MAT 2342 — First Midterm Exam Solutions — Version α

1. True (T) or false (F)? *Circle the correct answer next to the statement.*

   (a) If the characteristic polynomial of an $n \times n$ matrix A has fewer than n distinct real roots, then A is not diagonalizable in real numbers.

   Answer: False. A diagonalizable matrix may have roots of algebraic multiplicity greater than 1, as long as the algebraic multiplicity of each eigenvalue equals its geometric multiplicity.

   (b) If a row echelon form of a 5×5 matrix A has exactly 2 zero rows, then the general solution of the homogeneous system of linear equations with coefficient matrix A has exactly 2 parameters.

   Answer: True. The coefficient matrix has exactly 3 leading ones, so exactly 2 variables of the system are non-leading.

   (c) If the feasible region of a linear program is non-empty and bounded, then the linear program has an optimal solution.

   Answer: True (theorem from class).

   (d) The matrix A given below is diagonalizable.

$$A = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$

   Answer: False: A has eigenvalue 2 with algebraic multiplicity 2, but only one corresponding basic eigenvector (i.e. geometric multiplicity 1).

   (e) A Markov process with transition matrix P given below possesses a steady state vector.

$$P = \begin{bmatrix} 0.5 & 0.3 & 0.4 \\ 0.2 & 0.3 & 0.3 \\ 0.3 & 0.4 & 0.3 \end{bmatrix}$$

   Answer: True: $P^k > 0$ for $k = 1$, so P is a regular matrix.

In each of the following two questions, write your final answer in the answer box. No explanation is required.

2. A wolf family lives in one of the three dens, A, B, or C. Once a week, the female may move her pups from one den to another. The following is observed. If the family is staying in den A or C, then the week after they will stay in the same den with 50% probability, and move to den B with 30% probability. However, if they are staying in den B, then the week after they will stay in den B with 70% probability, and are equally likely to move to one of the other two dens. Give the transition matrix P for the Markov process described above.

$$P = \begin{bmatrix} 0.5 & 0.15 & 0.2 \\ 0.3 & 0.7 & 0.3 \\ 0.2 & 0.15 & 0.5 \end{bmatrix}$$

3. We have a sequence of real numbers $x_0, x_1, x_2, \ldots$ that satisfies the recurrence relation

$$x_{k+1} = 3x_k - 2x_{k-1} \qquad \text{for } k \geq 1.$$

The initial terms are $x_0 = 1$ and $x_1 = 4$. Set up a dynamical system of the form $V_k = AV_{k-1}$ that will allow you to solve this recurrence relation; here $V_k = \begin{bmatrix} x_k \\ x_{k+1} \end{bmatrix}$. That is, give the appropriate matrix A and initial vector V_0. *Do not solve the recurrence relation.*

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \end{bmatrix} \qquad V_0 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

4. A baker brings three kinds of breads to the Organic Farmers' Market: sourdough, rye, and gluten-free. She knows that she will be able to sell at most 60 loaves of bread in total, and that she should bring at least three times as many other loaves as gluten-free loaves. In addition, she remembers that she has a pre-order for 12 sourdough loaves (that is, she must bring at least 12 sourdough loaves). If sourdough loaves sell for \$5 a piece, rye loaves sell for \$6 a piece and gluten-free loaves sell for \$7 a piece, how many of each kind of loaves should the baker bring to the market to maximize her revenue?

   Write down a mathematical model for this problem in the form of a standard (canonical) linear program. Begin by defining your variables. Clearly indicate the objective function, the type of optimum you are seeking, and all constraints.

Variables:

$$
\begin{aligned}
x_1 &= \text{number of sourdough loaves} \\
x_2 &= \text{number of rye loaves} \\
x_3 &= \text{number of gluten-free loaves}
\end{aligned}
$$

LP in standard form:

Maximize $5x_1 + 6x_2 + 7x_3$

Subject to

$$
\begin{aligned}
x_1 + x_2 + x_3 &\leq 60 \\
-x_1 - x_2 + 3x_3 &\leq 0 \\
-x_1 &\leq -12 \\
x_1, x_2, x_3 &\geq 0
\end{aligned}
$$

Constraint $x_1 + x_2 \geq 3x_3$ is equivalent to $-x_1 - x_2 + 3x_3 \leq 0$ (standard form).

Constraint $x_1 \geq 12$ is equivalent to $-x_1 \leq -12$ (standard form).

5. Consider a simple economy with two industries whose input-output model is described by the consumption matrix

$$C = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix}.$$

Recall that c_{ij} is the number of units of product i consumed to produce one unit of product j.

Use the space below for your work (show all relevant steps!), and then write the final answers in the box.

(a) Suppose the economy manufactures 50 units of product 1 and 25 units of product 2. What is the intermediate demand for each of the two products?

Answer: 50 units for product 1, and 15 units for product 2.

(b) What final demand (for each product) can be met at this production level?

Answer: 0 units for product 1, and 10 units for product 2.

(c) How many units of each product must be manufactured to meet the final demand for 35 units of product 1 and 12 units of product 2?

Answer: 340 units of product 1, and 100 units of product 2.

Space for work:

(a) The production vector is given as $x = [50, 25]^T$. The corresponding intermediate demand is

$$Cx = \begin{bmatrix} \frac{3}{4} & \frac{1}{2} \\ \\ \frac{1}{5} & \frac{1}{5} \end{bmatrix} \begin{bmatrix} 50 \\ 25 \end{bmatrix} = \begin{bmatrix} 50 \\ 15 \end{bmatrix}.$$

(b)

$$d = x - Cx = \begin{bmatrix} 50 \\ 25 \end{bmatrix} - \begin{bmatrix} 50 \\ 15 \end{bmatrix} = \begin{bmatrix} 0 \\ 10 \end{bmatrix}.$$

(c) Solve the system $x = Cx + d$ with the given demand vector $d = [35, 12]^T$:

$$\begin{aligned} [\, I - C \mid d \,] \quad &\sim \quad \left[\begin{array}{cc|c} \frac{1}{4} & -\frac{1}{2} & 35 \\ -\frac{1}{5} & \frac{4}{5} & 12 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 140 \\ -1 & 4 & 60 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 140 \\ 0 & 2 & 200 \end{array}\right] \\ &\sim \quad \left[\begin{array}{cc|c} 1 & -2 & 140 \\ 0 & 1 & 100 \end{array}\right] \end{aligned}$$

We have a unique solution $x = [340, 100]^T$.

6. Consider a population with two age groups, senior and young. The population dynamics are described by the following equations:

$$\begin{aligned} y_k &= \frac{6}{5} y_{k-1} \\ s_k &= \frac{1}{5} y_{k-1} + \frac{2}{5} s_{k-1} \end{aligned}$$

where y_k and s_k denote the sizes of the young and senior population, respectively, after k years. The initial sizes of the young group and senior group are $y_0 = 100$ and $s_0 = 40$, respectively.

Use the space below for your work (show all relevant steps!), and then write the final answers in the box.

(a) What is the size of the senior group after 1 year?

Answer: 36

(b) Give an exact formula for the size of the senior group after k years (as a function of k).

Answer: $s_k = 25(\frac{6}{5})^k + 15(\frac{2}{5})^k$

(c) Give a formula that approximates (using an exponential function) the size of the senior group after k years for k very large.

Answer: $s_k \approx 25(\frac{6}{5})^k$

(d) Explain what happens with the size of the senior group in the long run.

Answer: The size of the senior group grows without bound (approximately as an exponential function in base $\frac{6}{5}$).

Space for work:

(a) $s_1 = \frac{1}{5} y_0 + \frac{2}{5} s_0 = \frac{1}{5} 100 + \frac{2}{5} 40 = 36$.

(b) We have a dynamical system $V_k = AV_{k-1}$ with matrix

$$A = \begin{bmatrix} \frac{6}{5} & 0 \\ \frac{1}{5} & \frac{2}{5} \end{bmatrix}$$

and initial population profile $V_0 = [100, 40]^T$. Here, of course, $V_k = [y_k, s_k]^T$. We need to find an explicit formula for V_k.

Diagonalize A first.

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - \frac{6}{5} & 0 \\ -\frac{1}{5} & \lambda - \frac{2}{5} \end{bmatrix} = \left(\lambda - \frac{6}{5}\right)\left(\lambda - \frac{2}{5}\right).$$

So the eigenvalues are $\lambda_1 = \frac{6}{5}$ and $\lambda_2 = \frac{2}{5}$.
To find a basic eigenvector for $\lambda_1 = \frac{6}{5}$, solve the system $[\frac{6}{5}I - A|0]$:

$$\left[\begin{array}{cc|c} 0 & 0 & 0 \\ -\frac{1}{5} & \frac{4}{5} & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -4 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The general solution is $[4t, t]^T$, so we may take the basic eigenvector $X_1 = [4, 1]$.
To find a basic eigenvector for $\lambda_2 = \frac{2}{5}$, solve the system $[\frac{2}{5}I - A|0]$:

$$\left[\begin{array}{cc|c} -\frac{4}{5} & 0 & 0 \\ -\frac{1}{5} & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 0 & 0 \end{array}\right].$$

The general solution is $[0, t]^T$, so we may take the basic eigenvector $X_2 = [0, 1]$.
Thus A diagonalizes as $A = PDP^{-1}$, where

$$P = \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} \frac{6}{5} & 0 \\ 0 & \frac{2}{5} \end{bmatrix}, \qquad P^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}$$

Finally,

$$\begin{aligned} V_k &= A^k V_0 = PD^kP^{-1}V_0 = \\ &= \begin{bmatrix} 4 & 0 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} (\frac{6}{5})^k & 0 \\ 0 & (\frac{2}{5})^k \end{bmatrix}\frac{1}{4}\begin{bmatrix} 1 & 0 \\ -1 & 4 \end{bmatrix}\begin{bmatrix} 100 \\ 40 \end{bmatrix} \\ &= \begin{bmatrix} 4(\frac{6}{5})^k & 0 \\ (\frac{6}{5})^k & (\frac{2}{5})^k \end{bmatrix}\begin{bmatrix} 25 \\ 15 \end{bmatrix} \\ &= \begin{bmatrix} 100(\frac{6}{5})^k \\ 25(\frac{6}{5})^k + 15(\frac{2}{5})^k \end{bmatrix} \end{aligned}$$

Thus $s_k = 25(\frac{6}{5})^k + 15(\frac{2}{5})^k$.

7. For k large, since $\left(\frac{\lambda_2}{\lambda_1}\right)^k \longrightarrow \infty$,

$$\begin{aligned} s_k &= 25\left(\frac{6}{5}\right)^k + 15\left(\frac{2}{5}\right)^k = \left(\frac{6}{5}\right)^k\left(25 + 15\left(\frac{2}{6}\right)^k\right) \\ &\approx 25\left(\frac{6}{5}\right)^k \end{aligned}$$