MATH 251 Linear Algebra I: Review

Ryan M. Gibara

Autumn 2013

This text is a brief survey of what is needed for exams of MATH 251. Basic definitions are given, along with main theorems and some results. Most importantly, however, this text contains some of the types of questions that a student should expect to see. Everything is taken from the textbook, past midterms, homework assignments, and final exams. There is no guarantee that these questions are representative of the exams *this* semester or that you will see these questions *exactly*. Take this review package as being *extra* help with your studying, not your sole source.

The following represents the notational conventions that will used throughout this text. They may be different from those used by your professor; the trick is to *understand* what they mean instead of *memorizing* what they mean.

- For a field, the symbol $\mathbb{F}$ will be used. Possible specific fields include $\mathbb{C}$, the set of all complex numbers; $\mathbb{R}$, the set of all real numbers; or $\mathbb{Q}$, the set of all rational numbers.
- The vector space of n-tuples of elements from a given field $\mathbb{F}$ is denoted $\mathbb{F}^n$.
- The vector space of all polynomials of degree less than or equal to n with coefficients from a field $\mathbb{F}$ is denoted $\mathcal{P}_n(\mathbb{F})$. For the space of all polynomials, the notation $\mathcal{P}(\mathbb{F})$ is used.
- The vector space of all $m \times n$ matrices with entries from a field $\mathbb{F}$ is denoted $\mathcal{M}_{m,n}(\mathbb{F})$. For square matrices (i.e. those for which $m = n$), the notation $\mathcal{M}_n(\mathbb{F})$ is used.
- The vector space of all functions from a non-empty set S to a field $\mathbb{F}$ is denoted $\mathcal{F}(S, \mathbb{F})$. For real-valued functions of a real variable, the notation $\mathcal{F}(\mathbb{R})$ is used.

Note that for the purpose of MATH 251, students may generally interpret the arbitrary field $\mathbb{F}$ as the familiar field $\mathbb{R}$, hence the vector space $\mathbb{F}^n$ as the vector space $\mathbb{R}^n$.

A little note on linear algebra. Students often have the idea that this material is too abstract to be useful or applicable. It turns out, however, that some of the fundamental notions presented to you in this course relate to many other branches of mathematics and has applications in many fields and applied branches of mathematics. For instance, many of these notions follow through (and are, in some sense, extended) when one considers other algebraic structures other than vector spaces (MATH 369 and MATH 472, for example). Furthermore, the consideration of vector spaces that are infinite-dimensional and endowed with additional structures is the main focus of a branch of analysis called functional analysis. On a more applied side, the subject of quantum mechanics can be considered nothing more than an application of functional analysis. Also, do not think that the subject of linear algebra ends here! Even with MATH 252, linear algebra is a vaste cornucopia of mathematics about which one can easily spend a lot longer than one year learning.

I hope you enjoy it as much as I do.

Section 1: Vector Spaces

Definitions

1. A set V is called a vector space (and its elements, vectors) over a field $\mathbb{F}$ if it is endowed with two binary operations (called addition and scalar multiplication) such that the following axioms hold:
   - $u + v = v + u$ for all $u, v \in V$
   - $(u + v) + w = u + (v + w)$ for all $u, v, w \in V$
   - there exists a zero element v such that $u + v = u$ for all $u \in V$
   - for each element $u \in V$ there exists an additive inverse $v \in V$ such that $u + v = \theta$
   - $1u = u$ for all $u \in V$
   - $(ab)u = a(bu)$ for all $a, b \in \mathbb{F}$ and $u \in V$
   - $a(u + v) = au + av$ for all $a \in \mathbb{F}$ and $u, v \in V$
   - $(a + b)u = au + bu$ for all $a, b \in \mathbb{F}$ and $u \in V$.
2. A subset W of a vector space V is called a subspace of V if it is, itself, a vector space under the same operations as V.

Theorems

Let V be a vector space over a field $\mathbb{F}$.

1. The following come directly from the axioms of a vector space:
   - If $u, v, w \in V$ such that $u + w = v + w$, then $u = v$.
   - The zero vector in V is unique and denoted θ_V.
   - The additive inverse of a vector $u \in V$ is unique and denoted $-u$.
2. Let W be a subset of V. Then, W is a subspace if and only if the following hold:
   - $\theta_V \in W$
   - $w_1 + w_2 \in W$ for all $w_1, w_2 \in W$
   - $aw \in W$ for all $a \in \mathbb{F}$ and $w \in W$.
3. Let W_1, W_2 be subspaces of V. Then, the following are true:
   - The intersection $W_1 \cap W_2$ is a subspace of V.
   - The sum $W_1 + W_2$ is a subspace of V.
   - The union $W_1 \cup W_2$ is not necessarily a subspace of V.

Problems

1. Is $W_1 = \{(a_1, a_2, a_3) \in \mathbb{F}^3 : a_1 + a_2 - a_3 = 0,\ a_1 - a_2 - a_3 = 0\}$ a subspace of $\mathbb{F}^3$?
2. Is $W_2 = \{(x, y) \in \mathbb{R}^2 : y = x^2\}$ a subspace of $\mathbb{R}^2$?
3. Is $W_3 = \{a_0 + a_1 x + a_2 x^2 \in \mathcal{P}_2(\mathbb{R}) : a_0, a_1, a_2 \in [0, \infty)\}$ a subspace of $\mathbb{P}_2(\mathbb{R})$?
4. Is $W_4 = \{A \in \mathcal{M}_{m,n}(\mathbb{F}) : a_{i,j} = 0\ \forall i > j\}$ a subspace of $\mathbb{M}_{m,n}(\mathbb{F})$?
5. Is $W_5 = \{A \in \mathcal{M}_n(\mathbb{Q}) : A^t = -A\}$ a subspace of $\mathcal{M}_n(\mathbb{Q})$?
6. Is $W_6 = \{f \in \mathcal{F}(\mathbb{R}) : \forall x \in \mathbb{R}, -f(x) = f(-x)\}$ a subspace of $\mathcal{F}(\mathbb{R})$?

Solutions

1. Yes, because the following are true:

   - $(0,0,0) \in W_1$ since $0+0-0=0$ and $0-0-0=0$;
   - for $x=(x_1,x_2,x_3) \in W_1$ and $y=(y_1,y_2,y_3) \in W_1$ we have, by the definition of W_1, that $x_1+x_2-x_3=0, x_1-x_2-x_3=0$ and $y_1+y_2-y_3=0, y_1-y_2-y_3=0$. Thus, $x+y=(x_1+y_1,x_2+y_2,x_3+y_3) \in W_1$ since $(x_1+y_1)+(x_2+y_2)-(x_3+y_3) = (x_1+x_2-x_3)+(y_1+y_2-y_3)=0+0=0$ and $(x_1+y_1)-(x_2+y_2)-(x_3+y_3) = (x_1-x_2-x_3)+(y_1-y_2-y_3)=0+0=0$;
   - for $x=(x_1,x_2,x_3) \in W_1$ we have, by the definition of W_1, that $x_1+x_2-x_3 = 0, x_1-x_2-x_3=0$. Thus, for a $c \in \mathbb{F}$, we have that $cx=(cx_1,cx_2,cx_3) \in W_1$ since $cx_1+cx_2-cx_3=c(x_1+x_2-x_3)=c(0)=0$ and $cx_1-cx_2-cx_3=c(x_1-x_2-x_3) = c(0)=0$.

   Therefore, W_1 is a subspace of $\mathbb{F}^3$.

2. No, because the following is true:

   - $(0,0) \in W_2$ because $0=0^2$;

   however, the following are not:

   - for $u=(x_1,y_1) \in W_2$ and $v=(x_2,y_2) \in W_2$ we have, by the definition of W_2, that $y_1=x_1^2$ and $y_2=x_2^2$. But, considering $u+v=(x_1+x_2,y_1+y_2)$, is not in W_2 since we have that $y_1+y_2=x_1^2+x_2^2 \neq (x_1+x_2)^2$;
   - for $u=(x_1,y_1) \in W_2$ and $c \in \mathbb{R}$, we have, by definition of W_2, that $y_1=x_1^2$. But, $cu=(cx_1,cy_1)$ is not in W_2 since $cy_1=cx_1^2 \neq (cx_1)^2$.

   Therefore, W_2 is not a subspace of $\mathbb{R}^2$.

3. No, because the following are true:

   - The zero polynomial is in W_3 because all its coefficients are in $[0,\infty)$ since they are all zero (i.e. $0 \in [0,\infty)$, clearly);
   - for $p_1(x)=\alpha_0+\alpha_1x+\alpha_2x^2 \in W_3$ and $p_2(x)=\beta_0+\beta_1x+\beta_2x^2 \in W_3$ we have, by the definition of W_3 that $\alpha_0,\alpha_1,\alpha_2 \in [0,\infty)$ and $\beta_0,\beta_1,\beta_2 \in [0,\infty)$. Thus, $(p_1+p_2)(x) = (\alpha_0+\beta_0)+(\alpha_1+\beta_1)x+(\alpha_2+\beta_2)x^2 \in W_3$ since $\alpha_0+\beta_0,\alpha_1+\beta_1,\alpha_2+\beta_2 \in [0,\infty)$ because the sum of two non-negative real numbers is a non-negative real number;

   however, the following is not:

   - for $p_1(x)=\alpha_0+\alpha_1x+\alpha_2x^2 \in W_3$ we have, by the definition of W_3, that $\alpha_0,\alpha_1,\alpha_2 \in [0,\infty)$; but, for $c \in \mathbb{R}$, $(cp_1)(x)=c\alpha_0+c\alpha_1x+c\alpha_2x^2$ may not be in W_3 since the product of a real number and a non-negative real number may not be a non-negative real number (pick $c=-1$, so $c\alpha_0=-\alpha_0 \in (-\infty,0]$).

   Therefore, W_3 is not a subspace of $\mathcal{P}_2(\mathbb{R})$.

4. Yes, because the following are true:

   - The zero matrix θ is in W_4 since $a_{i,j}=0$ for all $i>j$ because $a_{i,j}=0$ for all i,j by definition of the zero matrix;
   - for $A=\{a_{i,j}\} \in W_4$ and $B=\{b_{i,j}\} \in W_4$ we have, by the definition of W_4, that $a_{i,j}=0$ and $b_{i,j}=0$ for all $i>j$. Thus, $A+B=\{(a+b)_{i,j}\} \in W_4$ since $(a+b)_{i,j} = a_{i,j}+b_{i,j}=0+0=0$ for all $i>j$;

- for $A = \{a_{i,j}\} \in W_4$ we have, by the definition of W_4, that $a_{i,j} = 0$ for all $i > j$. Thus, for a $c \in \mathbb{F}$, we have that $cA = \{(ca)_{i,j}\} \in W_4$ since $(ca)_{i,j} = c(a_{i,j}) = c(0) = 0$ for all $i > j$.

Therefore, W_4 is a subspace of $\mathbb{M}_{m,n}(\mathbb{F})$.

5. Yes, because the following are true:

   - The zero matrix θ is in W_5 since $\theta^t = \theta = -\theta$;
   - for $A, B \in W_5$ we have, by definition of W_5, that $A^t = -A$ and $B^t = -B$. Thus, $A + B \in W_5$ since $(A + B)^t = A^t + B^t = -A - B = -(A + B)$;
   - for $A \in W_5$ and $c \in \mathbb{Q}$, we have, by definition of W_5, that $A^t = -A$. Thus, $(cA)^t = cA^t = c(-A) = -(cA)$, i.e. $cA \in W_5$.

   Therefore, W_5 is a subspace of $\mathcal{M}_n(\mathbb{Q})$.

6. Yes, because the following are true:

   - The zero function $\theta(x) \equiv 0$ is in W_6 since $-\theta(x) = -0 = 0 = \theta(-x)$ for all $x \in \mathbb{R}$;
   - for $f \in W_6$ and $g \in W_6$ we have, by the definition of W_6, that $-f(x) = f(-x)$ and $-g(x) = g(-x)$ for all $x \in \mathbb{R}$. Thus, $(f+g) \in W_6$ since $-(f+g)(x) = -(f(x)+g(x)) = -f(x) - g(x) = f(-x) + g(-x) = (f + g)(-x)$ for all $x \in \mathbb{R}$;
   - for $f \in W_6$ we have, by the definition of W_6, that $-f(x) = f(-x)$ for all $x \in \mathbb{R}$. Thus, for a $c \in \mathbb{R}$, we have that $(cf) \in W_6$ since $-(cf)(x) = -cf(x) = c(-f(x)) = c(f(-x)) = (cf)(-x)$ for all $x \in \mathbb{R}$.

   Therefore, W_6 is a subspace of $\mathcal{F}(\mathbb{R})$.

Section 2: Bases

Definitions

Let V be a vector space and $\beta = \{v_1, v_2 \cdots, v_n\}$ be a non-empty subset of V.

1. The span of β is the set of all linear combinations $a_1v_1 + a_2v_2 + \cdots + a_nv_n$ of the vectors in β.

2. The set β is called linearly independent if the only solution to the equation $a_1v_1 + a_2v_2 + \cdots + a_nv_n = \theta_V$ is the trivial solution $a_1 = a_2 = \cdots = a_n = 0$.

3. The set β is called a basis if its span is the entire space V (i.e. it generates V) and it is linearly independent.

4. If β is a basis for V, then the unique number of vectors in β (in this case n) is called the dimension of V.

Theorems

Let V be a vector space with subset $\beta = \{v_1, v_2 \cdots, v_n\}$ of V.

1. The set β is a basis if and only if each element of V can be uniquely written as a linear combination of elements of β. In other words, for all $v \in V$ there exist constants $a_1, \cdots, a_n$ such that
$$v = \sum_{i=1}^{n} a_i v_i.$$

2. If β is a finite basis for V, then every basis for V contains the same number of elements as β (in this case n).

3. If β is a basis for V (i.e. $\dim(V) = n$), then any generating set of V contains at least n vectors and any linearly independent set contains at most n vectors. Furthermore, any generating set can be reduced to a basis for V and any linearly independent set can be extended to a basis for V. Moreover, if a subset of V has exactly n vectors and is either a generating set for V or linearly independent in V, then it is a basis for V.

4. If W_1, W_2 are subspaces of finite-dimensional V, then $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

5. Let W be a subspace of V and let V be finite dimensional. Then, W is finite dimensional and $\dim(W) \leq \dim(V)$. Moreover, if $\dim(W) = \dim(V)$, then $W = V$.

6. If V is finite-dimensional, any basis for a subspace W can be extended to a basis for V.

Problems

1. Is the set $\{e^{rt}, e^{st}\}$ for some $r, s \in \mathbb{R}$ such that $r \neq s$ linearly independent in $\mathcal{F}(\mathbb{R})$?

2. Does the set $\{1, 1 + x, 1 + x + x^2\}$ generate $\mathcal{P}_2(\mathbb{F})$?

3. Find a basis for the subspace of $\mathbb{R}^4$ that is the solution space of the following system:
$$\begin{aligned} x_1 - x_2 + 5x_3 - x_4 &= 0 \\ 2x_1 + x_2 - 4x_3 - 7x_4 &= 0. \end{aligned}$$

4. Find a basis for W_1, W_2, $W_1 \cap W_2$, and $W_1 + W_2$, where

$$W_1 = \left\{ \begin{pmatrix} a & a \\ a & b \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) : a, b \in \mathbb{R} \right\}$$
$$W_2 = \left\{ \begin{pmatrix} 0 & d \\ d & e \end{pmatrix} \in \mathcal{M}_2(\mathbb{R}) : d, e \in \mathbb{R} \right\}$$

and verify that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

5. Determine which of the following are bases:

   (a) $\beta_1 = \{1 + x, 1 - x^2\}$ in $\mathcal{P}_2(\mathbb{R})$.

   (b) $\beta_2 = \left\{ \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$ in $\mathbb{R}^3$.

   (c) $\beta_3 = \left\{ \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \right\}$ in $\mathcal{M}_2(\mathbb{R})$.

Solutions

1. Consider the set $\{e^{rt}, e^{st}\}$ for $r, s \in \mathbb{R}$ such that $r \neq s$. Writing the zero function as a linear combination of these two vectors, we have that $ae^{rt} + be^{st} = \theta(t)$ for some $a, b \in \mathbb{R}$, where θ represents the zero function. Clearly, since we are considering the vector space $\mathcal{F}(\mathbb{R})$, that equation (which is an equation of functions, remember) must remain valid for all choices of t. Thus, for $t = 0$ we have $e^{r(0)} = e^{s(0)} = 1$ and $\theta(0) = 0$ (this is valid, in truth, for all t), giving us $a(1) + b(1) = 0 \Rightarrow a = -b$. Hence, we may modify our linear combination to read $ae^{rt} - ae^{st} = \theta(t)$. From here, we have that $a(e^{rt} - e^{st}) = \theta(t)$. In order for this to be true for all t we must either have that $a = 0$ or $e^{rt} - e^{st} = \theta(t)$ (recall the zero-product property of real numbers from high school); however, our assumption that $r \neq s$ implies that $e^{rt} - e^{st}$ is distinct from 0 for all values of t, i.e. not equal to $\theta(t)$. To see this more clearly, assume that $e^{rt} - e^{st} = \theta(t)$ is true for all $t \in \mathbb{R}$. Then, $e^{rt} = e^{st}$ for all $t \in \mathbb{R}$ and $1 = e^{(r-s)t}$ for all $t \in \mathbb{R}$ which is a contradiction since this can only be true when $r = s$. Therefore, we have that $a = b = 0$ and the set is linearly independent.

2. To determine whether the given set generates $\mathcal{P}_2(\mathbb{F})$ is the same as to determine whether the set spans $\mathcal{P}_2(\mathbb{F})$. So we attempt to find constants $a, b, c \in \mathbb{F}$ such that $a(1) + b(1 + x) + c(1 + x + x^2) = p(x)$ for all $p \in \mathcal{P}_2(\mathbb{F})$. Representing $p(x)$ as $\alpha + \beta x + \gamma x^2$, we are trying to solve $a(1) + b(1 + x) + c(1 + x + x^2) = \alpha + \beta x + \gamma x^2$ for the unknowns a, b, c to hopefully say that they exist for all possible α, β, γ. We can express this as an augmented matrix now, to find

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & \alpha \\ 0 & 1 & 1 & \beta \\ 0 & 0 & 1 & \gamma \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & \alpha - \beta \\ 0 & 1 & 0 & \beta - \gamma \\ 0 & 0 & 1 & \gamma \end{array}\right).$$

Therefore, we have that the set $\{1, 1+x, 1+x+x^2\}$ generates $\mathcal{P}_2(\mathbb{F})$ since we can write any element of $\mathcal{P}_2(\mathbb{F})$ as a linear combination of the elements of the set namely, $(\alpha - \beta)(1) + (\beta - \gamma)(1 + x) + \gamma(1 + x + x^2) = \alpha + \beta x + \gamma x^2$.

3. To find a basis for the solution space of a homogeneous system of linear equations (note that the solution space is guaranteed to be non-empty since the system homogeneous), we write out its solution:

$$\left(\begin{array}{cccc|c} 1 & -1 & 5 & -1 & 0 \\ 2 & 1 & -4 & -7 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & \frac{1}{3} & -\frac{8}{3} & 0 \\ 0 & 1 & -\frac{14}{3} & -\frac{5}{3} & 0 \end{array}\right) \Rightarrow \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = s \begin{pmatrix} -1 \\ 14 \\ 3 \\ 0 \end{pmatrix} + t \begin{pmatrix} 8 \\ 5 \\ 0 \\ 3 \end{pmatrix}$$

where we use the parametres $3s = x_3$ and $3t = x_4$. We see immediately that we can write the set of all solutions, X, as

$$X = \text{span}\left\{\begin{pmatrix} -1 \\ 14 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 5 \\ 0 \\ 3 \end{pmatrix}\right\}.$$

Therefore, a basis for the solution space is the set

$$\left\{\begin{pmatrix} -1 \\ 14 \\ 3 \\ 0 \end{pmatrix}, \begin{pmatrix} 8 \\ 5 \\ 0 \\ 3 \end{pmatrix}\right\}.$$

This is easily justified by showing that the two vectors are linearly independent, which is guaranteed since neither is a multiple of the other. That they span the entire subspace is guaranteed by our having written the subspace as a span of these very vectors.

4. We consider each of the four subspaces of $\mathcal{M}_2(\mathbb{R})$ separately. Firstly, for W_1 we have that

$$\begin{pmatrix} a & a \\ a & b \end{pmatrix} = \begin{pmatrix} a & a \\ a & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix} = a\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e.

$$W_1 = \text{span}\left\{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\},$$

making $\dim(W_1) = 2$. For W_2, we have that

$$\begin{pmatrix} 0 & d \\ d & e \end{pmatrix} = \begin{pmatrix} 0 & d \\ d & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & e \end{pmatrix} = d\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + e\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

making

$$W_2 = \text{span}\left\{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\},$$

so that $\dim W_2 = 2$. For any matrix in $W_1 \cap W_2$, we must have that it is contained in both W_1 and W_2, thus we must have

$$\begin{pmatrix} a & a \\ a & b \end{pmatrix} = \begin{pmatrix} 0 & d \\ d & e \end{pmatrix},$$

giving us that

$$W_1 \cap W_2 = \text{span}\left\{\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right\}.$$

Therefore, we have that $\dim(W_1 \cap W_2) = 1$. Lastly, we have that any matrix in $W_1 + W_2$ must be the sum of a matrix from each of W_1 and W_2, thus we must have

$$\begin{pmatrix} a & a \\ a & b \end{pmatrix} + \begin{pmatrix} 0 & d \\ d & e \end{pmatrix} = \begin{pmatrix} a & a+d \\ a+d & b+e \end{pmatrix},$$

giving us the form of any matrix in $W_1 + W_2$ as

$$\begin{pmatrix} a & a+d \\ a+d & b+e \end{pmatrix} = a\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + e\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$

i.e.

$$W_1 + W_2 = \text{span}\left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$$

and $\dim(W_1 + W_2) = 3$.

We see now that $\dim(W_1)+\dim(W_2) = 3 = 2+2-1 = \dim(W_1)+\dim(W_2)-\dim(W_1 \cap W_2)$. Alternatively, once the bases for W_1 and W_2 were found, we could have immediately said that a basis for $W_1 \cap W_2$ is the intersection of the bases of W_1 and W_2, while a basis for $W_1 + W_2$ is the union of the bases of W_1 and W_2.

5. (a) β_1 cannot be a basis for $\mathcal{P}_2(\mathbb{R})$ since $\dim(\mathcal{P}_2(\mathbb{R})) = 3 \neq 2 = |\beta_1|$.

   (b) Since $|\beta_2| = \dim(\mathbb{R}^3)$, we need to show either that β_2 is linearly independent or that it generates $\mathbb{R}^3$. Let's choose linearly independence. We begin with the linear combination

$$a \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

   and try to show that $a = b = c = 0$ by using the augmented matrix

$$\left(\begin{array}{ccc|c} 0 & 1 & -1 & 0 \\ 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right),$$

   which gives us the desired result. Therefore β_2 is linearly independent and, thus, a basis for $\mathbb{R}^3$.

   (c) Since $|\beta_3| = \dim(\mathcal{M}_2(\mathbb{R}))$, we need to show either that β_3 is linearly independent or that it generates $\mathcal{M}_2(\mathbb{R})$. Like, before, we check for linear independence. We begin with the linear combination

$$a \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + b \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} + c \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + d \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

   and try to show that $a = b = c = d = 0$ by using the augmented matrix

$$\left(\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 0 \end{array}\right) \sim \left(\begin{array}{cccc|c} 1 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

   which does not give us the desired result (for instance, the linear combination is satisfied for $a = 1, b = 0, c = -1, d = 1$). We see now that β_3 is not linearly independent and, therefore, not a basis for $\mathcal{M}_2(\mathbb{R})$.

Section 3: Linear Transformations

Definitions

Let V, W be vector spaces over a field $\mathbb{F}$.

1. A mapping $T : V \to W$ is called a linear transformation if
   - $T(v_1 + v_2) = T(v_1) + T(v_2)$ for all $v_1, v_2 \in V$,
   - $T(av) = aT(v)$ for all $a \in \mathbb{F}$ and $v \in V$.

   This is denoted by $T \in \mathcal{L}(V, W)$.

2. The nullspace (or kernel) of a linear transformation $T : V \to W$ is the set of all vectors in V that maps to θ_W, i.e.
$$N(T) = \{v \in V : T(v) = \theta_W\}.$$
   Also, the dimension of the nullspace is called the nullity of T.

3. The range of a linear transformation $T : V \to W$ is the set of all vectors in W to which something from V is mapped, i.e.
$$R(T) = \{w \in W : T(v) = w \text{ for some } v \in V\}.$$
   Also, the dimension of the range is called the rank of T.

4. A linear transformation $T : V \to W$ is said to be one-to-one if $T(v_1) = T(v_2) \Rightarrow v_1 = v_2$ for all $v_1, v_2 \in V$, and is said to be onto if $R(T) = W$.

5. A linear transformation $T : V \to W$ is said to be invertible if there exists another transformation $U : W \to V$ such that $TU = I_W$ and $UT = I_V$.

6. An isomorphism is an invertible linear transforamtion.

7. The vector spaces V, W are said to be isomorphic to each other, denoted $V \cong W$, if there exists an isomorphism between them.

Theorems

Let $T : V \to W$ be a linear transformation between vector spaces over a field $\mathbb{F}$ with dimensions n and m, respectively.

1. (Dimension Theorem) If $n < \infty$, then
$$\dim N(T) + \dim R(T) = n.$$

2. T is one-to-one if and only if $N(T) = \{\theta_V\}$.

3. If $n = m < \infty$, then T is one-to-one if and only if it is onto.

4. T is invertible if and only if it is one-to-one and onto.

5. $V \cong W$ if and only if $n = m$.

6. $\mathcal{L}(V, W) \cong M_{m,n}(\mathbb{F})$.

7. If $n = m$ and β is a basis for V, then T is an isomorphism if and only if $T(\beta)$ is a basis for W.

Results

1. For all linear transformations $T : V \to W$ between vector spaces, $T(\theta_V) = \theta_W$.
2. If T is invertible, then its inverse T^{-1} is unique and linear.

Problems

For each of the following mappings, show that it is a linear transformation, find a basis for its nullspace and range, and verify the Dimension Theorem if it applies. Is the linear transformation transformation one-to-one, onto, invertible?

1. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(a, b, c) = (a + 2b - c, b + c, a + b + 2c)$.
2. Define $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ by $T(f) = f(x-1) + x^2 f''(x)$.
3. Define $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}^3$ by $T(f) = \big(f(-1), f(0), f(1)\big)$.
4. Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $T(f) = \int_0^x f(t)\,\mathrm{d}t$.
5. Define $T : \mathcal{P}(\mathbb{R}) \to \mathcal{P}(\mathbb{R})$ by $T(f) = f'(x)$.

Solutions

1. To show that T is linear, we take $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$ and $c \in \mathbb{R}$. Then,

$$\begin{aligned} T(cx + y) &= T(cx_1 + y_1, cx_2 + y_2, cx_3 + y_3) \\ &= \big((cx_1 + y_1) + 2(cx_2 + y_2) - (cx_3 + y_3), (cx_2 + y_2) + (cx_3 + y_3), \\ &\qquad (cx_1 + y_1) + (cx_2 + y_2) + 2(cx_3 + y_3)\big) \\ &= c(x_1 + 2x_2 - x_3, x_2 + x_3, x_1 + x_2 + 2x_3) \\ &\qquad +(y_1 + 2y_2 - y_3, y_2 + y_3, y_1 + y_2 + 2y_3) \\ &= cT(x) + T(y). \end{aligned}$$

   To find its nullspace we set, for $x = (x_1, x_2, x_3)$ and the zero 3-vector θ,

$$T(x) = \theta \Leftrightarrow \left\{ \begin{array}{ll} 0 = & x_1 + 2x_2 - x_3 \\ 0 = & x_2 + x_3 \\ 0 = & x_1 + x_2 + 2x_3 \end{array} \right. \Leftrightarrow x_1 = x_2 = x_3 = 0.$$

   Thus, we have that $N(T) = \{\theta\}$. Therefore, a basis for $N(T)$ is $\emptyset$, the nullity of T is 0, and T is one-to-one.

   Since $\dim(\mathbb{R}^3) = 3 < \infty$ and T is between $\mathbb{R}^3$ and itself, we have that T is onto since its one-to-one, i.e. $R(T) = \mathbb{R}^3$. Hence, T is onto, a basis for $R(T)$ is the standard basis for $\mathbb{R}^3$, and the rank of T is 3.

   Since T is onto and one-to-one, it is invertible.

   The Dimension Theorem applies since $\dim(\mathbb{R}^3) < \infty$ and is easily verified: $\dim N(T) + \dim R(T) = 0 + 3 = 3 = \dim \mathbb{R}^3$.

2. To show that T is linear, we take $p_1(x)$ and $p_2(x)$ in $\mathcal{P}_2(\mathbb{R})$, and $c \in \mathbb{R}$. Then,

$$\begin{aligned} T(cp_1 + p_2) &= (cp_1 + p_2)(x-1) + x^2 (cp_1 + p_2)''(x) \\ &= c\big(p_1(x-1) + x^2 p_1''(x)\big) + \big(p_2(x-1) + x^2 p_2''(x)\big) \\ &= cT(p_1) + T(p_2). \end{aligned}$$

To find its nullspace we set, for $p(x) = a + bx + cx^2$ and the zero polynomial θ,

$$\begin{aligned} T(p) = \theta \quad &\Leftrightarrow \quad p(x-1) + x^2 p''(x) = \theta(x) \\ &\Leftrightarrow \quad (a - b + c) + (b - 2c)x + 3cx^2 = \theta(x) \\ &\Leftrightarrow \quad \begin{cases} 0 = & a - b + c \\ 0 = & b - 2c \\ 0 = & 3c \end{cases} \\ &\Leftrightarrow \quad a = b = c = 0. \end{aligned}$$

Thus, we have that $N(T) = \{\theta\}$. Therefore, a basis for $N(T)$ is $\emptyset$, the nullity of T is 0, and T is one-to-one.

Since $\dim(\mathcal{P}_2(\mathbb{R})) = 3 < \infty$ and T is between $\mathcal{P}_2(\mathbb{R})$ and itself, we have that T is onto since its one-to-one, i.e. $R(T) = \mathcal{P}_2(\mathbb{R})$. Hence, T is onto, a basis for $R(T)$ is the standard basis for $\mathcal{P}_2(\mathbb{R})$, and the rank of T is 3.

Since T is onto and one-to-one, it is invertible.

The Dimension Theorem applies since $\dim(\mathcal{P}_2(\mathbb{R})) < \infty$ and is easily verified: $\dim N(T) + \dim R(T) = 0 + 3 = 3 = \dim \mathcal{P}_2(\mathbb{R})$.

3. To show that T is linear, we take $p_1(x)$ and $p_2(x)$ in $\mathcal{P}_2(\mathbb{R})$, and $c \in \mathbb{R}$. Then,

$$\begin{aligned} T(cp_1 + p_2) &= \big((cp_1 + p_2)(-1), (cp_1 + p_2)(0), (cp_1 + p_2)(1)\big) \\ &= c\big(p_1(-1), p_1(0), p_1(1)\big) + \big(p_2(-1), p_2(0), p_2(1)\big) \\ &= cT(p_1) + T(p_2). \end{aligned}$$

To find its nullspace we set, for $p(x) = a + bx + cx^2$ with the zero 3-vector θ_1,

$$\begin{aligned} T(p) = \theta \quad &\Leftrightarrow \quad \big(p(-1), p(0), p(1)\big) = \theta \\ &\Leftrightarrow \quad (a - b + c, a, a + b + c) = \theta \\ &\Leftrightarrow \quad \begin{cases} 0 = & a - b + c \\ 0 = & a \\ 0 = & a + b + c \end{cases} \\ &\Leftrightarrow \quad a = b = c = 0. \end{aligned}$$

Thus, we have that $N(T) = \{\theta_2\}$, where θ_2 is the zero polynomial. Therefore, a basis for $N(T)$ is $\emptyset$, the nullity of T is 0, and T is one-to-one.

Since $\dim(\mathcal{P}_2(\mathbb{R})) = \dim(\mathbb{R}^3) = 3 < \infty$, we have that T is onto since its one-to-one, i.e. $R(T) = \mathbb{R}^3$. Hence, T is onto, a basis for $R(T)$ is the standard basis for $\mathbb{R}^3$, and the rank of T is 3.

Since T is onto and one-to-one, it is invertible.

The Dimension Theorem applies since $\dim(\mathcal{P}_2(\mathbb{R})) < \infty$ and is easily verified: $\dim N(T) + \dim R(T) = 0 + 3 = 3 = \dim \mathcal{P}_2(\mathbb{R})$.

4. To show that T is linear, we take $p_1(x)$ and $p_2(x)$ in $\mathcal{P}(\mathbb{R})$, and $c \in \mathbb{R}$. Then,

$$\begin{aligned} T(cp_1 + p_2) &= \int_0^x (cp_1 + p_2)(t)\,\mathrm{d}t \\ &= c\int_0^x p_1(t)\,\mathrm{d}t + \int_0^x p_2(t)\,\mathrm{d}t \\ &= cT(p_1) + T(p_2). \end{aligned}$$

To find its nullspace we set, for $p(x) = a_0 + a_1x + a_2x^2 + \cdots$ and the zero polynomial θ,

$$\begin{aligned} T(p) = \theta \quad &\Leftrightarrow \quad \int_0^x p(t)\,\mathrm{d}t = \theta(x) \\ &\Leftrightarrow \quad p(x) = \theta(x), \end{aligned}$$

using the Fundamental Theorem of Calculus.

Thus, we have that $N(T) = \{\theta\}$. Therefore, a basis for $N(T)$ is $\emptyset$, the nullity of T is 0, and T is one-to-one.

Now, however, we cannot use the familiar theorem to say anything about $R(T)$. In fact, T is not onto because $R(T) \neq \mathcal{P}(\mathbb{R})$. This can most easily be seen by example: we show that there exists no polynomial p such that $\int_0^x p(t)dt = I(x)$, where I is the identity polynomial.

By the Fundamental Theorem of Calculus, we have that $p(x) = \theta(x)$, but if this is so, $\int_0^x p(t)dt = \int_0^x \theta(t)dt = \theta(x)$, which is a contradiction.

Since T is one-to-one but not onto, it is not invertible.

In terms of the Dimension Theorem, it does not apply since we are considering an infinite-dimensional vector space.

5. To show that T is linear, we take $p_1(x)$ and $p_2(x)$ in $\mathcal{P}(\mathbb{R})$, and $c \in \mathbb{R}$. Then,

$$\begin{aligned} T(cp_1 + p_2) &= (cp_1 + p_2)'(x) \\ &= cp_1'(x) + p_2'(x) \\ &= cT(p_1) + T(p_2). \end{aligned}$$

To find its nullspace we set, for $p(x) = a_0 + a_1x + a_2x^2 + \cdots$ and the zero polynomial θ,

$$\begin{aligned} T(p) = \theta \quad &\Leftrightarrow \quad p'(x) = \theta(x) \\ &\Leftrightarrow \quad p(x) = a_0, \end{aligned}$$

meaning that the nullspace $N(T)$ has basis $\{1\}$, i.e. dimension 1. So, we see that T is not one-to-one.

Now, however, we cannot use the familiar theorem to say anything about $R(T)$. That being said, T is actually onto. This is because for all $p(x) = a_0 + a_1x + a_2x^2 + \cdots \in \mathcal{P}(\mathbb{R})$ there exists a $q(x) = a_0x + \frac{a_1}{2}x^2 + \frac{a_2}{3}x^3 + \cdots \in \mathcal{P}(\mathbb{R})$ such that $T(q) = p$.

Since T is onto but not one-to-one, it is not invertible.

In terms of the Dimension Theorem, it does not apply since we are considering an infinite-dimensional vector space.

Section 4: Matrices of Transformations

Definitions

Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces V, W with ordered bases $\alpha = \{v_1, v_2, \cdots v_n\}$ and $\beta = \{w_1, w_2, \cdots, w_m\}$, respectively.

1. The coordinate vector of $v \in V$ with respect to the basis α is defined as

$$[v]_\alpha = \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}$$

   where the c_i are the unique constants such that $v = c_1 v_1 + c_2 v_2 + \cdots c_n v_n$.

2. The matrix representation of the linear transformation T with respect to the ordered bases α and β is defined to be the $m \times n$ matrix

$$[T]_\alpha^\beta = \left(\begin{array}{c|c|c|c} [T(v_1)]_\beta & [T(v_2)]_\beta & \cdots & [T(v_n)]_\beta \end{array} \right),$$

   where we simplify the notation to $[T]_\alpha$ if $V = W$ and $\alpha = \beta$.

Theorems

Let V, W, Z be vector spaces with respective ordered bases α, β, and γ, respectively.

1. The collection of all linear transformations running from V to W, denoted $\mathcal{L}(V, W)$, is a vector space.

2. If $T, U \in \mathcal{L}(V, W)$ and $a \in \mathbb{F}$, then

$$[T + U]_\alpha^\beta = [T]_\alpha^\beta + [U]_\alpha^\beta$$

   and

$$[aT]_\alpha^\beta = a[T]_\alpha^\beta.$$

3. If $T \in \mathcal{L}(V, W)$ and $U \in \mathcal{L}(W, Z)$, then

$$[UT]_\alpha^\gamma = [U]_\beta^\gamma [T]_\alpha^\beta.$$

4. If $T \in \mathcal{L}(V, W)$ and $v \in V$, then

$$[T(v)]_\beta = [T]_\alpha^\beta [v]_\alpha.$$

5. Let $T \in \mathcal{L}(V, W)$. Then, T is invertible if and only if $[T]_\alpha^\beta$ is invertible. Furthermore, $[T^{-1}]_\beta^\alpha = \left([T]_\alpha^\beta\right)^{-1}$.

Problems

1. Define $T : \mathcal{P}_2(\mathbb{R}) \to \mathbb{R}$ as $T(f) = f(2)$. Find $[T]_\alpha^\beta$ if $\alpha = \{x^2 + 3x - 2, 2x^2 + 5x - 3, -x^2 - 4x + 4\}$ and $\beta = \{1\}$, find $[f]_\alpha$ if $f(x) = 3 - 6x + x^2$, and find $[T(f)]_\beta$ if $f(x) = 4 + 2x^2$.

2. Define $T : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ as $T(A) = A^t$. Find $[T]_\gamma$ if $\gamma = \left\{ \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \right\}$, find $[A]_\gamma$ if $A = \begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$, and find $[T(A)]_\gamma$ if $A = \begin{pmatrix} 0 & 1 \\ -1 & 8 \end{pmatrix}$.

3. Define $T : \mathbb{R}^3 \to \mathbb{R}^2$ as $T(a_1, a_2, a_3) = (a_1 + 2a_2, a_2 + 2a_3)$. Find $[T]_{\sigma_1}^{\sigma_2}$ for $\sigma_1 = \{(1,1,1), (1,1,0), (1,0,0)\}$ and $\sigma_2 = \{(1,1), (1,2)\}$, find $[x]_{\sigma_1}$ if $x = (2,1,0)$, and find $[T(x)]_{\sigma_2}$ if $x = (2,1,0)$. Verify that the equation $[T(x)]_{\sigma_2} = [T]_{\sigma_1}^{\sigma_2}[x]_{\sigma_1}$ holds for this choice of x.

4. Define $T : V \to W$, where V and W are vector spaces, as $T(f) = f(x)$. Find $[T]_\alpha^\beta$ if $\alpha = \{1, \cos(2x)\}$ and $\beta = \{\cos^2(x), \sin^2(x)\}$, find $[f]_\alpha$ if $f(x) = \cos^2(x)$, and find $[T(f)]_\beta$ if $f(x) = \cos^2(x)$. Verify that the equation $[T(f)]_\beta = [T]_\alpha^\beta [f]_\alpha$ holds for this choice of f.

Solutions

1. We first need to find the transformations of each element of the basis α and write is as cordinate with respect to the basis β. Denoting the elements of α as $a_1(x), a_2(x), a_3(x)$ we have that

$$T(a_1) = a_1(2) = 8 \qquad T(a_2) = a_2(2) = 15 \qquad T(a_3) = a_3(2) = -8,$$

and the coordinates with respect to β are read off straightforwardly, giving us

$$\begin{pmatrix} 8 & 15 & -8 \end{pmatrix}.$$

To find $[3 - 6x + x^2]_\alpha$ we must successfully write f as a linear combinaion of the elements of α, i.e.

$$\left(\begin{array}{ccc|c} -2 & -3 & 4 & 3 \\ 3 & 5 & -4 & -6 \\ 1 & 2 & -1 & 1 \end{array}\right) \sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & -29 \\ 0 & 1 & 0 & 13 \\ 0 & 0 & 1 & -4 \end{array}\right)$$

so we have that $[3 - 6x + x^2]_\alpha = \begin{pmatrix} -29 \\ 13 \\ -4 \end{pmatrix}$.

To find $[T(4 + 2x^2)]_\beta$ is actually quite easy. Since $T(4 + 2x^2) = 4 + 2(2)^2 = 12$ we have that $[12]_\beta = (12)$.

2. To find the matrix of the linear operator with respect to this basis, we must find the coordinates of the the transformation of each vector in γ with respect to γ:

$$\left[T\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right]_\gamma = \left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^t\right]_\gamma = \left[\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right]_\gamma = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

$$\left[T\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\right]_\gamma = \left[\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}^t\right]_\gamma = \left[\begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}\right]_\gamma = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$$\left[T\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\right]_\gamma = \left[\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}^t\right]_\gamma = \left[\begin{pmatrix}0 & 1\\ 1 & 1\end{pmatrix}\right]_\gamma = \begin{pmatrix}0\\0\\1\\0\end{pmatrix},$$

$$\left[T\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}\right]_\gamma = \left[\begin{pmatrix}1 & 0\\ 1 & 1\end{pmatrix}^t\right]_\gamma = \left[\begin{pmatrix}1 & 1\\ 0 & 1\end{pmatrix}\right]_\gamma = \begin{pmatrix}0\\1\\0\\0\end{pmatrix},$$

which gives us the matrix

$$[T]_\gamma = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & 1 & 0\\ 0 & 1 & 0 & 0\end{pmatrix}.$$

Note that finding the coordinates required no computation since the transformed vectors were themselves a part of the basis.

Now, to find the coordinates of a vector, we need to find the linear combination of elements of γ that add up to it:

$$\left(\begin{array}{cccc|c}1 & 1 & 0 & 1 & 2\\ 1 & 1 & 1 & 0 & 3\\ 1 & 0 & 1 & 1 & -1\\ 0 & 1 & 1 & 1 & 0\end{array}\right) \sim \left(\begin{array}{cccc|c}1 & 0 & 0 & 0 & 4/3\\ 0 & 1 & 0 & 0 & 7/3\\ 0 & 0 & 1 & 0 & -2/3\\ 0 & 0 & 0 & 1 & -5/3\end{array}\right) \Rightarrow \left[\begin{pmatrix}2 & 3\\ -1 & 0\end{pmatrix}\right]_\gamma = \begin{pmatrix}4/3\\7/3\\-2/3\\-5/3\end{pmatrix}$$

We repeat the same process for the second matrix.

$$\left(\begin{array}{cccc|c}1 & 1 & 0 & 1 & 0\\ 1 & 1 & 1 & 0 & -1\\ 1 & 0 & 1 & 1 & 1\\ 0 & 1 & 1 & 1 & 8\end{array}\right) \sim \left(\begin{array}{cccc|c}1 & 0 & 0 & 0 & -16/3\\ 0 & 1 & 0 & 0 & 5/3\\ 0 & 0 & 1 & 0 & 8/3\\ 0 & 0 & 0 & 1 & 11/3\end{array}\right) \Rightarrow \left[T\begin{pmatrix}0 & 1\\ -1 & 8\end{pmatrix}\right]_\gamma = \begin{pmatrix}-16/3\\5/3\\8/3\\11/3\end{pmatrix}.$$

3. Like previous questions, we repeat the same steps. Except now, the finding of coordinates is not trivial, so we need to solve a system of equations for each coordinate we want to find. To help reduce redundant calculations, we can do them all at the same time:

$$T\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}3\\3\end{pmatrix},\ T\begin{pmatrix}1\\1\\0\end{pmatrix} = \begin{pmatrix}3\\1\end{pmatrix},\ T\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix},$$

so we calculate the coordinates

$$\left(\begin{array}{cc|ccc}1 & 1 & 3 & 3 & 1\\ 1 & 2 & 3 & 1 & 0\end{array}\right) \sim \left(\begin{array}{cc|ccc}1 & 0 & 3 & 5 & 2\\ 0 & 1 & 0 & -2 & -1\end{array}\right)$$

and end up with the matrix representation of the transformation

$$[T]_{\sigma_1}^{\sigma_2} = \begin{pmatrix}3 & 5 & 2\\ 0 & -2 & -1\end{pmatrix}.$$

Now we find $[(2,1,0)]_{\sigma_1}$:

$$\left(\begin{array}{ccc|c}1 & 1 & 1 & 2\\ 1 & 1 & 0 & 1\\ 1 & 0 & 0 & 0\end{array}\right) \sim \left(\begin{array}{ccc|c}1 & 0 & 0 & 0\\ 0 & 1 & 0 & 1\\ 0 & 0 & 1 & 1\end{array}\right) \Rightarrow \left[\begin{pmatrix}2\\1\\0\end{pmatrix}\right]_{\sigma_1} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

and $[T(2,1,0)]_{\sigma_2} = [(4,1)]_{\sigma_1}$:

$$\left(\begin{array}{cc|c} 1 & 1 & 4 \\ 1 & 2 & 1 \end{array}\right) \sim \left(\begin{array}{cc|c} 1 & 0 & 7 \\ 0 & 1 & -3 \end{array}\right) \Rightarrow \left[\begin{pmatrix} 4 \\ 1 \end{pmatrix}\right]_{\sigma_2} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}.$$

And now we verify:

$$\begin{pmatrix} 3 & 5 & 2 \\ 0 & -2 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}.$$

4. This question is less computational than others and simply requires that we remember our trigonometric identities. We have:

$$T(1) = 1 = 1\cos^2(x) + 1\sin^2(x) \qquad T(\cos(2x)) = \cos(2x) = 1\cos^2(x) - 1\sin^2(x)$$

so the matrix is

$$[T]_\alpha^\beta = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

In terms of $f(x) = \cos^2(x)$, $\cos^2(x) = (1/2)1 + (1/2)\cos(2x)$, giving us $[\cos^2(x)]_\alpha = \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix}$ and we have that $T(\cos^2(x)) = \cos^2(x) = 1\cos^2(x) + 0\sin^2(x)$, giving us $[T(\cos^2(x))]_\beta = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. Now, we verify:

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

Section 5: Diagonalizability

Definitions

Let $T \in \mathcal{L}(V)$, where $\dim V = n < \infty$.

1. A scalar λ is called an eigenvalue of T if $T(v) = \lambda v$ for some non-zero vector $v \in V$ called an eigenvector corresponding to λ.

2. The characteristic polynomial of T is defined to be $f(t) = \det(T - tI_V)$.

3. A polynomial $f(t)$ splits over a field $\mathbb{F}$ if there exist scalars $c, a_0, a_1, \cdots, a_n \in \mathbb{F}$ such that

$$f(t) = c(t - a_0)(t - a_1) \cdots (t - a_n),$$

   i.e. $f(t)$ can be factored into the product of linear factors with respect to the field $\mathbb{F}$.

4. The algebraic multiplicity of an eigenvalue is the number of times it is a root of the operator's characteristic polynomial.

5. The eigenspace of T corresponding to the eigenvalue λ is defined as $E_\lambda = \{v \in V : T(v) = \lambda v\} = N(T - \lambda I_V)$.

6. The linear operator T is called diagonalizable if there exists an ordered basis β for V such that $[T]_\beta$ is a diagonal matrix.

Theorems

Let $T \in \mathcal{L}(V)$, where $\dim V = n < \infty$.

1. A scalar λ is an eigenvalue of T if and only if $f(\lambda) = 0$, i.e. it is a root of the characteristic polynomial of T.

2. The linear operator T can have at most n distinct eigenvalues.

3. A vector v is an eigenvector of T corresponding to the eigenvalue λ if and only if $v \neq \theta$ and $v \in N(T - \lambda I_V)$.

4. If $v_1, v_2, \cdots, v_k$ are eigenvectors corresponding to the distinct eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_k$, then the set $\{v_1, v_2, \cdots, v_k\}$ is linearly independent.

5. If λ is an eigenvalue of multiplicity m, then $1 \leq \dim(E_\lambda) \leq m$.

6. The linear operator T is diagonalizable if and only if there exists an ordered basis β for V consisting of eigenvectors of T.

7. The linear operator T is diagonalizable if and only if its characteristic polynomial splits and the multiplicity of each eigenvalue λ_i equals $\dim(E_{\lambda_i})$.

8. If the linear operator T is diagonalizable and β_i is an ordered basis for E_{λ_i}, then $\beta = \bigcup_i \beta_i$ is a basis for V consisting of eigenvectors of T.

Problems

1. For each of the following linear operators, determine if it is diagonalizable. If it is, find a basis β such that $[T]_\beta$ is diagonalizable.

   - $T : \mathcal{P}_2(\mathbb{R}) \to \mathcal{P}_2(\mathbb{R})$ defined by $T(f) = xf'(x) + f(2)x + f(3)$,
   - $T : \mathcal{P}_3(\mathbb{R}) \to \mathcal{P}_3(\mathbb{R})$ defined by $T(f) = f'(x) + f''(x)$,
   - $T : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ defined by $T(A) = A^t$,
   - $T : \mathcal{M}_2(\mathbb{R}) \to \mathcal{M}_2(\mathbb{R})$ defined by $T\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} c & d \\ a & b \end{pmatrix}$,
   - $T : \mathbb{R}^3 \to \mathbb{R}^3$ defined by $T(a, b, c) = (7a - 4b + 10c, 4a - 3b + 8c, -2a + b - 2c)$.

Section 6: Proofs

1. Let V be a vector space over a field $\mathbb{F}$. Prove that $W \subseteq V$ is a subspace if and only if $W \neq \emptyset$; $ax \in W$ for all $a \in \mathbb{F}$ and all $x \in W$; and, $x + y \in W$ for all $x, y \in W$.

2. Let $T : V \to W$ be a one-to-one linear transformation between two vector spaces and let $S \subseteq V$. Prove that S is a linearly independent subset of V if and only if $T(S)$ is a linearly independent subset of W.

3. Let V be a finite-dimensional vector space and $\beta = \{u_1, u_2, \cdots, u_n\}$ be a subset of V. Show that β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors on β.

4. Let V be a finite-dimensional vector spaces with bases α and β. Show that the number of elements in α is the same as the number of elements in β.

5. Let W_1, W_2 be finite-dimensional subspaces of a vector space V. Prove that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2) - \dim(W_1 \cap W_2)$.

6. Let $T : V \to W$ be a linear transformation between two vector spaces. Prove that $N(T)$ and $R(T)$ are subspaces and indicate the vector space of which each is a subspace.

7. Let $T : V \to V$ be a linear operator for a vector space V. Prove that $R(T) \subseteq N(T)$ if and only if $T^2 = T_0$.

8. Let $T : V \to W$ be a linear transformation between two finite-dimensional vector spaces. Prove that if $\dim(V) > \dim(W)$, then T cannot be one-to-one; and, if $\dim(V) < \dim(W)$, then T cannot be onto.

9. Let $T : V \to W$ and $U : W \to Z$ be linear transformations on finite-dimensional vector spaces. Show that $\text{rank(UT)} \leq \text{rank(U)}$. What condition(s) must be imposed for equality to hold?

10. Show that the system $Ax = b$ has a solution if and only if $b \in R(L_A)$.

11. Let $T : V \to V$ be a linear operator defined on a finite-dimensional vector space. Show that T is invertible if and only if 0 is not an eigenvalue of T.

12. Let $T : V \to V$ be a linear operator defined on a finite-dimensional vector space with eigenvector v and corresponding eigenvalue λ. Show that for any natural number m, v is an eigenvector of the linear operator T^m with corresponding eigenvalue λ^m.

13. Let A, B be square matrices with characteristic polynomials $f(t)$ and $g(t)$, respectively. Show that if A is similar to B, then $f(t) = g(t)$. Use this to conclude that A and B have the same eigenvalues.

14. Let $f(t)$ be the characteristic polynomial of a square matrix A and $g(t)$ be the characteristic polynomial of A^t. Show that $f(t) = g(t)$ and use this to conclude that A and A^t have the same eigenvalues.

15. Let T be a linear operator on an n-dimensional vector space V. Show that if T has n distince eigenvalues, then T is diagonalizable.

Do not forget to use your time wisely for studying and for writing your exam. Remember that mathematics is about two things: breathing and thinking. Study well, breathe deeply, and think as much as you need before rashly computing things.

Good luck!