ECON 2400A Summer 2012

Department of Economics

Carleton University

Assignment 2 Solution

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Q1 (20 points). Use Jacobian method to find $\frac{\partial Y^*}{\partial I_0}$, $\frac{\partial C^*}{\partial a}$, $\frac{\partial T^*}{\partial d}$ without calculating Y^*, C^* and T^*. (see ch8 for simultaneous equations) for the national income model:

$Y = C + I_0 + G_0$

$C = a + b(Y - T)(a > 0, 0 < b < 1)$

$T = d + tY(d > 0, 0 < t < 1)$

Solution:

Rewrite the model:

$Y - C - I_0 - G_0 = 0$

$C - a - b(Y - T) = 0$

$T - d - tY = 0$

(The following step is not neccessary, but helpful for finding which column of Jacobian determinant to be replaced.)

Take Y to be y_1, C to be y_2, T to be y_3,

I_0 to be x_1, G_0 to be x_2, a to be x_3,

b to bex_4, d to be x_5, t to be x_6

then the model becomes a set of simultaneous equations:

$F^1(y_1, y_2, y_3, x_1, x_2, x_3, x_4, x_5, x_6) = 0$

$F^2(y_1, y_2, y_3, x_1, x_2, x_3, x_4, x_5, x_6) = 0$

$F^3(y_1, y_2, y_3, x_1, x_2, x_3, x_4, x_5, x_6) = 0$

Jacobian determinant is

$$|J| = \begin{vmatrix} \frac{\partial F^1}{\partial Y} & \frac{\partial F^1}{\partial C} & \frac{\partial F^1}{\partial T} \\ \frac{\partial F^2}{\partial Y} & \frac{\partial F^2}{\partial C} & \frac{\partial F^2}{\partial T} \\ \frac{\partial F^3}{\partial Y} & \frac{\partial F^3}{\partial C} & \frac{\partial F^3}{\partial T} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix} = 1 - b + bt$$

$\frac{\partial Y^*}{\partial I_0} = \frac{\partial y_1}{\partial x_1}$, $\frac{\partial C^*}{\partial a} = \frac{\partial y_2}{\partial x_3}$, $\frac{\partial T^*}{\partial d} = \frac{\partial y_3}{\partial x_5}$, by the sub-number of y, we know this column in Jacobian should be replaced.

For example, to calculate $\frac{\partial Y^*}{\partial I_0}$, it equals $\frac{\partial y_1}{\partial x_1}$, here the sub-number of y is 1, to obtain J_1, the first column of the Jacobian should be replaced by the negative derivatives w.r.t. I_0 from the simultaneous equations.

To calculate $\frac{\partial C^*}{\partial a}$, it equals $\frac{\partial y_2}{\partial x_3}$, here the sub-number of y is 2, to obtain J_2, the second column of the Jacobian should be replaced by the negative derivatives w.r.t. a from the simultaneous equations.

To calculate $\frac{\partial T^*}{\partial d}$, it equals $\frac{\partial y_3}{\partial x_5}$, here the sub-number of y is 3, to obtain J_3, the third column of the Jacobian should be replaced by the negative derivatives w.r.t. d from the simultaneous equations.

$\frac{\partial F^1}{\partial x_1} = -1$, $\frac{\partial F^2}{\partial x_1} = 0$, $\frac{\partial F^3}{\partial x_1} = 0$,

$$|J_1| = \begin{vmatrix} -\frac{\partial F^1}{\partial x_1} & -1 & 0 \\ -\frac{\partial F^2}{\partial x_1} & 1 & b \\ -\frac{\partial F^3}{\partial x_1} & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & -1 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$\frac{\partial Y^*}{\partial I_0} = \frac{|J_1|}{J} = \frac{1}{1-b+bt}$,
$\frac{\partial F^1}{\partial x_3} = 0$, $\frac{\partial F^2}{\partial x_3} = -1$, $\frac{\partial F^3}{\partial x_3} = 0$,

$$|J_2| = \begin{vmatrix} 1 & -\frac{\partial F^1}{\partial x_3} & 0 \\ -b & -\frac{\partial F^2}{\partial x_3} & b \\ -t & -\frac{\partial F^3}{\partial x_3} & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ -b & 1 & b \\ -t & 0 & 1 \end{vmatrix} = 1$$

$\frac{\partial C^*}{\partial a} = \frac{|J_2|}{J} = \frac{1}{1-b+bt}$,
$\frac{\partial F^1}{\partial x_5} = 0$, $\frac{\partial F^2}{\partial x_5} = -1$, $\frac{\partial F^3}{\partial x_5} = 0$,

$$|J_3| = \begin{vmatrix} 1 & -1 & -\frac{\partial F^1}{\partial x_5} \\ -b & 1 & -\frac{\partial F^2}{\partial x_5} \\ -t & 1 & -\frac{\partial F^3}{\partial x_5} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & -1 & 0 \\ -b & 1 & 0 \\ -t & 1 & 1 \end{vmatrix} = 1 - b$$

$\frac{\partial T^*}{\partial d} = \frac{|J_3|}{J} = \frac{1-b}{1-b+bt}$

Q2 (5 points). Find the stationary value and use the N-th derivative test to determine the nature of the stationary value: $f(x) = (3-x)^6 + 7$.

Solution:

$f^{'}(x) = -6(3-x)^5 = 0 \Rightarrow x^* = 3, y^* = 7$

$f^{''}(x) = 30(3-x)^4, f^{''}(x^*) = 0.$

$f^{'''}(x) = -120(3-x)^3, f^{'''}(x^*) = 0$

$f^{(4)}(x) = 360(3-x)^2, f^{(4)}(x^*) = 0$

$f^{(5)}(x) = -720(3-x), f^{(5)}(x^*) = 0$

$f^{(6)}(x) = 720, f^{(6)}(x^*) = 0$

Because 6 is an even number and $f^{(6)}(x^*)$ is positive, so according to the Nth derivative test, $x^* = 3$ and $y^* = 7$ is a local minimum.

Q3 (5 points). Find the first 5 terms of Maclaurin Series and Taylor series with $x_0 = 2$, $f(x) = \frac{1+x}{1-x}$.

Solution:

$f^{'}(x) = \frac{2}{(1-x)^2}$

$f^{''}(x) = \frac{4}{(1-x)^3}$

$f^{'''}(x) = \frac{12}{(1-x)^4}$

$f^{(4)}(x) = \frac{48}{(1-x)^5}$

$f(0) = 1, f^{'}(0) = 2, f^{''}(0) = 4, f^{'''}(0) = 12, f^{(4)} = 48$

Maclaurin Series (because it is a nonpolynomial function, we should have a remainder.):

$f(x) = \frac{f(0)}{0!} + \frac{f^{'}(0)}{1!} * x + \frac{f^{''}(0)}{2!} * x^2 + \frac{f^{'''}(0)}{3!} * x^3 + \frac{f^{(4)}(0)}{4!} * x^4 + R_4$

$= 1 + 2x + 2x^2 + 2x^3 + 2x^4 + R_4$

$f(2) = -3, f^{'}(2) = 2, f^{''}(2) = -4, f^{'''}(2) = 12, f^{(4)}(2) = -48$

Taylor Series (because it is a nonpolynomial function, we should have a remainder.):

$f(x) = \frac{f(2)}{0!} + \frac{f^{'}(2)}{1!} * (x-2) + \frac{f^{''}(2)}{2!} * (x-2)^2 + \frac{f^{'''}(2)}{3!} * (x-2)^3 + \frac{f^{(4)}(2)}{4!} * (x-2)^4 + R_4$

$= -3 + 2(x-2) - 2(x-2)^2 + 2(x-2)^3 - 2(x-2)^4 + R_4$

$= -2x^4 + 18x^3 - 62x^2 + 98x - 63 + R_4$

Q4 (10 points). Use determinantal test to find the extreme value:

$z = f(x_1, x_2, x_3) = 3x_1^2 - 2x_1 + 4x_1x_3 + 5x_2^2 + 4x_3^2 - 2x_2x_3.$

Solution:

$f_1 = 6x_1 - 2 + 4x_3$

$f_2 = 10x_2 - 2x_3$

$f_3 = 4x_1 + 8x_3 - 2x_2$

$f_{11} = 6, f_{22} = 10, f_{33} = 8$

$f_{12} = 0, f_{13} = 4, f_{23} = -2$

F.O.C.

$f_1 = f_2 = f_3 = 0$

$6x_1 - 2 + 4x_3 = 0$

$10x_2 - 2x_3 = 0$

$4x_1 + 8x_3 - 2x_2 = 0$

$x_1^* = \frac{19}{37}, x_2^* = -\frac{2}{37}, x_3^* = -\frac{10}{37}$

$z^* = 3(x_1^*)^2 - 2x_1^* + 4x_1^*x_3^* + 5(x_2^*)^2 + 4(x_3^*)^2 - 2x_2^*x_3^*$

$= 3 * (\frac{19}{37})^2 - 2 * \frac{19}{37} + 4 * \frac{19}{37} * (-\frac{10}{37}) + 5 * (-\frac{2}{37})^2 + 4 * (-\frac{10}{37})^2 - 2 * (-\frac{2}{37}) * (-\frac{10}{37})$

$= -\frac{19}{37}$

Hessian Matrix is

$$H = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix} = \begin{pmatrix} 6 & 0 & 4 \\ 0 & 10 & -2 \\ 4 & -2 & 8 \end{pmatrix}$$

Leading principal minors:

$|H_1| = 6,$

$$H_2 = \begin{vmatrix} 6 & 0 \\ 0 & 10 \end{vmatrix} = 60$$

$$H_3 = \begin{vmatrix} 6 & 0 & 4 \\ 0 & 10 & -2 \\ 4 & -2 & 8 \end{vmatrix} = 296$$

All leading principal minors of Hessian Matrix are positive, so the Hessian Matrix is positive definite. z reaches local minimum $-\frac{19}{37}$ at point $x_1^* = \frac{19}{37}$, $x_2^* = -\frac{2}{37}$, $x_3^* = -\frac{10}{37}$.

Q5 (20 points). Assume a monopolistic firm produces 3 products Q_1, Q_2, Q_3, $Q_1 + Q_2 + Q_3 = Q$. According to the following inverse demand functions $P_1 = 63 - 4Q_1$, $P_2 = 105 - 5Q_2$, $P_3 = 75 - 6Q_3$, and the total-cost function is $C(Q) = 20 + 15Q + Q^2$. Find the equilibrium quantities and prices and verify that the second-order sufficient condition is met.

Solution:

Total revenue function: $R(Q) = R_1(Q_1) + R_2(Q_2) + R_3(Q_3)$

$= P_1Q_1 + P_2Q_2 + P_3Q_3$

$= (63 - 4Q_1)Q_1 + (105 - 5Q_2)Q_2 + (75 - 6Q_3)Q_3$

Profit function: $\pi = R(Q) - C(Q)$

$= R_1(Q_1) + R_2(Q_2) + R_3(Q_3) - C(Q)$

$= (63-4Q_1)Q_1 + (105-5Q_2)Q_2 + (75-6Q_3)Q_3 - 20 - 15(Q_1+Q_2+Q_3) - (Q_1 + Q_2 + Q_3)^2$

F.O.C.

$\frac{\partial\pi}{\partial Q_1} = 63 - 8Q_1 - 15 - 2(Q_1 + Q_2 + Q_3) = 0 \Rightarrow 24 - 5Q_1 - Q_2 - Q_3 = 0$

$\frac{\partial\pi}{\partial Q_2} = 105 - 10Q_2 - 15 - 2(Q_1 + Q_2 + Q_3) = 0 \Rightarrow 45 - 6Q_2 - Q_1 - Q_3 = 0$

$\frac{\partial\pi}{\partial Q_3} = 75 - 12Q_3 - 15 - 2(Q_1 + Q_2 + Q_3) = 0 \Rightarrow 30 - 7Q_3 - Q_1 - Q_2 = 0$

$\Rightarrow Q_1^* = \frac{282}{97}$, $Q_2^* = \frac{633}{97}$, $Q_3^* = \frac{285}{97}$

$Q^* = Q_1^* + Q_2^* + Q_3^* = \frac{1200}{97}$

S.O.C.

$\pi_{11} = -5$,$\pi_{12} = -1$, $\pi_{13} = -1$

$\pi_{21} = -1$,$\pi_{22} = -6$, $\pi_{23} = -1$

$\pi_{31} = -1$,$\pi_{32} = -1$, $\pi_{33} = -7$

Hessian Matrix:

$$H = \begin{pmatrix} \pi_{11} & \pi_{12} & \pi_{13} \\ \pi_{21} & \pi_{22} & \pi_{23} \\ \pi_{31} & \pi_{32} & \pi_{33} \end{pmatrix} = \begin{pmatrix} -5 & -1 & -1 \\ -1 & -6 & -1 \\ -1 & -1 & -7 \end{pmatrix}$$

Leading principal minors:

$|H_1| = -5$,

$$H_2 = \begin{vmatrix} -5 & -1 \\ -1 & -6 \end{vmatrix} = 29$$

$$H_3 = \begin{vmatrix} -5 & -1 & -1 \\ -1 & -6 & -1 \\ -1 & -1 & -7 \end{vmatrix} = -194$$

Because odd number leading pricipal minors are negative and even number leading pricipal minor is positive, so the Hessian Matrix is negative definite and profit is maximized at $Q_1^* = \frac{282}{97}$, $Q_2^* = \frac{633}{97}$, $Q_3^* = \frac{285}{97}$.

Equilibirium Prices:

$P_1^* = 63 - 4Q_1^* = 63 - 4 * \frac{282}{97} = \frac{4983}{97}$

$P_2^* = 105 - 5Q_2^* = 105 - 5 * \frac{633}{97} = \frac{7020}{97}$

$P_3^* = 75 - 6Q_3^* = 75 - 6 * \frac{285}{97} = \frac{5565}{97}$

(It is OK if you do not have following steps because I did not ask in the question.)

$R^* = P_1^*Q_1^* + P_2^*Q_2^* + P_3^*Q_3^* = \frac{4983}{97} * \frac{282}{97} + \frac{7020}{97} * \frac{633}{97} + \frac{5565}{97} * \frac{285}{97} = \frac{7434891}{9409}$

$C^* = 20 + 15 * Q^* + (Q^*)^2 = 20 + 15 * \frac{1200}{97} + (\frac{1200}{97})^2 = \frac{3374180}{9409}$

$\pi^* = R^* - C^* = \frac{7434891}{9409} - \frac{3374180}{9409} = \frac{41863}{97}$

Q6 (5 points).Use the lagrange-multiplier method to find the stationary value of z and use the bordered Hessian to check whether it is a maximum or a minimum.

$x = f(x, y) = x - 3y - xy$

s.t.$x + y = 6$.

Solution:

Lagrangian function:

$L = (x - 3y - xy) + \lambda(6 - x - y)$

F.O.C.

$\frac{\partial L}{\partial x} = 1 - y - \lambda = 0$

$\frac{\partial L}{\partial y} = -3 - x - \lambda = 0$
$\frac{\partial L}{\partial \lambda} = 6 - x - y = 0$
$\Rightarrow x^* = 1, y^* = 5, \lambda^* = -4$
$z^* = f(x^*, y^*) = x^* - 3y^* - x^*y^* = 1 - 3 * 5 - 5 * 1 = -19$
Bordered Hessian:

$$|\overline{H}| = \begin{vmatrix} 0 & g_x & g_y \\ g_x & L_{xx} & L_{xy} \\ g_y & L_{xy} & L_{yy} \end{vmatrix}$$
$$= \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{vmatrix}$$
$$= -2 < 0$$

Thus $z^* = -19$ is a minimum.

Q7 (5 points). Is the following production homogeneous?
$Q(K, L) = A(\delta K^{-\rho} + (1 - \delta)L^{-\rho})^{-\frac{\gamma}{\rho}}$
Find the elasticity of substitution between K and L for the production function.

Solution:
$Q(tK, tL) = A(\delta(tK)^{-\rho} + (1 - \delta)(tL)^{-\rho})^{-\frac{\gamma}{\rho}}$
$= t^\gamma A(\delta K^{-\rho} + (1 - \delta)L^{-\rho})^{-\frac{\gamma}{\rho}}$
$= t^\gamma Q(K, L)$
Thus production function is homogeneous of degree γ.
$Q_K \equiv \frac{\partial Q}{\partial K}$
$= (-\frac{\gamma}{\rho})A(\delta K^{-\rho} + (1 - \delta)L^{-\rho})^{-\frac{\gamma}{\rho}-1}\delta(-\rho)K^{-\rho-1}$
$= \delta\gamma A(\delta K^{-\rho} + (1 - \delta)L^{-\rho})^{-\frac{\gamma+\rho}{\rho}}K^{-(1+\rho)}$
$Q_L \equiv \frac{\partial Q}{\partial L}$
$= A(-\frac{\gamma}{\rho})(\delta K^{-\rho} + (1 - \delta)L^{-\rho})^{-\frac{\gamma}{\rho}-1}(1 - \delta)(-\rho)L^{-\rho-1}$
$= (1 - \delta)\gamma A(\delta K^{-\rho} + (1 - \delta)L^{-\rho})^{-\frac{\gamma+\rho}{\rho}}L^{-(1+\rho)}$
Marginal rate of substitution between K and L
$MRS_{KL} \equiv \frac{Q_L}{Q_K} = \frac{(1-\delta)}{\delta}(\frac{K}{L})^{1+\rho}$
$\Rightarrow (\frac{K}{L}) = ((\frac{\delta}{1-\delta})MRS_{KL})^{\frac{1}{1+\rho}}$
$\ln(\frac{K}{L}) = \frac{1}{1+\rho}\ln(\frac{\delta}{1-\delta}) + \frac{1}{1+\rho}\ln(MRS_{KL})$.

Elasticity of substitution between K and L

$\sigma \equiv \frac{d\ln(\frac{K}{L})}{d\ln(MRS_{KL})} = \frac{1}{1+\rho} = constant$

This production function is a CES production function.

Q8 (5 points). Use L'HôPital's Rule to find:

i) $\lim_{x\to 0^+} x\ln x$

ii)$\lim_{x\to\infty} \frac{x^n}{e^x}$

Solution:

(i)

$$\lim_{x\to 0^+} xlnx = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{(-\frac{1}{x^2})} = \lim_{x\to 0^+}(-x) = 0.$$

(ii)

$$\lim_{x\to\infty} \frac{x^n}{e^x} = \lim_{x\to\infty} \frac{nx^{n-1}}{e^x} = \lim_{x\to\infty} \frac{n(n-1)x^{n-2}}{e^x} = ... = \lim_{x\to\infty} \frac{n!x^0}{e^x} = \lim_{x\to\infty} \frac{n!}{e^x} = 0.$$

Q9 (5 points).Use bordered determinant to check the following function for quasiconcarity and quasiconvesity.

$z = f(x,y) = -x^2 - y^2 (x, y > 0)$

Solution:

$f_x = -2x,\ f_y = -2y,\ f_{xy} = -2,\ f_{xy} = 0,\ f_{yy} = -2$

Bordered determinant

$$|B| = \begin{vmatrix} 0 & f_x & f_y \\ f_x & f_{xx} & f_{xy} \\ f_y & f_{yx} & f_{yy} \end{vmatrix}$$
$$= \begin{vmatrix} 0 & -2x & -2y \\ -2x & -2 & 0 \\ -2y & 0 & -2 \end{vmatrix}$$

Leading principal minors:

$$|B_1| = \begin{vmatrix} 0 & -2x \\ -2x & -2 \end{vmatrix} = -4x^2$$

$$|B_2| = \begin{vmatrix} 0 & -2x & -2y \\ -2x & -2 & 0 \\ -2y & 0 & -2 \end{vmatrix} = 8x^2 + 8y^2$$

Because $|B_1| < 0$ and $|B_2| > 0$, the function is strictly quasiconcave.

Q10 (20 points) A consumer lives on an island where she produces two goods, x and y, according to the production possibility frontier $x^2 + y^2 \leq 200$, and she consumes all the goods herself. Her utility function is: $U = xy^3$, The consumer also faces an environmental constraint on her total output of both goods. the environmental constraint is given $x + y \leq 20$.

(a) Write ont the Kuha-Tucker first-order conditions.

(b) Find the consumer's optional x and y identity which constraints are binding.

Solution:

(a) $\max U = xy^3$

s.t. $x^2 + y^2 \leq 200$

$x + y \leq 20$

The Lagrangian for the problem is:

$L = xy^3 + \lambda_1(200 - x^2 - y^2) + \lambda_2(20 - x - y)$

Kuhn-Tucker conditions:

$L_x = y^3 - 2\lambda_1 x - \lambda_2 \leq 0$, $x \geq 0$, and $xL_x = 0$

$L_y = 3xy^2 - 2\lambda_1 y - \lambda_2 \leq 0$, $y \geq 0$, and $yL_y = 0$

$L_{\lambda_1} = 200 - x^2 - y^2 \geq 0$, $\lambda_1 \geq 0$, and $\lambda_1 L_{\lambda_1} = 0$

$L_{\lambda_2} = 20 - x - y \geq 0$, $\lambda_2 \geq 0$, and $\lambda_2 L_{\lambda_2} = 0$

Use Trial and Error Method to solve Kuhn-Tucker conditions.

Step 1: Assume the 1st constraint is nonbinding and the 2nd constraint is binding in the solution then $\lambda_1 = 0, \lambda_2 > 0$. Let $x > 0, y > 0$, then by complementary slackness conditions, the Kuhn-Tucker conditions become:

$L_x = y^3 - \lambda_2 = 0 \Rightarrow y^3 = \lambda_2$..............(1)

$L_y = 3xy^2 - \lambda_2 = 0 \Rightarrow 3xy^2 = \lambda_2$...........(2)

$L_{\lambda_2} = 20 - x - y = 0$..(3)

$\frac{(1)}{(2)} : \frac{y}{3x} = 1 \Rightarrow y = 3x$........................(4)

Substitute (4) into (3):

$20 - x - 3x = 0 \Rightarrow x^* = 5$ and $y^* = 3x^* = 15$

Check whether the solution satisfies the 1st constraint:

$x^{*2} + y^{*2} = 250 > 200.$

The solution does not satisfy the first constraint, the solution should be rejected.

Step 2: Assume that the 1st constraint is binding and the 2nd constraint is non-binding in the solution, then $\lambda_1 > 0, \lambda_2 = 0$. Let$x > 0, y > 0$, then by compementary slackness conditions, the Kuhn-Tucker conditions become:

$L_x = y^3 - 2\lambda_1 x = 0$ then $y^3 = 2\lambda_1 x$ ....(1)

$L_y = 3xy^2 - 2\lambda_1 y = 0$ then $3xy = 2\lambda_1$ ....(2)

$L_{\lambda_1} = 200 - x^2 - y^2 = 0$ ....(3)

$\frac{(1)}{(2)}$: $\frac{y^3}{3xy} = x$ then $y^2 = 3x^2$..........(4)

Substitute (4) into (3), $200 - x^2 - 3x^2 = 0$, $x^* = \sqrt{50}$ $y^* = \sqrt{150}$.

From Equation (2),

$(\lambda_1)^* = \frac{3x^* y^*}{2} = \frac{3 * \sqrt{50} * \sqrt{150}}{2} = 75\sqrt{3}$

Check whether the solution satisfies the 2nd constraint:.

$x^* + y^* = \sqrt{50} + \sqrt{150} \approx 19.3 < 20$.

So the solution satisfies both constraints, accept it.