Lecture 2:

Limits and Continuity

September 12, 2016

Overview

1. Continuity
2. Continuity at a point
3. One-sided Limits
4. Two-sided Limits and Continuity at a point
5. How to evaluate a limit:
    - Options
    - Methods
    - Examples

Continuity

Definition

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It is intuitively clear that it must be a function without any *holes* or/and *jumps* at the point.

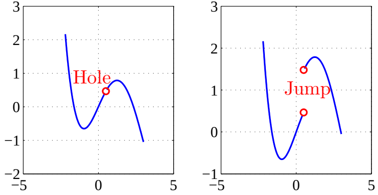

Figure: Examples of functions with discontinuity

Limits

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Limits from the right (see the text 2.1)

The function f has a limit from the right at $x = x_0$ whose value is L, and we denote it symbolically by

$$f(x_0 + 0) = \lim_{x \to x_0^+} f(x) = L$$

if BOTH of the following statements are satisfied:

- Let $x > x_0$ and x be very close to $x = x_0$.
- As x approaches x_0 ("from the right"), the value of $f(x)$ approaches the value L

Epsilon-delta definition of the right-hand limit

We say that the function f has a limit from the right at $x = x_0$ whose value is L and denote this by

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if for all positive ε there exists a positive δ such that for all x in the open interval $(x_0, x_0 + \delta)$ the absolute value of the difference $f(x) - L$ is less than ε. (**Remark:** given in advance ε)

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Or the same using *quantifiers* notation

$$\lim_{x \to x_0^+} f(x) = L \Leftrightarrow \forall \varepsilon > 0\ \exists \delta > 0 : \forall x \in (x_0, x_0 + \delta) \Rightarrow |f(x) - L| < \varepsilon$$

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Left-hand limits in epsilon-delta notation

$$\lim_{x \to x_0^-} f(x) = L \Leftrightarrow \forall \varepsilon > 0 \; \exists \delta > 0 : \forall x \in (x_0 - \delta, x_0) \Rightarrow |f(x) - L| < \varepsilon$$

One-sided limits. Examples.

Note To apply the concept of limits, the function need not even be defined at x_0!

$$f(x) = \begin{cases} -x^3, & x < 0 \\ x^2+1, & 1 > x \geqslant 0 \\ x+1, & x \geqslant 1, \end{cases}$$

$$1)\ \lim_{x\to 0^-} f(x), \quad 2)\ \lim_{x\to 0^+} f(x), \quad 3)\ \lim_{x\to 1^-} f(x), \quad 4)\ \lim_{x\to 1^+} f(x) \quad ???$$

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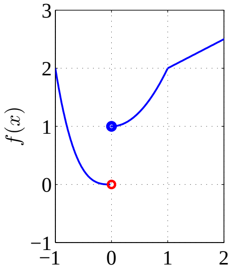

answers:

1) $-0^3 = 0$,

2) $0^2 + 1 = 1$,

3) $1^2 + 1 = 2$,

4) $1 + 1 = 2$.

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Consider the function $f(x) = \sin\left(\frac{1}{x}\right)$ and try to evaluate the limits: $\lim_{x\to 0^-} f(x)$ and $\lim_{x\to 0^+} f(x)$.

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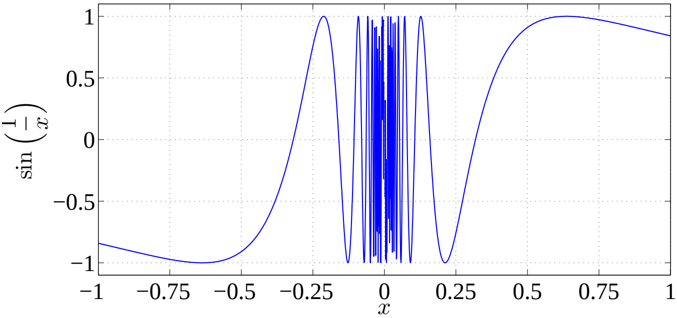
Well... all we can say is: "It is *something* between -1 and 1"



(Two-sided) limits

Definition of the limit

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Remark1: L may be infinite.
Remark2: There are several equivalent definitions of the limit.
Keep in mind: When one says a "limit" it means a "two-sided limit".

Continuity at the point

Now we are ready to provide a

Definition of continuity at $x = x_0$ (see the text 2.2)

We say that f is continuous at $x = x_0$ if ALL the following conditions apply:

- f is defined at $x = x_0$, so $f(x_0)$ is *finite*.
- There exists a *two-sided* limit $\lim_{x \to x_0} f(x) = L$
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- There exists a *two-sided* limit $\lim\limits_{x \to x_0} f(x) = L$
- $L = f(x_0)$.

In other words, the function is continuous at the given point if a (two-sided) limit exists at this point and coincides with the finite value of the function at the same point.

Epsilon-delta definition of continuity

A function $f(x)$ is continuous at $x_0 \in (a,b)$ if

$$\forall \varepsilon > 0 \, \exists \delta > 0 \, : \, \forall x \in [a,b], \, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$$

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Believe it or not... but this ε-δ definition encompasses all the three conditions considered above. And if at least one of them is not satisfied, we say that $f(x)$ is **discontinuous** at x_0.

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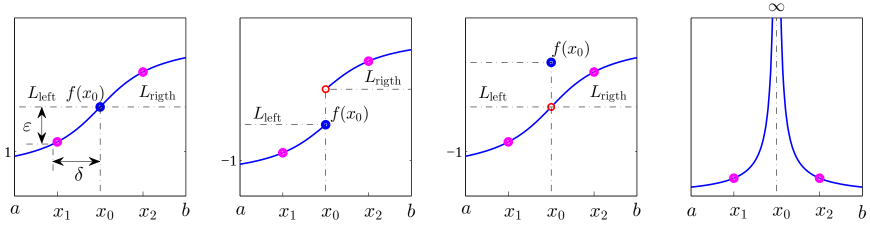
Let us proceed to the illustrations of these definitions.

Continuity at the point: Pictures



<u>$f(x)$ is continuous at x_0:</u>

$L = L_{\text{left}} = L_{\text{right}} = f(x_0)$

OR

for any ε one can always pick up δ, such that when $|x - x_0| < \delta$ it follows $|f(x) - f(x_0)| < \varepsilon$

<u>function $f(x)$ is discontinuous at x_0 because:</u>

$L_{\text{left}} \neq L_{\text{right}}$, hence

there is no two-sided limit

OR

for any $x > x_0$ we cannot choose ε less than $\varepsilon_{\max}$,

$\varepsilon_{\max} = |L_{\text{right}} - f(x_0)|$

$L = L_{\text{left}} = L_{\text{right}} \neq f(x_0)$

OR

for any $x < x_0$ we cannot choose ε less than $\varepsilon_{\max}$,

$\varepsilon_{\max} = |L - f(x_0)|$;

for $x > x_0$: $|x - x_0| \to 0 \Rightarrow$

$|f(x) - f(x_0)| \to |L - f(x_0)|$

$L = L_{\text{left}} = L_{\text{right}} = \infty$

but $f(x_0)$ is infinite

OR

as x approaches x_0

$|f(x) - f(x_0)|$ approaches to ∞

The text, Section 2.7, Ex. 16:

$$g(x) = \begin{cases} x^2 + 1, & x < 0 \\ 1 - |x|, & 1 > x \geqslant 0 \\ x, \quad x \geqslant 1, & \end{cases}$$

Evaluate the limits:

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and make conclusion about continuity or discontinuity at the points $x = 0$ and $x = 1$.

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Which means that $g(x)$ is discontinuous at $x = 1$.

Theorems About Continuous Functions

Intermediate Value Theorem

Let f be *continuous* on a closed interval $[a, b]$.

- Assume $f(a) \neq f(b)$;
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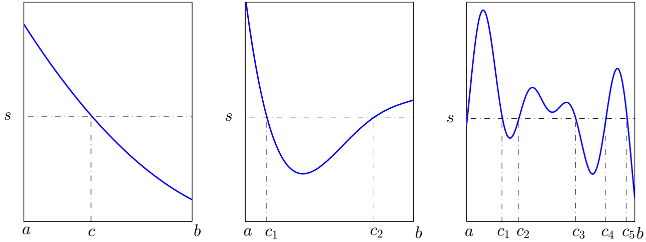

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Weierstrass' Extreme Value Theorem

Let f be *continuous* on a closed interval $[a, b]$. Then f is bounded on this interval and reaches on it its minimum and maximum values, each at least once. In other words, $[a, b]$ contains the points x_m and x_M, such that:

$$f(x_M) \geqslant f(x) \geqslant f(x_m), \quad \text{for all} \quad x \in [a, b].$$

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Limits equal to $\pm\infty$

$$\begin{array}{llll} \text{case} & a > 0 & \text{then} & \frac{a}{0} = \infty, \\ \text{case} & a < 0 & \text{then} & \frac{a}{0} = -\infty. \end{array}$$

Concept of the Extended Real Numbers (see Table 2.16)

The extended real number set is obtained from the real number set by adding two “elements”: $+\infty$ and $-\infty$.

Limits equal to $\pm\infty$.

Examples (see Section 2.6):
Do the following limits exist?
If so, guess their values.
1) One-sided limits

$$(i)\ \lim_{x\to 0^+}\frac{2x+1}{x}, \qquad (ii)\ \lim_{x\to 8^+}\frac{x}{x-8}, \qquad (iii)\ \lim_{x\to \pi^-}(3x+\cot x).$$

2) Two-sided limits

$$(i)\ \lim_{x\to 0}\frac{x+1}{|x|}, \qquad (ii)\ \lim_{x\to 0}\frac{3-x}{x}, \qquad (iii)\ \lim_{x\to \infty}2^x\sin x.$$

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Try to factor some terms, use trigonometric identities, multiply and divide by the same (non-zero!) symbols, use substitutions, apply the Sandwich Theorem etc. *Your goal is to get something appropriate...* if possible.

Sandwich Theorem

Let f, g, and h be functions defined on a certain interval (which may be even infinite). Suppose that for every x in the interval (except possibly for the point a)

$$g(x) \leqslant f(x) \leqslant h(x)$$

and also suppose that

$$\lim_{x \to a} g(x) = L, \qquad \lim_{x \to a} h(x) = L.$$

Then f also has a limit at a and

$$\lim_{x \to a} f(x) = L.$$

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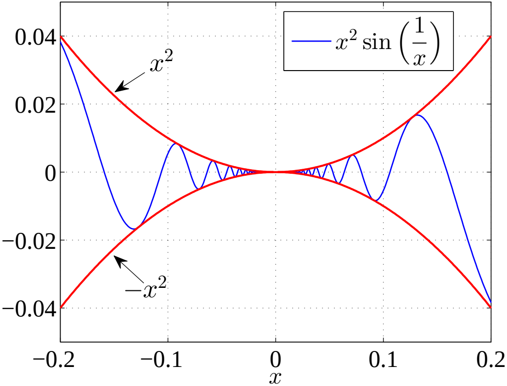

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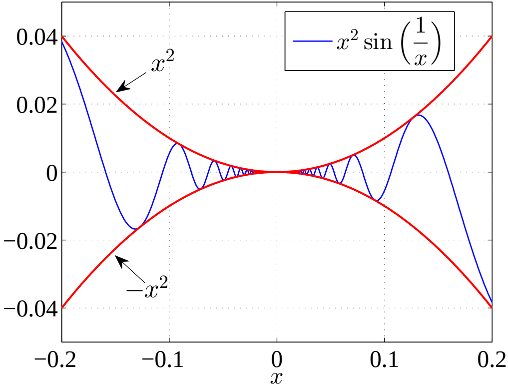

Thus, $\lim_{x\to 0} x^2 \sin\left(\frac{1}{x}\right) = 0.$

Two Remarkable Limits to Learn By Heart

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The First Remarkable Limit

$$\lim_{x\to 0}\frac{\sin x}{x} = 1$$

For example, this expression can be proved using the Sandwich Theorem, see the textbook, section 2.2.

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$$\lim_{g(x)\to 0}\frac{\sin g(x)}{g(x)}=1$$

The Second Remarkable Limit

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^{x}=e,$$

where $e\approx 2.71828182845 90$, so-called *Euler's Number*

How to “play around”

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$$(2)\quad \lim_{x\to\infty} (\sqrt{x^2+1}-x)$$

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Examples:

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\begin{aligned}
&(1) \quad \lim_{x\to 4} \frac{x-4}{x^2-2x-8}\\
&(2) \quad \lim_{x\to\infty} (\sqrt{x^2+1}-x)\\
&(3) \quad \lim_{x\to\infty} (\sin^2 x + \cos^2 x)
\end{aligned}
$$

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&(2) \quad \lim_{x\to\infty} (\sqrt{x^2+1}-x) \\
&(3) \quad \lim_{x\to\infty} (\sin^2 x+\cos^2 x) \\
&(4) \quad \lim_{x\to\infty} \frac{5x^2-4x+2}{7x^2+1}
\end{aligned}$$

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&(1) \quad \lim_{x\to 4} \frac{x-4}{x^2-2x-8} \\
&(2) \quad \lim_{x\to\infty} (\sqrt{x^2+1}-x) \\
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&(4) \quad \lim_{x\to\infty} \frac{5x^2-4x+2}{7x^2+1} \\
&(5) \quad \lim_{x\to 0} \frac{\sin 5x}{8x}
\end{aligned}
$$