\begin{aligned} \int_a^1 \arctan(x^{-1})\,dx &= x\arctan(x^{-1})\ \big|_a^1 + \int_a^1 \frac{x}{x^2+1}\,dx \\ &= \arctan(1) - a\arctan(a^{-1}) + \frac{1}{2}\ln(x^2+1)\ \big|_a^1 \\ &= \frac{\pi}{4} - a\arctan(a^{-1}) + \frac{1}{2}\ln(2) - \frac{1}{2}\ln(a^2+1). \end{aligned}$$

EXAMPLE **6.** Calculate

$$\int \frac{\sin^3 t + \cos^3 t}{\cos t(2+\sin(2t))^2}\, dt.$$

SOLUTION**.** We can attempt to simplify the integrand, using the double angle formula for sine, to get

$$\int \frac{\sin^3 t + \cos^3 t}{\cos t(2+\sin(2t))^2}\, dt = \int \frac{\sin^3 t + \cos^3 t}{\cos t(2+2\sin t\cos t)^2}\, dt = \frac{1}{4}\int \frac{\sin^3 t + \cos^3 t}{\cos t(1+\sin t\cos t)^2}\, dt.$$

There is no obvious antiderivative, nor can we immediately use a substitution like $x = \sin(t)$ or $x = \cos(t)$ since we do not have an extra term to account for the change of variables. However, notice that we can get a $\tan(t)$ term appearing in the denominator if we divide through by $\cos^5 t$, yielding a $\sec^2 t$ term in the numerator. We then save a copy of $\sec^2 t$, and express the remainder in terms of $\tan t$:

$$\begin{aligned}\int \frac{\sin^3 t + \cos^3 t}{\cos t(2+\sin(2t))^2}\, dt &= \frac{1}{4}\int \frac{\frac{\sin^3 t+\cos^3 t}{\cos^3 t}\frac{1}{\cos^2 t}}{(\frac{1+\sin t\cos t}{\cos^2 t})^2}\, dt \\ &= \frac{1}{4}\int \frac{(\tan^3 t+1)\sec^2 t}{(\sec^2 t+\tan t)^2}\, dt \\ &= \frac{1}{4}\int \frac{(\tan^3 t+1)\sec^2 t}{(\tan^2 t+\tan t+1)^2}\, dt.\end{aligned}$$

From here it is clear that a good substitution would be $x = \tan t$, with $dx = \sec^2 t\, dt$. Then

$$\int \frac{\sin^3 t + \cos^3 t}{\cos t(2+\sin(2t))^2}\, dt = \frac{1}{4}\int \frac{x^3+1}{(x^2+x+1)^2}\, dx.$$

We can use a partial fraction decomposition now, say

$$\frac{x^3+1}{(x^2+x+1)^2} = \frac{Ax+B}{x^2+x+1} + \frac{Cx+D}{(x^2+x+1)^2}.$$

which, after multiplying by $(x^2+x+1)^2$, yields

$$\begin{aligned}x^3+1 &= (Ax+B)(x^2+x+1) + (Cx+D) \\ &= Ax^3 + (A+B)x^2 + (A+B+C)x + (B+D).\end{aligned}$$

Comparing coefficients gives

$$A = 1, \qquad A+B = 0, \qquad A+B+C = 0, \qquad B+D = 1.$$

Solving,

$$A = 1, \qquad B = -1, \qquad C = 0, \qquad D = 2.$$

Then,

$$\int \frac{\sin^3 t + \cos^3 t}{\cos t(2+\sin(2t))^2}\, dt = \frac{1}{4}\int \frac{x-1}{x^2+x+1} + \frac{2}{(x^2+x+1)^2}\, dx$$

For the first integral, we complete the square and make a shift $u = x + \frac{1}{2}$:

$$\begin{aligned}\int \frac{x-1}{x^2+x+1}\,dx = \int \frac{(x+\frac{1}{2})-\frac{3}{2}}{(x+\frac{1}{2})^2+\frac{3}{4}}\,dx = \int \frac{u-\frac{3}{2}}{u^2+\frac{3}{4}}\,du \\ = \int \frac{u}{u^2+\frac{3}{4}}\,du - \frac{3}{2}\int \frac{1}{u^2+\frac{3}{4}}\,du \\ = \frac{1}{2}\ln(u^2+\tfrac{3}{4}) - \frac{3}{2}\frac{1}{\frac{\sqrt{3}}{2}}\arctan\left(\frac{u}{\frac{\sqrt{3}}{2}}\right) \\ = \frac{1}{2}\ln(x^2+x+1) - \sqrt{3}\arctan\left(\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right) \\ = \frac{1}{2}\ln(x^2+x+1) - \sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right).\end{aligned}$$

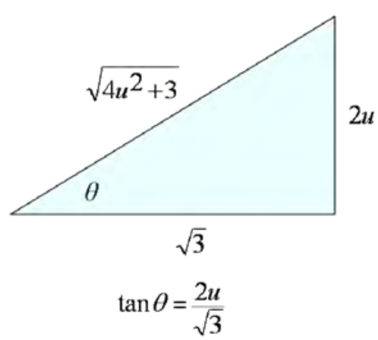
For the second integral, we complete the sqaure, make a shift $u = x + \frac{1}{2}$, and use the trigonometric substitution $u = \frac{\sqrt{3}}{2}\tan\theta$. We have $du = \frac{\sqrt{3}}{2}\sec^2\theta$, and we use the identity $\tan^2\theta + 1 = \sec^2\theta$.

$$\begin{aligned}\int \frac{1}{(x^2+x+1)^2}\,dx = \int \frac{1}{(u^2+\frac{3}{4})^2}\,du = \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{(\frac{3}{4}\tan^2\theta+\frac{3}{4})^2}\,d\theta \\ = \int \frac{\frac{\sqrt{3}}{2}\sec^2\theta}{\frac{9}{16}(\tan^2\theta+1)^2}\,d\theta = \frac{\sqrt{3}}{2}\frac{16}{9}\int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta \\ = \frac{8\sqrt{3}}{9}\int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta = \frac{8\sqrt{3}}{9}\int \cos^2\theta\,d\theta.\end{aligned}$$

Since we have an even power of sine and cosine, we try to use the double angle formula, $\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$.

$$\int \frac{1}{(x^2+x+1)^2}\,dx = \frac{8\sqrt{3}}{9}\frac{1}{2}\left(\int d\theta + \int \cos 2\theta\,d\theta\right) = \frac{4\sqrt{3}}{9}\left(\theta + \frac{1}{2}\sin 2\theta\right).$$

To return to our original variable u, we need to construct the triangle defined by $u = \frac{\sqrt{3}}{2}\tan\theta$.



We see that

$$\sin\theta = \frac{2u}{\sqrt{4u^2+3}}, \qquad \cos\theta = \frac{\sqrt{3}}{\sqrt{4u^2+3}}$$

so that

$$\sin 2\theta = 2\sin\theta\cos\theta = 2\frac{2u}{\sqrt{4u^2+3}}\frac{\sqrt{3}}{\sqrt{4u^2+3}} = \frac{4u\sqrt{3}}{4u^2+3}.$$

Then

$$\begin{aligned}\int \frac{1}{(x^2+x+1)^2}\,dx &= \frac{4\sqrt{3}}{9}\left(\arctan\left(\frac{2u}{\sqrt{3}}\right)+\frac{1}{2}\frac{4u\sqrt{3}}{4u^2+3}\right)\\ &= \frac{4\sqrt{3}}{9}\arctan\left(\frac{2(x+\frac{1}{2})}{\sqrt{3}}\right)+\frac{2\sqrt{3}\sqrt{3}}{9}\frac{4(x+\frac{1}{2})\sqrt{3}}{4(x+\frac{1}{2})^2+3}\\ &= \frac{4\sqrt{3}}{9}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{4(2x+1)}{3(4x^2+4x+4)}\\ &= \frac{4\sqrt{3}}{9}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{(2x+1)}{3(x^2+x+1)}.\end{aligned}$$

We have

$$\begin{aligned}\int \frac{\sin^3 t+\cos^3 t}{\cos t(2+\sin(2t))^2}\,dt &= \frac{1}{4}\left(\frac{1}{2}\ln(x^2+x+1)-\sqrt{3}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{8\sqrt{3}}{9}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{2(2x+1)}{3(x^2+x+1)}\right)\\ &= \frac{1}{4}\left(\frac{1}{2}\ln(x^2+x+1)-\frac{\sqrt{3}}{9}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{2(2x+1)}{3(x^2+x+1)}\right)\\ &= \frac{1}{8}\ln(x^2+x+1)-\frac{\sqrt{3}}{36}\arctan\left(\frac{2x+1}{\sqrt{3}}\right)+\frac{(2x+1)}{6(x^2+x+1)}\\ &= \frac{1}{8}\ln(\tan^2 t+\tan t+1)-\frac{\sqrt{3}}{36}\arctan\left(\frac{2\tan t+1}{\sqrt{3}}\right)+\frac{(2\tan t+1)}{6(\tan^2 t+\tan t+1)}+K\end{aligned}$$

where K is a constant.

EXAMPLE **7.** Calculate

$$\int_{-3}^{3}\frac{dx}{x+2}.$$

SOLUTION**.** By inspection or using the substitution $u=x+2$, we get

$$\int_{-3}^{3}\frac{dx}{x+2}\,dx = \ln|x+2|\big|_{-3}^{3} = \ln(5)-\ln(1)=\ln(5).$$

Wait! The Fundamental Theorem of Calculus only gives this relationship between integrals and derivatives if the integrand is continuous on the closed interval $[-3,3]$, which is not true here. This is a trick question: we must use the definition of improper integrals to attempt this problem, which is coming up next.

EXERCISE **1.** Calculate

$$\int_0^1 \ln(x^2+1)\,dx.$$

SOLUTION**.** The strategy is not immediately obvious. There is no clear simplification to be made, no trigonometric functions, no radicals, and no rational functions. We might be able to use integration by parts, using the "1" trick.

$$u = \ln(x^2+1) \qquad dv = dx$$
$$du = \frac{2x}{x^2+1}\,dx \qquad v = x$$

Then

$$\int_0^1 \ln(x^2+1)\,dx = x\ln(x^2+1)\,\big|_0^1 - 2\int_0^1 \frac{x^2}{x^2+1}\,dx$$

Notice $\frac{x^2}{x^2+1}$ is a rational function, so we can try using integration by parts. By our algorithm, we must first apply polynomial division since the degree of the numerator is at least the degree of the denominator. We get

$$\frac{x^2}{x^2+1} = 1 - \frac{1}{x^2+1}.$$

Notice the denominator is already factored into 1 term, so we have the partial fraction decomposition.

$$\begin{aligned}\int_0^1 \ln(x^2+1)\,dx &= x\ln(x^2+1)\,\big|_0^1 - 2\int_0^1 1 - \frac{1}{x^2+1}\,dx\\ &= \ln(2) - 0 - 2(x)\,\big|_0^1 + 2(\arctan x)\,\big|_0^1\\ &= \ln(2) - 2(1-0) + 2\left(\frac{\pi}{4} - 0\right) = \ln(2) - 2 + \frac{\pi}{2}.\end{aligned}$$

EXERCISE **2.** Calculate

$$\int_1^2 \frac{(\ln x)^2}{x^5}\,dx.$$

SOLUTION**.** This is a simple integration by parts:

$$u = (\ln x)^2 \qquad dv = \frac{1}{x^5}dx$$
$$du = 2(\ln x)\frac{1}{x}\,dx \qquad v = -\frac{1}{4x^4}$$

so that

$$\int_1^2 \frac{(\ln x)^2}{x^5}\,dx = -\left.\frac{(\ln x)^2}{4x^4}\right|_1^2 + \frac{2}{4}\int_1^2 \frac{\ln x}{x^5}\,dx$$

Using another integration by parts,

$$u = \ln x \qquad dv = \frac{1}{x^5}dx$$
$$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{4x^4}$$

so that

$$\begin{aligned}\int_1^2 \frac{(\ln x)^2}{x^5}\,dx &= -\left.\frac{(\ln x)^2}{4x^4}\right|_1^2 + \frac{1}{2}\left(-\left.\frac{\ln x}{4x^4}\right|_1^2 + \frac{1}{4}\int_1^2 \frac{1}{x^5}\,dx\right)\\ &= -\frac{2\ln 2}{64} + \frac{1}{2}\left(-\frac{\ln 2}{64} + \frac{1}{4}\left.\left(-\frac{1}{4x^4}\right)\right|_1^2\right)\\ &= -\frac{(\ln 2)^2}{64} - \frac{\ln 2}{128} + \frac{1}{8}\left(-\frac{1}{64} + \frac{1}{4}\right)\\ &= \frac{15}{512} - \frac{\ln 2}{128} - \frac{(\ln 2)^2}{64}.\end{aligned}$$

EXERCISE **3.** Calculate

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx$$

in two ways, by making the substitution $u = 1 - x^2$ and by a suitable trigonometric substitution.

SOLUTION**.** Making the substitution $u = 1 - x^2$, we get $du = -2x\,dx$ and $x^2 = 1 - u^2$. Then

$$\begin{aligned}\int \frac{x^3}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{1-u}{\sqrt{u}}\,du &= \frac{1}{2}\int u^{1/2} - u^{-1/2}\,du\\ &= \frac{1}{2}\left(\frac{3}{2}u^{3/2} - 2u^{1/2}\right) + C = \frac{\sqrt{u}}{3}(u-3) + C\\ &= \frac{\sqrt{1-x^2}}{3}(1-x^2-3) + C = -\frac{\sqrt{1-x^2}}{3}(2+x^2) + C.\end{aligned}$$

Alternatively, we could make the trigonometric substitution $x = \sin\theta$ so that $dx = \cos\theta\,d\theta$. Then

$$\begin{aligned}\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{\sin^3\theta\cos\theta}{\sqrt{1-\sin^2\theta}}\,d\theta &= \int \sin\theta\sin^2\theta\,d\theta\\ &= \int \sin\theta(1-\cos^2\theta)\,d\theta.\end{aligned}$$

Now we make the substitution $u = \cos\theta$ so that $du = -\sin\theta\,d\theta$.

$$\begin{aligned}\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int -(1-u^2)\,du &= -u + \frac{1}{3}u^3 + C\\ &= \frac{1}{3}\cos^3\theta - \cos\theta + C = \frac{\cos\theta}{3}(\cos^\theta - 3) + C.\end{aligned}$$

To return to our original variable, we must construct the triangle generated by $\sin\theta = \frac{x}{1}$, yielding $\cos\theta = \sqrt{1-x^2}$. Then

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \frac{\sqrt{1-x^2}}{3}(1-x^2-3) + C = -\frac{\sqrt{1-x^2}}{3}(2+x^2) + C.$$

As expected, our answers coincide.

EXERCISE **4.** Calculate

$$\int \frac{dx}{\sqrt{4x^2 - 4x - 3}}\, dx.$$

SOLUTION. Resist the urge to factor $4x^2 - 4x - 3$ and instead complete the square:

$$4x^2 - 4x - 3 = (2x-1)^2 - 4.$$

Make the substitution $u = 2x - 1$ so that $du = 2x$, and so

$$\int \frac{dx}{\sqrt{4x^2 - 4x - 3}}\, dx = \frac{1}{2}\int \frac{du}{\sqrt{u^2 - 4}}\, dx.$$

Now, we are in a position to use the trigonometric substitution $u = 2\sec\theta$, so that $du = 2\sec\theta\tan\theta\, d\theta$, and so

$$\begin{aligned}\int \frac{dx}{\sqrt{4x^2 - 4x - 3}}\, dx &= \frac{1}{2}\int \frac{2\sec\theta\tan\theta}{\sqrt{4\sec\theta^2 - 4}}\, d\theta \\ &= \frac{1}{2}\int \sec\theta\, d\theta = \frac{1}{2}\ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

To return to our variable u, we must construct the triangle generated by $\sec\theta = \frac{u}{2}$, yielding $\tan\theta = \frac{1}{2}\sqrt{u^2-4}$. Then

$$\int \frac{dx}{\sqrt{4x^2 - 4x - 3}}\, dx = \frac{1}{2}\ln|\tfrac{1}{2}u + \tfrac{1}{2}\sqrt{u^2-4}| + C = \frac{1}{2}\ln|u + \sqrt{u^2-4}| + \ln(\tfrac{1}{2}) + C = \frac{1}{2}\ln|u + \sqrt{u^2-4}| + C,$$

where we absorb the constant $\ln(\frac{1}{2})$ into C. Finally, converting back to our original variable x gives

$$\int \frac{dx}{\sqrt{4x^2 - 4x - 3}}\, dx = \frac{1}{2}\ln|2x - 1 + \sqrt{4x^2 - 4x - 3}| + C.$$

EXAMPLE **8.** Determine if the following integral converges, and if so, find its value.

$$\int_{-\infty}^{0} e^x \sin(x+e^x)\sin(x-e^x)\,dx.$$

SOLUTION. By definition, if the integral converges, then

$$\int_{-\infty}^{0} e^x \sin(x+e^x)\sin(x-e^x)\,dx = \lim_{t\to-\infty}\int_{t}^{0} e^x \sin(x+e^x)\sin(x-e^x)\,dx.$$

By Example 3,

$$\begin{aligned}\int_{-\infty}^{0} e^x \sin(x+e^x)\sin(x-e^x)\,dx &= \lim_{t\to-\infty}\left(\frac{1}{4}\sin(2e^x) - \frac{e^x}{10}(2\sin(2x)+\cos(2x))\right)\Big|_t^0\\ &= \lim_{t\to-\infty}\frac{1}{4}\sin(2) - \frac{1}{10} - \frac{1}{4}\sin(2e^t) + \frac{e^t}{10}(2\sin(2t)+\cos(2t))\\ &= \frac{1}{4}\sin(2) - \frac{1}{10}.\end{aligned}$$

DEFINITION **4.2.** Type II improper integrals.

(i) If f is continuous on $[a,b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t\to b^-}\int_a^t f(x)\,dx,$$

provided this limit exists.

(ii) If f is continuous on $(a,b]$ and is discontinuous at a, then

$$\int_a^b f(x)\,dx = \lim_{t\to a^+}\int_t^b f(x)\,dx,$$

provided this limit exists.

The improper integrals $\int_a^b f(x)\,dx$ is called *convergent* if the corresponding limit exists (finite is included in this definition) and *divergent* if the limit does not exist.

Finally, if f has a discontinuity at c, where $a<c<b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

EXAMPLE **9.** Determine if the following integral converges, and if so, find its value.

$$\int_{-3}^{3} \frac{dx}{x+2} .$$

SOLUTION. Notice $x = -2$ is a vertical asymptote of the integrand, which lies in the interval $(-3, 3)$. Then if the integral exists, by definition we have

$$\int_{-3}^{3} \frac{dx}{x+2} = \int_{-3}^{-2} \frac{dx}{x+2} + \int_{-2}^{3} \frac{dx}{x+2} .$$

By inspection or using the substitution $u = x + 2$, we get that if the integral exists, then

$$\begin{aligned}\int_{-3}^{-2} \frac{dx}{x+2} &= \lim_{t \to -2^-} \int_{-3}^{t} \frac{dx}{x+2} := \lim_{t \to -2^-} \ln|x+2|\Big|_{-3}^{t} \\ &= \lim_{t \to -2^-} (\ln|t+2| - \ln(1)) = \lim_{t \to -2^-} \ln|t+2| = -\infty.\end{aligned}$$

Thus, $\int_{-3}^{-2} \frac{dx}{x+2}\, dx$ is divergent, and hence $\int_{-3}^{3} \frac{dx}{x+2}\, dx$ is divergent.

EXERCISE **5.** Determine if the following integral converges, and if so, find its value.

$$\int_0^1 \arctan(x^{-1})\, dx$$

(Recall Example 5)

SOLUTION. By Example 5,

$$\int_a^1 \arctan(x^{-1})\, dx = \frac{\pi}{4} - a \arctan(a^{-1}) + \frac{1}{2}\ln(2) - \frac{1}{2}\ln(a^2+1).$$

Then

$$\begin{aligned}\int_0^1 \arctan(x^{-1})\, dx &= \lim_{a \to 0^+} \int_a^1 \arctan(x^{-1})\, dx \\ &= \lim_{a \to 0^+} \frac{\pi}{4} - a \arctan(a^{-1}) + \frac{1}{2}\ln(2) - \frac{1}{2}\ln(a^2+1) \\ &= \frac{\pi}{4} + \frac{1}{2}\ln(2),\end{aligned}$$

so the integral converges.

TIP. Here is a useful improper integral to remember:

$$\int_1^{\infty} \frac{1}{x^p}\, dx$$

is convergent if $p > 1$ and divergent if $p \leq 1$. I leave the proof as an exercise, although it is covered on page 511 in the textbook. Similarly,

$$\int_0^1 \frac{1}{x^p}\,dx$$

is convergent if $p < 1$ and divergent if $p \geq 1$.

Sometimes, it is difficult to explicitly solve the integral and then examine the limits. However, it is still possible to determine convergence, by comparing the function to a dominating function.

THEOREM **4.3** [COMPARISON THEOREM]. *Suppose that f and g are continuous functions such that $0 \leq f(x) \leq g(x)$ for $x \geq a$.*

(a) $\int_a^\infty g(x)\,dx$ *convergent* $\Longrightarrow \int_a^\infty f(x)\,dx$ *convergent.*

(b) $\int_a^\infty f(x)\,dx$ *divergent* $\Longrightarrow \int_a^\infty g(x)\,dx$ *divergent.*

A similar result holds for the other forms of improper integrals and their convergence/divergence.

Note that the converse is not true: for example, it is possible for $\int_a^\infty g(x)\,dx$ to be divergent but $\int_a^\infty f(x)\,dx$ to be convergent, or vice versa.

EXAMPLE **10.** Determine if $\displaystyle\int_0^2 \frac{dx}{e^{x-1}x^2}$ is convergent.

SOLUTION. On $(0,1]$, $e^{x-1} < 1$ since e^t is an increasing function, so that $e^{x-1}x < 1$ on $(0,1]$. In particular, for $0 < x \leq 1$ we have

$$\frac{1}{e^{x-1}x^2} > \frac{1}{x}.$$

We know $\int_0^1 \frac{dx}{x}$ is divergent, so by the comparison test, $\displaystyle\int_0^2 \frac{dx}{e^{x-1}x^2}$ is divergent. Notice we do not need to check what happens on $[1,2]$, by the definition of convergence of improper integrals.

EXAMPLE **11.** Determine if $\displaystyle\int_{1/2}^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

SOLUTION. Since $|\sin x| \leq 1$ for all x, we have that on $[1,\infty)$,

$$0 \leq \frac{\sin^2 x}{x^2} \leq \frac{1}{x^2}.$$

We know $\int_1^\infty \frac{1}{x^2}$ is convergent, so $\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent by the comparison test. But

$$\int_{1/2}^\infty \frac{\sin^2 x}{x^2}\,dx = \int_{1/2}^1 \frac{\sin^2 x}{x^2}\,dx + \int_1^\infty \frac{\sin^2 x}{x^2}\,dx,$$

and this first integral is simply a finite number, as it is a proper definite integral. Thus, $\int_{1/2}^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent.

EXERCISE **6.** Which of the following integrals are improper and why?

(i) $\int_0^1 \frac{\sin x}{x^2-4}\,dx$

(ii) $\int_0^2 \frac{\sin x}{x^2-4}\,dx$

(iii) $\int_{-3}^3 \frac{\sin x}{x^2-4}\,dx$

(iv) $\int_0^\infty \frac{\sin x}{x^2-4}\,dx$

SOLUTION. (i) The integrand is continuous on $[0,1]$, a finite interval, so this integral is proper.

(ii) The integrand is discontinuous at 2, and must be evaluated as a Type II improper integral.

(iii) The integrand is discontinuous at -2 and at 2, and must be evaluated as a sum of two Type II improper integrals.

(iv) The integrand is discontinuous at 2 and the integral is over an infinite interval, and must be evaluated as sum of a Type I and a Type II improper integral.

EXERCISE **7.** Find the values of p for which the integral $\displaystyle\int_0^1 x^p \ln x\,dx$ converges and evaluate the integral for those values of p.

SOLUTION. Notice there is a discontinuity at 0, we we evaluate the integral as a Type II improper integral. If $p=-1$, we get by integration by parts that for $0<t<1$,

$$\int_t^1 x^{-1}\ln x\,dx = (\ln x)(\ln x)\big|_t^1 - \int_t^1 x^{-1}\ln x\,dx,$$

so that

$$\int_t^1 x^{-1}\ln x\,dx = \tfrac{1}{2}\ (\ln x)(\ln x)\big|_t^1 = -\tfrac{1}{2}(\ln t)^2.$$

Then

$$\int_0^1 x^p \ln x\,dx = \lim_{t\to 0^+}(-\tfrac{1}{2}(\ln t)^2) = -\infty,$$

so the integral is not convergent. If $p\neq -1$, then by integration by parts,

$$\begin{aligned}\int_t^p x^{-1}\ln x\,dx &= (\ln x)\left(\frac{x^{p+1}}{p+1}\right)\Bigg|_t^1 - \frac{1}{p+1}\int_t^1 x^p\,dx \\ &= -fract^{p+1}\ln tp+1 - \frac{1}{p+1}\ \left(\frac{x^{p+1}}{p+1}\right)\Bigg|_t^1.\end{aligned}$$

If $p<-1$, then $t^{p+1}\ln t\to -\infty$, while if $p>-1$, then $t^{p+1}\ln t\to 0$. Thus the integral converges if $p>-1$, and in this case,

$$\int_0^1 x^p \ln x\,dx = \lim_{t\to 0^+}\frac{1}{p+1}\ \left(\frac{x^{p+1}}{p+1}\right)\Bigg|_t^1 = -\frac{1}{(p+1)^2}.$$

EXAMPLE **12.** Find the volume of the solid obtained by rotating the region bounded by the curves $y = x$ and $y = \sqrt{x}$ about the specified line using the washer method.

(a) About the x-axis.

(b) About the y-axis.

(c) About $x = 2$.

SOLUTION. First, we calculate the intersection points. We want $x = \sqrt{x}$, so that $x^2 = x$, and so $x = 0$ $(y = 0)$or $x = 1$ $(y = 1)$.

(a) Here, the outer function is $y = \sqrt{x}$ and the inner function is $y = x$. Then

$$V = \pi \int_0^1 \sqrt{x}^2 - x^2\,dx = \pi \int_0^1 x - x^2\,dx$$
$$= \pi \left(\frac{1}{2}x^2 - \frac{1}{3}x^3 \right)\Big|_0^1 = \pi \left(\frac{1}{2} - \frac{1}{3} \right) = \frac{\pi}{6}.$$

(b) Here, the outer function is $x = y$ and the inner function is $x = y^2$. Then

$$V = \pi \int_0^1 y^2 - (y^2)^2\,dy = \pi \int_0^1 y^2 - y^4\,dy$$
$$= \pi \left(\frac{1}{3}y^3 - \frac{1}{5}y^5 \right)\Big|_0^1 = \pi \left(\frac{1}{3} - \frac{1}{5} \right) = \frac{2\pi}{15}.$$

(c) Here, the outer function is $x = y^2$ and the inner function is $x = y$. However, the outer radius is given by $2 - y^2$ and the inner radius is given by $2 - y$. Then

$$V = \pi \int_0^1 (2 - y^2)^2 - (2 - y)^2\,dy = \pi \int_0^1 4 - 4y^2 + y^4 - 4 + 4y - y^2\,dy$$
$$= \pi \int_0^1 4y - 5y^2 + y^4\,dy = \pi \left(2y^2 - \frac{5}{3}y^3 + \frac{1}{5}y^5 \right)\Big|_0^1$$
$$= \pi \left(2 - \frac{5}{3} + \frac{1}{5} \right) = \pi \left(\frac{1}{3} + \frac{1}{5} \right) = \frac{8\pi}{15}.$$

EXAMPLE **13.** Find the volume of the solid obtained by rotating the region bounded by the curves y = x and y = px about the specified line using the cylindrical shell method.
(a) About the x-axis.

(b) About the y-axis.

(c) About $x = 2$.

SOLUTION. First, we calculate the intersection points. We want $x = \sqrt{x}$, so that $x^2 = x$, and so $x = 0$ ($y = 0$) or $x = 1$ ($y = 1$).

(a) Here, the top function is $x = y$ and the bottom function is $x = y^2$. Then

$$\begin{aligned} V &= 2\pi \int_0^1 y(y - y^2)\,dy = 2\pi \int_0^1 y^2 - y^3\,dy \\ &= 2\pi \left(\frac{1}{3}y^2 - \frac{1}{4}y^4 \right)\Big|_0^1 = 2\pi \left(\frac{1}{3} - \frac{1}{4} \right) = \frac{\pi}{6}. \end{aligned}$$

(b) Here, the top function is $y = \sqrt{x}$ and the bottom function is $y = x$. Then

$$\begin{aligned} V &= 2\pi \int_0^1 x(\sqrt{x} - x)\,dx = 2\pi \int_0^1 x^{3/2} - x^2\,dx \\ &= 2\pi \left(\frac{2}{5}x^{5/2} - \frac{1}{3}x^3 \right)\Big|_0^1 = 2\pi \left(\frac{2}{5} - \frac{1}{3} \right) = \frac{2\pi}{15}. \end{aligned}$$

(c) Here, the top function is $y = \sqrt{x}$ and the bottom function is $y = x$. However, the radius is given by $2 - x$. Then

$$\begin{aligned} V &= 2\pi \int_0^1 (2 - x)(\sqrt{x} - x)\,dx = 2\pi \int_0^1 2x^{1/2} - 2x - x^{3/2} + x^2\,dx \\ &= 2\pi \left(\frac{4}{3}x^{3/2} - x^2 - \frac{2}{5}x^{5/2} + \frac{1}{3}x^3 \right)\Big|_0^1 \\ &= 2\pi \left(\frac{4}{3} - 1 - \frac{2}{5} + \frac{1}{3} \right) = 2\pi \left(\frac{5}{3} - \frac{7}{5} \right) = \frac{8\pi}{15}. \end{aligned}$$

You may notice that the solutions we arrived at in Examples 12 and 13 are identical, which is good - we wouldn't want to have an ill-defined concept of volume.

TIP. With this in mind, you can check your answer by using the alternate method.

In the previous two examples, we were lucky that the outer/inner functions and top/bottom heights did not change. This is not always the case, so we must be careful.

EXAMPLE **14.** Find the volume of the solid obtained by rotating the region bounded by the curves $y = x$, $y = \sqrt{x}$, for $x \leq \frac{1}{4}$ about the y-axis using both the washer method and the cylindrical shell method.

SOLUTION. First, we calculate the intersection points. We want $x = \sqrt{x}$, so that $x^2 = x$, and so $x = 0$ ($y = 0$); we do not need the other intersection since we have the restriction $x \leq \frac{1}{4}$. Finally, we get intersection points between the first two curves and the line $x = \frac{1}{4}$ at $(\frac{1}{4}, \frac{1}{4})$ and $(\frac{1}{4}, \frac{1}{2})$.

If we use the washer method, the outer radius is $x = y$ and the inner radius is $x = y^2$ only for $0 \leq y \leq \frac{1}{4}$. For $\frac{1}{4} \leq y \leq \frac{1}{2}$, the outer radius is $\frac{1}{4}$ and the inner radius remains as $x = y^2$. Then

$$\begin{aligned}
V &= \pi \int_0^{1/4} y^2 - (y^2)^2\, dy + \pi \int_{1/4}^{1/2} \left(\frac{1}{4}\right)^2 - (y^2)^2\, dy \\
&= \pi \int_0^{1} y^2 - y^4\, dy + \pi \int_{1/4}^{1/2} \frac{1}{16} - y^4\, dy \\
&= \pi \left(\frac{1}{3}y^3 - \frac{1}{5}y^5\right)\Bigg|_0^{1/4} + \pi \left(\frac{1}{16}y - \frac{1}{5}y^5\right)\Bigg|_{1/4}^{1/2} \\
&= \pi \left(\frac{1}{3}\frac{1}{64} - \frac{1}{5}\frac{1}{4^5}\right) + \pi \left(\frac{1}{16}\frac{1}{2} - \frac{1}{5}\frac{1}{32} - \frac{1}{16}\frac{1}{4} + \frac{1}{5}\frac{1}{4^5}\right) = \frac{7\pi}{480}.
\end{aligned}$$

If we use the cylindrical shell method, the top height is always $y = \sqrt{x}$ and the bottom height is always $y = x$. Then

$$\begin{aligned}
V &= 2\pi \int_0^{1/4} x(\sqrt{x} - x)\, dx = 2\pi \int_0^{1/4} x^{3/2} - x^2\, dx \\
&= 2\pi \left(\frac{2}{5}x^{5/2} - \frac{1}{3}x^3\right)\Bigg|_0^{1/4} = 2\pi \left(\frac{2}{5}\frac{1}{32} - \frac{1}{3}\frac{1}{64}\right) = \frac{7\pi}{480}.
\end{aligned}$$

TIP. Sometimes, the washer method will be easier than the clyndrical shell method, and sometimes the converse is true. There are two things to look for here: an simpler integrand, or fewer partitions of the region.

EXERCISE **8.** Find the volume of the solid obtained by rotating the region bounded by the curves $y = 0$, $y = xe^{-x}$, and $x = 2$ about the y-axis.

SOLUTION. Using the washer method is very difficult, since we would need to partition the region into $0 \leq y \leq 2e^{-2}$ and $2e^{-2} \leq y \leq 1$, and we would need to find an expression of x in terms of y. Using the cylindrical shell method however, we simply need to calculate

$$2\pi \int_0^2 x(xe^{-x} - 0)\, dx = 4\pi(1 - 5e^{-x}),$$

which we can easily do using integration by parts twice.