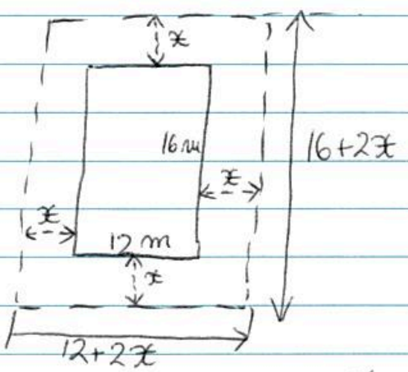
Math 2111 Chapter 1

(1) 

The new area is

$$(12+2x)(16+2x)=285$$

$$192+24x+32x+4x^2=285$$

$$4x^2+56x-93=0$$

$$x_{1,2}=\frac{-56\pm\sqrt{(56)^2-4\cdot(-93)}}{2\cdot 4}=\frac{-56\pm\sqrt{4624}}{8}$$

$$=\frac{-56\pm 68}{8} \to x_1=\frac{-56+68}{8}=\frac{12}{8}=\frac{3}{2}=1.5>0 \text{ the only viable solution}$$

$$x_2=-15.5<0$$

Solution: the width of the pathway should be $x=1.5$ m

(2) Let $y(t)$ = number of sales per month. Since this follows the exponential model of growth, we have that

$y'=ky$ which has solution $y(t)=Ae^{kt}$

We know $y(0)=100\,000$, $y(4)=80\,000$. We want to find $y(6)$.

We'll solve for A, k: $y(0)=100\,000 \Leftrightarrow A=100\,000$

$$y(4)=80\,000 \Leftrightarrow 100\,000\,e^{4k}=80\,000$$

$$\Leftrightarrow e^{4k}=\frac{8}{10}=0.8$$

$$\Leftrightarrow 4k=\ln 0.8$$

$$\Leftrightarrow k=\frac{\ln 0.8}{4}$$

Then $y(t)=100\,000\cdot e^{\frac{\ln 0.8}{4}t}$

We can now calculate $y(6)=100\,000\,e^{\frac{6\cdot\ln(0.8)}{4}}\simeq 71\,500$ units.

③ We need to show LHS = RHS. To do that we need to calculate $y'(x), y''(x)$
By applying the chain and product rule we have

$$y'(x) = -\sin(\ln x)\frac{1}{x}\ln(\cos(\ln x)) + \cancel{\cos(\ln x)}\cdot\frac{1}{\cancel{\cos(\ln x)}}\cdot(-\sin(\ln x))\frac{1}{x}$$

$$+\frac{1}{x}\sin(\ln x) + \ln x\cdot\cos(\ln x)\cdot\frac{1}{x}$$

$$y'(x) = -\frac{\sin(\ln x)\ln(\cos(\ln x))}{x} - \cancel{\frac{\sin(\ln x)}{x}} + \cancel{\frac{\sin(\ln x)}{x}} + \frac{\ln x\cdot\cos(\ln x)}{x}$$

$$y'(x) = -\frac{\sin(\ln x)\ \ln(\cos(\ln x))}{x} + \frac{\ln x\cdot\cos(\ln x)}{x}$$

Calculate y'':

$$y''(x) = -\frac{\left[\cos(\ln x)\frac{1}{x}\ln(\cos(\ln x)) + \sin(\ln x)\cdot\frac{1}{\cos(\ln x)}(-\sin(\ln x))\frac{1}{x}\right]x - \sin(\ln x)\cdot\ln(\cos}{x^2}$$

$$+\frac{\left[\frac{1}{x}\cos(\ln x) + \ln x\frac{(-\sin(\ln x))}{x}\right]x - \ln x\cos(\ln x)}{x^2}$$

$$y''(x) = -\frac{\left(\cos(\ln x)\ln(\cos(\ln x)) - \frac{\sin^2(\ln x)}{\cos(\ln x)}\right) - \sin(\ln x)\ln(\cos(\ln x))}{x^2}$$

$$+\frac{\cos(\ln x) - \ln x\sin(\ln x) - \ln x\cos(\ln x)}{x^2}$$

Calculate the LHS

$$LHS = x^2 y'' + x y' + y$$

$$= \cancel{x^2}\,\frac{\left[-\cos(\ln x)\ln(\cos(\ln x)) + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \sin(\ln x)\ln(\cos(\ln x))\right]}{\cancel{x^2}}$$

$$+ \cancel{x^2} \cdot \frac{\left[\cos(\ln x) - \ln x \sin(\ln x) - \ln x \cos(\ln x)\right]}{\cancel{x^2}}$$

$$+ \cancel{x} \cdot \frac{\left[-\sin(\ln x)\ln(\cos(\ln x)) + \ln x \cos(\ln x)\right]}{\cancel{x}} + \cos(\ln x)\ln(\cos(\ln x)) +$$

$$+ \ln x \sin(\ln x)$$

$$= -\cos(\ln x)\cancel{\ln(\cos(\ln x))} + \sin(\ln x)\cancel{\ln(\cos(\ln x))} + \frac{\sin^2(\ln x)}{\cos(\ln x)} + \cos(\ln x) -$$

$$- \ln x \cancel{\sin(\ln x)} - \ln x \cancel{\cos(\ln x)} - \sin(\ln x)\cancel{\ln(\cos(\ln x))} + \ln x \cancel{\cos(\ln x)}$$

$$+ \cos(\ln x)\cancel{\ln(\cos(\ln x))} + \ln x \cancel{\sin(\ln x)}$$

$$= \frac{\sin^2(\ln x) + \cos^2(\ln x)}{\cos(\ln x)} = \frac{1}{\cos(\ln x)} = \sec(\ln x)$$

equality

$$RHS = \sec(\ln x)$$

④ (i) Calculate $y'(x) = -\frac{2x}{(x^2+c)^2}$ and plug it into the LHS:

$$\left.\begin{aligned} LHS &= y'(x) + 2xy^2 = -\frac{2x}{(x^2+c)^2} + \frac{2x}{(x^2+c)^2} = 0 \\ RHS &= 0 \end{aligned}\right\} \text{equality}$$

This shows that $y(x) = \frac{1}{x^2+c}$ is a solution to the given ode

(ii) We must find c such that $y(-2) = \frac{1}{2} \Leftrightarrow \frac{1}{4+c} = \frac{1}{2}$

$\Leftrightarrow 4 + c = 2 \Leftrightarrow \boxed{c = -2}$

The particular solution that will satisfy $y(-2) = \frac{1}{2}$ is

$$\boxed{y(x) = \frac{1}{x^2-2}}$$