DEPARTMENT OF MATHEMATICS

MIDTERM TEST #1

MTH 240 – Calculus II

Last Name (Print):________________. First Name:________________. Student Number: ____________

Signature:______________.

Date: May 31, 2012, 6:00 pm

Duration: 90 minutes

Section (circle one)

Dr. B. Tasić :	1 2 3 7
Dr. A. Alvarez :	4 5 6 8

Instructions:

1. This is a closed-book test. **Notes, calculators and other aids are not permitted.**
2. Verify that your test has pages 1-6.
3. (a) Unless otherwise instructed, **make sure you include all significant steps in your solution, presented in the correct order. Unjustified answers will be given little or no credit. Cross out or erase all rough work not relevant to your solution.** Put a box around your final **answer.**

   (b) Write your solutions in the space provided. If you need more space, use the back of the page. Indicate this fact on the original page, making sure that your solution cannot be confused with any rough work which may be there. Marks (out of 50) are shown in brackets.
4. Do not separate the sheets.
5. Have your student card available on your desk.

For Instructor's use only.

Page	Mark
2	/
3	/
4	/
5	/
6	/
Total	/50

1. [7 marks] Evaluate the following integral.

$$\int_0^\pi t^2 \sin t \, dt$$

Put $t^2 = u$ and $\sin t \, dt = dv$. Then $2t\,dt = du$ and $v = -\cos t$

$$\int_0^\pi t^2 \sin t \, dt = -t^2 \cos t \Big|_0^\pi + 2\int_0^\pi t \cos t \, dt$$

Now put $t = u$ and $\cos t\,dt = dv$ which gives $dt = du$ and $v = \sin t$

$$= -t^2 \cos t \Big|_0^\pi + 2\left(t \sin t \Big|_0^\pi - \int_0^\pi \sin t \, dt\right)$$

$$= -t^2 \cos t \Big|_0^\pi + 2t \sin t \Big|_0^\pi + 2\cos t \Big|_0^\pi$$

$$= -\pi^2 \cdot (-1) + 0 + 2(-1-1)$$

$$= \underbrace{\pi^2 - 4}$$

2. [3 marks] Write out the form of the partial fraction decomposition of the function:

$$\frac{x^3 + x^2 + 1}{x(x-1)(x^2+x+1)(x^2+1)^3}$$

$$\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} + \frac{Ex+F}{x^2+1} + \frac{Gx+H}{(x^2+1)^2} + \frac{Ix+J}{(x^2+1)^3}$$

3. [10 marks] Evaluate the following integral.

$$\int \tan^2 x \sec^3 x \, dx$$

$$\int \tan^2 x \sec^3 x \, dx = \int (\sec^2 x - 1)\sec^3 x \, dx = \int \sec^5 x \, dx - \int \sec^3 x \, dx$$
$$= \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x \, dx - \int \sec^3 x \, dx =$$
$$= \frac{1}{4}\sec^3 x \tan x - \frac{1}{4}\int \sec^3 x \, dx$$
$$= \frac{1}{4}\sec^3 x \tan x - \frac{1}{8}\sec x \tan x - \frac{1}{8}\ln|\sec x + \tan x| + C$$

$$\int \sec^3 x \, dx = \sec x \tan x - \int \sec x \tan^2 x \, dx$$
$\sec x = u$, $\sec^2 x \, dx = dv \Rightarrow \sec x \tan x \, dx = du$, $v = \tan x$
$$= \sec x \tan x - \int \sec x (\sec^2 x - 1)\, dx =$$
$$= \sec x \tan x - \int \sec^3 x \, dx + \int \sec x \, dx$$
$$2\int \sec^3 x \, dx = \sec x \tan x + \int \sec x \, dx = \sec x \tan x + \ln|\sec x + \tan x|$$
. Hence

Finally $\int \sec^3 x \, dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x|$

$$\int \sec^5 x \, dx = \sec^3 x \tan x - 3\int \sec^3 x \tan^2 x \, dx =$$
$\sec^3 x = u$, $\sec^2 x \, dx = dv \Rightarrow 3\sec^2 x \cdot \sec x \tan x \, dx = du$, $v = \tan x$
$$= \sec^3 x \tan x - 3\int \sec^3 x (\sec^2 x - 1)\, dx =$$
$$= \sec^3 x \tan x - 3\int \sec^5 x \, dx + 3\int \sec^3 x \, dx$$
$$4\int \sec^5 x \, dx = \sec^3 x \tan x + 3\int \sec^3 x \, dx$$
. Hence
$$\Rightarrow \int \sec^5 x \, dx = \frac{1}{4}\sec^3 x \tan x + \frac{3}{4}\int \sec^3 x \, dx$$

4. [10 marks] Evaluate the following integral.

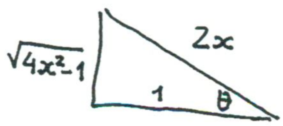
$$\int \frac{\sqrt{4x^2-1}}{x^3}\,dx$$

Put $x = \frac{1}{2}\sec\theta \Rightarrow dx = \frac{1}{2}\sec\theta\tan\theta\,d\theta$, $0 \leq \theta < \frac{\pi}{2}$ OR $\pi \leq \theta < \frac{3\pi}{2}$

$$\int \frac{\sqrt{4x^2-1}}{x^3}dx = \int \frac{\sqrt{\sec^2\theta - 1}}{\frac{1}{8}\sec^3\theta}\,\frac{1}{2}\sec\theta\tan\theta\,d\theta = 4\int \frac{\tan\theta}{\sec^3\theta}\cdot\sec\theta\tan\theta\,d\theta =$$

$$= 4\int \frac{\tan^2\theta}{\sec^2\theta}d\theta = 4\int \sin^2\theta\,d\theta = 4\int \frac{1-\cos 2\theta}{2}d\theta = 2\left(\int d\theta - \int \cos 2\theta\,d\theta\right)$$

$$= 2\theta - \sin 2\theta + C$$



$$= 2\sec^{-1}(2x) - \frac{\sqrt{4x^2-1}}{2x^2} + C$$

OR

$$= 2\cos^{-1}\left(\frac{1}{2x}\right) - \frac{\sqrt{4x^2-1}}{2x^2} + C$$

$\sec\theta = 2x \Rightarrow \cos\theta = \frac{1}{2x}$

$\Rightarrow \sin\theta = \frac{\sqrt{4x^2-1}}{2x}$

Hence $\sin 2\theta = 2\sin\theta\cos\theta = \frac{\sqrt{4x^2-1}}{2x^2}$

$\theta = \sec^{-1}(2x)$ OR $\theta = \cos^{-1}\left(\frac{1}{2x}\right)$

5. [10 marks] Evaluate the following integral

$$\int \frac{4x}{x^3+x^2+x+1}dx$$

$$\frac{4x}{x^3+x^2+x+1} = \frac{4x}{x^2(x+1)+x+1} = \frac{4x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$= \frac{Ax^2+A+Bx^2+Bx+Cx+C}{(x+1)(x^2+1)} = \frac{(A+B)x^2+(B+C)x+A+C}{(x+1)(x^2+1)}$$

$$A+B=0 \Rightarrow B=-A$$
$$B+C=4 \qquad -A+C=4$$
$$A+C=0 \qquad A+C=0$$
$$2C=4 \Rightarrow \boxed{C=2} \quad \boxed{A=-2} \quad \boxed{B=2}$$

$$\int \frac{4x}{x^3+x^2+x+1}dx = \int\left(\frac{-2}{x+1}+\frac{2x+2}{x^2+1}\right)dx = -2\int\frac{dx}{x+1}+\int\frac{2x\,dx}{x^2+1}+2\int\frac{dx}{x^2+1}$$

$$= -2\ln|x+1| + \ln|x^2+1| + 2\arctan x + C$$

6. [10 marks] Determine whether the following integral is convergent or divergent. If it is convergent, find its value.

$$\int_0^9 \frac{dx}{\sqrt[3]{x-1}}$$

Function $f(x) = \frac{1}{\sqrt[3]{x-1}}$ has a discontinuity at $x=1 \in [0,9]$

So $\int_0^9 \frac{dx}{\sqrt[3]{x-1}} = \int_0^1 \frac{dx}{\sqrt[3]{x-1}} + \int_1^9 \frac{dx}{\sqrt[3]{x-1}}$

$= -\frac{3}{2} + 6 = \frac{9}{2}$

$\int_0^1 \frac{dx}{\sqrt[3]{x-1}} = \lim_{t\to 1^-} \int_0^t \frac{dx}{\sqrt[3]{x-1}} = \lim_{t\to 1^-} \int_{-1}^{t-1} \frac{du}{\sqrt[3]{u}} = \lim_{t\to 1^-} \int_{-1}^{t-1} u^{-\frac{1}{3}} du = \lim_{t\to 1^-} \frac{3}{2} u^{\frac{2}{3}} \Big|_{-1}^{t-1}$

Put $x-1=u \Rightarrow dx = du$

$= \frac{3}{2} \lim_{t\to 1^-} \left[(t-1)^{\frac{2}{3}} - (-1)^{\frac{2}{3}} \right] = \frac{3}{2}\cdot(-1) = -\frac{3}{2}$

$\int_1^9 \frac{dx}{\sqrt[3]{x-1}} = \lim_{t\to 1^+} \int_t^9 \frac{dx}{\sqrt[3]{x-1}} = \lim_{t\to 1^+} \int_{t-1}^{8} u^{-\frac{1}{3}} du = \lim_{t\to 1^+} \frac{3}{2} u^{\frac{2}{3}} \Big|_{t-1}^{8} =$

$= \frac{3}{2} \lim_{t\to 1^+} \left[8^{\frac{2}{3}} - (t-1)^{\frac{2}{3}} \right] = \frac{3}{2}\cdot 4 = 6$