COMP 232 Mathematics for Computer Science
Fall 2016
Midterm Exam

Name: ______________________ Total Points:

ID: ______________________ ________ / 50

Instructions. *This is a closed book exam. The only allowed tool is an ENCS approved calculator. Provide all answers in this booklet.* *Use pen, not pencil.* *Do not detach any pages from this exam!*

(1pt ea.) **1.** True or False? You get 1 points for each correct answer.   [ ] 9 pts

(a) The following is a proposition: "The restaurant closes at 10 p.m."

☑ True   ☐ False   ☐ Don't know!

(b) The following is a proposition: "Close the restaurant at 11 p.m."

☐ True   ☑ False   ☐ Don't know!

(c) We can determine whether a proposition is a contradiction by constructing a truth table for it.

☐ False   ☑ True   ☐ Don't know!

(d) The proposition $\neg p \to \neg q$ is the contrapositive of the proposition $p \to q$.

☑ False   ☐ True   ☐ Don't know!

(e) The statement $p \to q$ means "q is a sufficient condition for p."

☑ False   ☐ True   ☐ Don't know!

(f) $\neg(p \leftrightarrow q)$ and $p \leftrightarrow \neg q$ are logically equivalent propositions.

☑ True   ☐ False   ☐ Don't know!

(g) The propositions $\neg p \to (q \to p)$ and $q \to (p \vee q)$ are not logically equivalent.

☐ True   ☑ False   ☐ Don't know!

(h) $\forall x(P(x)) \wedge \exists x(Q(x)) \equiv \forall x(P(x)) \wedge \exists y(Q(y))$

☐ False   ☑ True   ☐ Don't know!

(i) $\neg \forall x P(x) \equiv \neg \exists x \neg P(x)$

☐ True   ☑ False   ☐ Don't know!

[ ] 9 pts

(1$^{\text{pt}}_{\text{ea.}}$) **2.** Suppose $P(x, y)$ is a predicate and the Universe of Discourse for variables x and y is $\{1, 2, 3\}$. Suppose $P(1, 3), P(2, 1), P(2, 2), P(2, 3), P(3, 1), P(3, 2)$ are true, and $P(x, y)$ is false otherwise. Determine whether the following statements are true. You get 1 points for each correct answer.

5 pts

(a) $\forall x \exists y P(x, y)$

- [x] True
- [ ] False
- [ ] Don't know!

(b) $\exists x \forall y P(x, y)$

- [ ] False
- [x] True
- [ ] Don't know!

(c) $\neg\Big[\exists x \exists y \Big(P(x, y) \wedge \neg P(y, x)\Big)\Big]$

- [ ] True
- [x] False
- [ ] Don't know!

(d) $\forall x \exists x \Big(P(x, y) \rightarrow P(y, x)\Big)$

- [ ] False
- [x] True
- [ ] Don't know!

(e) $\forall y \exists x \Big(x \leq y \ \wedge \ P(x, y)\Big)$

- [x] False
- [ ] True
- [ ] Don't know!

5 pts

6 pts

($3^{\text{pts}}_{\text{ea.}}$) **3.** Here you are to prove propositional equivalences using the laws given in the crib-sheet.

(a) In the table below, construct a proof of the equivalence

$$p \to (q \to r) \equiv (p \wedge q) \to r.$$

Solution:

Step			Law applied
$p \to (q \to r)$	$\equiv$	$\neg p \vee (q \to r)$	Implication
	$\equiv$	$\neg p \vee (\neg q \vee r)$	Implication
	$\equiv$	$(\neg p \vee \neg q) \vee r$	Associativity
	$\equiv$	$\neg(p \wedge q) \vee r$	de Morgan
	$\equiv$	$(p \wedge q) \to r$	Implication

(b) In the table below, construct a proof of the equivalence

$$(r \vee p) \to (r \vee q) \equiv r \vee (p \to q).$$

Solution:

Step			Law applied
$(r \vee p) \to (r \vee q)$	$\equiv$	$\neg(r \vee p) \vee (r \vee q)$	Implication
	$\equiv$	$\big((\neg r) \wedge (\neg p)\big) \vee (r \vee q)$	de Morgan
	$\equiv$	$\Big(\big((\neg r) \wedge (\neg p)\big) \vee r\Big) \vee q$	Associativity
	$\equiv$	$\Big(r \vee \big((\neg r) \wedge (\neg p)\big)\Big) \vee q$	Commutativity
	$\equiv$	$\Big(\big(r \vee (\neg r)\big) \wedge \big(r \vee (\neg p)\big)\Big) \vee q$	Distributivity
	$\equiv$	$\Big(\big(r \vee (\neg p)\big) \wedge \big(r \vee (\neg r)\big)\Big) \vee q$	Commutativity
	$\equiv$	$\Big(\big(r \vee (\neg p)\big) \wedge T\Big) \vee q$	Excluded middle
	$\equiv$	$\big(r \vee (\neg p)\big) \vee q$	Identity
	$\equiv$	$r \vee \big((\neg p) \vee q\big)$	Associativity
	$\equiv$	$r \vee (p \to q)$	Implication

6 pts

(2$^{\text{pts}}_{\text{ea.}}$) **4.** We know that $\{\wedge, \neg\}$ forms a functionally complete set of operators, meaning that any other operator can be defined in terms of $\{\wedge, \neg\}$ only, for example

6 pts

$$p \vee q \;=_{\texttt{def}}\; \neg(\neg p \wedge \neg q)$$
$$p \to q \;=_{\texttt{def}}\; \neg(p \wedge \neg q)$$

The **S**haffer stroke $\uparrow$ is a binary operator that has the following truth table:

p	q	$p \uparrow q$
T	T	F
T	F	T
F	T	T
F	F	T

Show that the Shaffer stroke by itself is functionally complete, by defining in the space below, the following operators by using the Shaffer stroke only:

(a) $\neg p \;=_{\texttt{def}}\; p \uparrow p$

(b) $p \wedge q \;=_{\texttt{def}}\; (p \uparrow q) \uparrow (p \uparrow q)$

(c) $p \vee q \;=_{def}\; (p \uparrow p) \uparrow (q \uparrow q)$

(4$^{\text{pts}}$) **5.** Let the Universe of Discourse be $\mathbb{Z}$. Consider the assertion

4 pts

$$\exists x\Big(P(x) \wedge Q(x)\Big) \Leftrightarrow \Big(\exists x(P(x))\Big) \wedge \Big(\exists x(Q(x))\Big)$$

Which of the following statements correctly describes the assertion?

- ☐ The assertion is false. As a counterexample, let $P(x)$ mean "*x is divisible by 6*," and $Q(x)$ mean "*x is divisible by 3*."
- ☑ The assertion is false. For a counterexample, let $P(x)$ mean '" $x < 0$," and $Q(x)$ mean "$x \geq 0$."
- ☐ The assertion is true. The proof follows from the distributive laws for $\wedge$
- ☐ The assertion is true. To see why, let $P(x)$ mean "*x is divisible by 6*," and $Q(x)$ mean "*x is divisible by 3*." If $x = 6$, then x is divisible by both 3 and 6, so both side of the equivalence have the same truth value for this x.
- ☐ The assertion is false. For a counterexample, let $P(x)$ mean '"*$x < 0$ is a square*" and $Q(x)$ mean "*x is odd*."

10 pts

(1pt ea.) **6.** The *symmetric difference* between sets A and B is defined as

4 pts

$$A \oplus B = (A - B) \cup (B - A).$$

For each of the proposed identity involving $\oplus$ below, state whether the identity is true or false. You get 1 point for each correct answer.

(a) $(A \oplus B) \oplus C = A \oplus (B \oplus C)$.

☑ True ☐ False ☐ Don't know!

(b) $(A \oplus B) \oplus (C \oplus D) = (A \oplus C) \oplus (B \oplus D)$.

☐ False ☑ True ☐ Don't know!

(c) $((A \oplus B) \oplus C) \cup (A \cap B \cap C) = A \cup B \cup C$.

☐ True ☑ False ☐ Don't know!

(d) $(A \oplus B) \cup B = A \cup B$

☑ True ☐ False ☐ Don't know!

4 pts

(4 pts ea.) **7.** (a) Consider the assertion "*Let a and b be integers. If $a < b$ then $a < \frac{a+b}{2} < b$.*" Give a direct proof of the assertion. Solution:

16 pts

$$\tfrac{a+b}{2} < \tfrac{b+b}{2} = \tfrac{2b}{2} = b \;\Rightarrow\; \tfrac{a+b}{2} < b$$

$$\tfrac{a+b}{2} > \tfrac{a+a}{2} = \tfrac{2a}{2} = a \;\Rightarrow\; \tfrac{a+b}{2} > a.$$

(b) Consider the assertion "*Let $n \in \mathbb{Z}$. If $n^5 + 7$ is even, then n is odd.*" Give a indirect proof of the assertion.

Solution: We need to prove $even(n) \Rightarrow odd(n^5 + 7)$

$even(n) \Rightarrow n = 2k$, for some $k \in \mathbb{Z}$

$$\Rightarrow n^5 + 7 = (2k)^5 + 7 = 32k^5 + 7 = 32k^5 + 6 + 1 = 2(16k^5 + 3) + 1 \;\Rightarrow odd(n^5 + 7)$$

(c) Consider the assertion "*Let $x, y \in \mathbb{R}$ with $x \geq 0$ and $y \geq 0$. Then $\frac{x+y}{2} \geq \sqrt{xy}$.*" Prove the assertion by contradiction.

Solution: Suppose to the contrary that $\frac{x+y}{2} < \sqrt{xy}$.

Then $\frac{x+y}{2} \cdot \frac{x+y}{2} < \sqrt{xy} \cdot \sqrt{xy} = xy$

$\Rightarrow (x+y)^2 < 4xy$

$\Rightarrow x^2 + 2xy + y^2 < 4xy$

$\Rightarrow x^2 - 2xy + y^2 < 0$

$\Rightarrow (x-y)^2 < 0$; a contradiction, since a square always is ≥ 0.

(d) Consider the assertion "*For all intgers n it holds that $n^2 + n$ is even.*" Give a proof by cases of the assertion.

Solution:

- *Case 1:* n is even.

$\Rightarrow n = 2k$, for some $k \in \mathbb{Z} \Rightarrow n^2 + n = 4k^2 + 2k = 2(2k^2 + k) \;\Rightarrow n^2 + n$ is even

- *Case 2:* n is odd.

$\Rightarrow n = 2k + 1$, for some $k \in \mathbb{Z}$

$\Rightarrow n^2 + n = (2k+1)^2 + (2k+1) = (4k^2 + 4k + 1) + (2k + 1) = 4k^2 + 6k + 2$

$\Rightarrow n^2 + n = 2(2k^2 + 3k + 1)$

$\Rightarrow n^2 + n$ is even

. — End of Exam — .

16 pts