Differential Equations and Laplace Transforms

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Chapter 1 — First-Order Ordinary Differential Equations

- Check if the DE is separable ($g(y)y' = f(x)$ or $g(y)dy = f(x)dx$)
  - To solve, we integrate on both sides.
- If the DE $M(x,y)dx + N(x,y)dy$ is not separable, check if they're both homogeneous of the same degree.
  - If they are, then making either substitution $u = \frac{y}{x}$ or $u = \frac{x}{y}$ will make the DE separable.
- If the DE $M(x,y)dx + N(x,y)dy$ is not homogeneous of the same degree, then check if it is exact ($M_y = N_x$).
  - If it is, then solve knowing $M(x,y) = \frac{\partial F}{\partial x}$.
    - Find $F(x,y)$ by solving $F(x,y) = \int M(x,y)dx + g(y)$, then find $g(y)$ (after finding $g'(y)$) by solving $\frac{\partial F}{\partial y} = N(x,y)$.
  - Another possibility is to solve knowing $N(x,y) = \frac{\partial F}{\partial y}$.
    - Find $F(x,y)$ by solving $F(x,y) = \int N(x,y)dy + g(x)$, then find $g(x)$ (after finding $g'(x)$) by solving $\frac{\partial F}{\partial x} = M(x,y)$.
- If the DE $M(x,y)dx + N(x,y)dy$ is not exact, then check if we can make it exact by multiplying the DE by an integrating factor $\mu(x,y)$.
  - If $\frac{M_y - N_x}{N} = g(x)$, then the integrating factor is $\mu(x) = e^{\int g(x)dx}$. Multiply the original DE by $\mu(x)$ and check if it's exact, then solve.
  - If $\frac{M_y - N_x}{M} = g(y)$, then the integrating factor is $\mu(y) = e^{-\int g(y)dy}$. Multiply the original DE by $\mu(y)$ and check if it's exact, then solve.
- If a DE can be written in the form $y' + f(x)y = r(x)$, it is called first-order linear.
  - Its general solution is $y = \frac{1}{\mu(x)}\left[\int \mu(x)r(x)dx + C\right]$, where $\mu(x) = e^{\int f(x)dx}$.
- If a DE can be written in the form $y' + p(x)y = g(x)y^a$, where a is a real constant, it is a nonlinear Bernoulli equation. If $a = 0$ or $a = 1$, it is linear.
  - Let $u(x) = [y(x)]^{1-a}$, then $u' + (1-a)p(x)u = (1-a)g(x)$; it is

linear in $u(x)$.

Chapter 2 — Second-Order Ordinary Differential Equations

- To find the solutions of $y'' + ay' + by = 0$, we have to solve the characteristic equation $\lambda^2 + a\lambda + b = 0$.
  - If $\lambda_1 \neq \lambda_2$ are real distinct roots, then the general solution is $y(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$.
  - If $\lambda_{1,2} = \alpha \pm \mathbb{i}\beta$ is a complex conjugate pair, then the general solution is $y(x) = C_1 e^{\alpha x}\cos(\beta x) + C_2 e^{\alpha x}\sin(\beta x)$.
  - If $\lambda_1 = \lambda_2 = \lambda$ are real repeated roots, then the general solution is $y(x) = C_1 e^{\lambda x} + C_2 x e^{\lambda x}$.
- To find the solutions of a second-order homogeneous Euler-Cauchy equation $x^2 y'' + axy' + by = 0,\ a, b \in \mathbb{Z}$, we have to solve the characteristic equation $m(m-1) + am + b = 0$.
  - If $m_1 \neq m_2$ are real, distinct roots, then the general solution is $y(x) = C_1 x^{m_1} + C_2 x^{m_2}$.
  - If $m_{1,2} = \alpha \pm \mathbb{i}\beta$ is a complex conjugate pair, then the general solution is $y(x) = C_1 x^{\alpha}\cos(\beta\ln(x)) + C_2 x^{\alpha}\sin(\beta\ln(x))$.
  - If $m_1 = m_2$ are real repeated roots, then the general solution is $y(x) = C_1 x^m + C_2 x^m \ln(x)$.

Chapter 3 — Linear Differential Equations of Arbitrary Order

- For any n^{th} order linear homogeneous DE, find the characteristic equation and solve for λ.
- If all roots are real and distinct ($\lambda_1 \neq \lambda_2 \neq \cdots \neq \lambda_n$), let $y_j(x) = e^{\lambda_j x}$;
  - The general solution is $y(x) = C_1 y_1(x) + C_2 y_2(x) + \cdots + C_n y_n(x)$ $= C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x} + \cdots + C_n e^{\lambda_n x}$.
- If all roots are real and repeated;
  - $\lambda_j = \lambda_{j+1}$ is a real, repeated root, we'd put $C_j e^{\lambda_j x}$ and $C_{j+1} x e^{\lambda_j x}$ in the general solution.
  - If $\lambda_j = \lambda_{j+1} = \lambda_{j+2}$ is a real, triple root, we'd put $C_j e^{\lambda_j x}$, $C_{j+1} x e^{\lambda_j x}$ and $C_{j+2} x^2 e^{\lambda_j x}$ in the general solution.
  - If $\lambda_j = \lambda_{j+1} = \lambda_{j+2} = \cdots = \lambda_{j+m-1}$ is a real, root of multiplicity m, we'd put $C_j e^{\lambda_j x}$, $C_{j+1} x e^{\lambda_j x}$, $C_{j+2} x^2 e^{\lambda_j x}$, …, $C_{j+m-1} x^{m-1} e^{\lambda_j x}$ in the general solution.
- If all the roots are complex;

- $\lambda_{j,j+1} = \alpha \pm \mathbb{i}\beta$ are complex conjugates, we'd put $C_j e^{\alpha x}\cos(\beta x)$ and $C_{j+1}e^{\alpha x}\sin(\beta x)$ in the general solution.
- If $\lambda_{j,j+1} = \alpha_1 \pm \mathbb{i}\beta_1$ and $\lambda_{j+2,j+3} = \alpha_2 \pm \mathbb{i}\beta_2$, where $(\alpha_1, \beta_1) \neq (\alpha_2, \beta_2)$ (i.e. distinct conjugate pair), we'd put $C_j e^{\alpha_1 x}\cos(\beta_1 x)$, $C_{j+1}e^{\alpha_1 x}\sin(\beta_1 x)$, $C_{j+2}e^{\alpha_2 x}\cos(\beta_2 x)$ and $C_{j+3}e^{\alpha_2 x}\sin(\beta_2 x)$ in the general solution.
- If $\lambda_{j,j+1} = \alpha + \mathbb{i}\beta, \lambda_{j+2,j+3} = \alpha - \mathbb{i}\beta$ i.e. a repeated conjugate pair, we'd put $C_j e^{\alpha x}\cos(\beta x)$, $C_{j+1}e^{\alpha x}\sin(\beta x)$, $C_{j+2}xe^{\alpha x}\cos(\beta x)$ and $C_{j+3}xe^{\alpha x}\sin(\beta x)$ in the general solution.

- A third-order homogeneous Euler-Cauchy equation has the form $x^3y''' + ax^2y'' + bxy' + cy = 0, a, b, c \in \mathbb{Z}$.
  - To solve it, we need to first solve the charactaristic equation $m(m-1)(m-2) + am(m-1) + bm + c = 0$. The general solution is then $y(x) = C_1x^{m_1} + C_2x^{m_2} + C_3x^{m_3}$.
  - If any root is repeated, we multiply them by $\ln(x)$ as many times as needed (once for the second repeated root and twice for the third).
- If the DE is nonhomogeneous, then it has the form $y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = r(x), \qquad r(x) \not\equiv 0$
  - If the left-hand side only has constant coefficients and $r(x)$ has a finite number of independent derivatives (so polynomials with positive integer powers of x, $\sin(x)$, $\cos(x)$ and exponentials only; any power of x that is not positive or an integer, every logarithmic functions and everything else is not okay), then we can (and should) use the method of undetermined coefficients.
    - First, find $y_h(x)$, which is the general solution of the corresponding homogeneous equation.
    - Then, we make a guess for $y_p(x)$ based on $r(x)$.

If this is in $r(x)$	**You put this in $y_p(x)$**
A polynomial of degree n	$\alpha_n x^n + a_{n-1}x^{n-1} + \cdots + \alpha_2 x^2 + a_1 x + a_0$
An exponential e^{kx}	αe^{kx}
$\sin(\omega x)$ <u>or</u> $\cos(\omega x)$	$\alpha\cos(\omega x) + \beta\sin(\omega x)$
$e^{kx}\cos(\omega x)$ <u>or</u> $e^{kx}\sin(\omega x)$	$\alpha e^{kx}\cos(\omega x) + \beta e^{kx}\sin(\omega x)$
$x^n e^{kx}$	$(\alpha_n x^n + \cdots + \alpha_1 x + \alpha_0)e^{kx}$
$x^n\cos(\omega x)$ <u>or</u> $x^n\sin(\omega x)$	$(\alpha_n x^n + \cdots + \alpha_1 x + \alpha_0)\cos(\omega x) + (\beta_n x^n + \cdots + \beta_1 x + \beta_0)\sin(\omega x)$

    - If a guess in $y_p(x)$ is already in $y_h(x)$, then we use the multiplication rule and multiply the whole part by x as many times as needed.

        - We then differentiate $y_p(x)$ and plug those back in the original nonhomogeneous DE, solve for our variables and finally we can say what the general solution is.
    - If neither of these are true, then we have to use the variation of paramaters method.
        - First, solve the corresponding homogeneous DE to find $y_h(x)$.
        - Then we have to solve $u_1'y_1 + u_2'y_2 = 0$ and $u_1'y_1' + u_2'y_2' = r(x)$ for u_1' and u_2'.
        - The particular solution is $y_p(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$ and finally the general solution is $y_g(x) = y_h(x) + y_p(x)$.
        - A third-order nonhomogeneous DE has the form $y''' + a_2(x)y'' + a_1(x)y' + a_0(x)y = r(x)$, and to solve it using variation of parameters we need to solve $u_1'y_1 + u_2'y_2 + u_3'y_3 = 0$, $u_1'y_1' + u_2'y_2' + u_3'y_3' = 0$ and $u_1'y_1'' + u_2'y_2'' + u_3'y_3'' = r(x)$.

Chapter 4 — Systems of Differential Equations

- A system of DEs has the form $\vec{y}' = A\vec{y} + \vec{f}$. If $\vec{f}(x) \equiv \vec{0}$, the system is homogeneous. To solve a homogeneous system, we need to solve $\det(A - \lambda I) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0.$
  Let's consider the cases when the system is homogeneous.
    - If $\lambda_1 \neq \lambda_2$ are real distinct roots, then the general solution is $\vec{y}(x) = C_1\vec{y}^{(1)}(x) + C_2\vec{y}^{(2)}(x) = C_1e^{\lambda_1 x}\vec{v}_1 + C_2e^{\lambda_2 x}\vec{v}_2$.
        - For each λ, solve for $\left(A - \lambda_j I\right)\vec{v}_j = \vec{0}$ and find any nonzero solution for $\vec{v}_j$. We can then say what the general solution is.
    - We will not consider the case where $\lambda_{1,2} = \alpha \pm \mathbb{i}\beta$.
    - If $\lambda_1 = \lambda_2 = \lambda$ is a repeated, real eigenvalue, then the general solution is $\vec{y}(x) = C_1e^{\lambda x}\vec{v} + C_2e^{\lambda x}(x\vec{v} + \vec{u})$, where $\vec{u}$ satisfies $(A - \lambda I)\vec{u} = \vec{v}$.
- A nonhomogeneous system of DEs has the form $\vec{y}' = A\vec{y} + \vec{f}$, where $\vec{f}(x) \not\equiv \vec{0}$. The general solution has the form $\vec{y}_g(x) = \vec{y}_h(x) + \vec{y}_p(x)$, where $\vec{y}_h(x)$ is the general solution of the corresponding homogeneous system $\vec{y}' = A\vec{y}$ and $\vec{y}_p(x)$ is any particular solution of the nonhomogeneous system.
    - To find $\vec{y}_h(x)$, we simply solve the corresponding homogeneous

system $\vec{y}' = A\vec{y}$.

- Then we make a guess for $\vec{y}_p(x)$ based on $\vec{f}$, differentiate our guess and plug everything in $\vec{y}' = A\vec{y} + \vec{f}$. After that we solve for every variable and finally we sum $\vec{y}_h(x) + \vec{y}_p(x)$ to find our general solution $\vec{y}_g(x)$.
  - The method we use to find $\vec{y}_p(x)$ will depend on the components of $\vec{f}$; we will try to use the method of undetermined coefficients but we might have to use variation of parameters.

Chapter 5 — Laplace Transform

- Consider a function $f(t)$ defined for $t \geq 0$. Its Laplace transform is $F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt$.
- With this, we can define a lot of Laplace transforms:
  - $\mathcal{L}\{t^n\} = \dfrac{n!}{s^{n+1}}$
  - $\mathcal{L}\{e^{at}\} = \dfrac{1}{s-a}$
    - Because of this one, partial fractions can be necessary to solving an inverse Laplace transform, $\mathcal{L}^{-1}\{F(s)\}$.
  - $\mathcal{L}\{\sin(\omega t)\} = \dfrac{\omega}{s^2+\omega^2}, \qquad s > 0$
  - $\mathcal{L}\{\cos(\omega t)\} = \dfrac{s}{s^2+\omega^2}, \qquad s > 0$
  - $\mathcal{L}\{\sinh(at)\} = \dfrac{a}{s^2-a^2}, \qquad s > |a|$
  - $\mathcal{L}\{\cosh(at)\} = \dfrac{s}{s^2-a^2}, \qquad s > |a|$
  - And many more ($\mathcal{L}\{\delta(t-a)\}$, $\mathcal{L}\{u(t-a)\}$, $\mathcal{L}\{e^{at}f(t)\}$, $\mathcal{L}\{u(t-a)f(t-a)\}$, $\mathcal{L}\left\{\int_0^t f(x)\,dx\right\}$, $\mathcal{L}\left\{\frac{f(t)}{t}\right\}$, $\mathcal{L}\left\{\frac{d^n}{dt^n}f(t)\right\}$ and $\mathcal{L}\{(f * g)(t)\}$), all of which will be on the exam's formula sheet.
- To solve an initial value problem using Laplace transforms, start by finding the Laplace transform of each term. Then, solve for $Y(s)$ and find its inverse Laplace transform. Most of the time partial fractions will be needed.
- The first shifting theorem says that $\mathcal{L}\{e^{at}f(t)\} = F(s-a)$ and the second shifting theorem says that $\mathcal{L}\{u(t-a)f(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$.

- To solve an initial value problem with the unit step function, you do like you did before; find the Laplace transform of everything, solve for $Y(s)$ and find its inverse Laplace transform.
- To solve an initial value problem with the Dirac delta function, everything is the same but you have to know that $\mathcal{L}\{\delta(t-a)\} = e^{-as}$.
- It can be useful to know that
  $$\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}(\mathcal{L}\{f(t)\}) = (-1)^n F^{(n)}(s).$$
- The n^{th} Laguerre polynomial is defined as being $L_n(t) = \mathcal{L}^{-1}\left\{\frac{(s-1)^n}{s^{n+1}}\right\}$.
- The convolution of two functions $f(t)$ and $g(t)$ is
  $$(f * g)(t) = \int_0^t f(x)g(t-x)\,dx.$$
  - $\mathcal{L}\{f * g\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\} = F(s)G(s)$
  - $\mathcal{L}^{-1}\{F(s)G(s)\} = \mathcal{L}^{-1}\{F(s)\} * \mathcal{L}^{-1}\{G(s)\} = f * g$
  - For the $g(t)$ in $f(t) * g(t)$, it's usually easier to put exponentials first. If there's no exponentials it is easier to put a trigonometric function, and if there's only polynomials it's easier to put the one of the smaller degree.

Chapter 6 — Power Series Solutions

- Won't be on the exam.

Numerical Methods

August 5, 2015 8:02 PM

Chapter 7 — Solutions of Nonlinear Equations

- Fixed-point iteration
  - Find your $g(x)$ (usually done by solving for the lowest-degree x).
  - Check that $|g'(x)| < 1$ on your interval.
  - Then choose an x_0 that's close to a root, and then plug $g(x_0)$ in your $g(x)$, and keep repeating until the first n decimals are the same.
- Newton's method
  - $x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
  - We have to know this formula by heart.
- For an iterative scheme $x_{n+1} = g(x_n)$ for finding root p, the order of the method is the lowest-ordered derivative of $g(x)$ that is not zero at p.

Chapter 8 — Interpolation and Extrapolation

- To find a polynomial that passes through $n+1$ data points, we find

  $p_n(x)$, where $p_n(x) = \sum_{j=0}^{n} L_j(x) f_j$ with $L_j(x)$

  $= \prod_{\substack{i=0 \\ i \neq j}}^{n} \dfrac{(x - x_i)}{(x_j - x_i)}.$
- If we are using $p_n(x)$ to <u>inter</u>polate a value at x, then the error is

  $|\varepsilon_n(x)| = |f(x) - p_n(x)|$

  $= \left|(x - x_0)(x - x_i) \dots (x - x_n) \dfrac{f^{(n+1)}(t)}{(n+1)!}\right|$

  where t is in $[x_0, x_n]$.

- All formulas ($p_n(x) = \cdots$, $L_j(x) = \cdots$ and $|\varepsilon_n(x)| = \cdots$) will be on the exam's formula sheet.

Chapter 9 — Numerical Differentiation and Integration

- Consider 3 equally spaced data points $(x_0, f_0), (x_1, f_1), (x_2, f_2)$, where $\Delta x = x_1 - x_0 = x_2 - x_1 = h$. It can be shown that
  - $f_0' = f'(x_0) \approx p_2'(x_0) = \dfrac{1}{2h}(-3f_0 + 4f_1 - f_2)$
  - $f_1' = f'(x_1) \approx p_2'(x_1) = \dfrac{1}{2h}(-f_0 + f_2)$
  - $f_2' = f'(x_2) \approx p_2'(x_2) = \dfrac{1}{2h}(f_0 - 4f_1 + 3f_2)$
- We can estimate the area under a curve using different ways.
  - The midpoint rule:
    $$\int_a^b f(x)\,dx \approx h \sum_{j=1}^{n} f(x_j^*)$$
  - The trapezoidal rule:
    $$\int_a^b f(x)\,dx \approx \frac{h}{2} \sum_{j=1}^{n} \big(f(x_{j-1}) + f(x_j)\big)$$
  - Simpson's rule:
    $$\int_a^b f(x)\,dx$$
    $$\approx \frac{h}{3} \sum_{j=0}^{n-1} \big(f(x_{2j}) + 4f(x_{2j+1}) + f(x_{2j+2})\big)$$
  - Those three rules will be given on the exam's formula sheet.
  - Finally, we can use the Gaussian quadrature to estimate a definite integral. First we must convert $x = \frac{1}{2}\big(a(1-t) + b(t+1)\big)$ and $dx = \frac{1}{2}(b-a)\,dt$.

Therefore $\int_a^b f(x)\,dx$ becomes

$$\int_{-1}^{1} f\left(\frac{1}{2}\big(a(1-t)+b(t+1)\big)\right)\left(\frac{1}{2}(b-a)\right)\,dt,$$

which can be approximated as being

$$\left(\frac{1}{2}(b-a)\right)\sum_{j=1}^{n} A_j\, f\left(\frac{1}{2}\big(a(1-t_j)+b(t_j+1)\big)\right).$$

The values for the A_j and the t_j can be found in a table that will be on the exam's formula sheet.

Chapter 10 — Numerical Solutions of Differential Equations

- There are different methods to approximate points of a function, given its derivative and a point on the graph.
  - Euler's method:
    - $y_{n+1} = y_n + hf(x_n, y_n)$
  - Improved Euler's method:
    - $y_{n+1}^P = y_n^C + hf(x_n, y_n^C)$
    - $y_{n+1}^C = y_n^C + \frac{1}{2}h[f(x_n, y_n^C) + f(x_{n+1}, y_{n+1}^P)]$
  - Fourth-order Runge-Katta method:
    - $y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$
    - $k_1 = hf(x_n, y_n)$
    - $k_2 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_1\right)$
    - $k_3 = hf\left(x_n + \frac{1}{2}h, y_n + \frac{1}{2}k_2\right)$
    - $k_4 = hf(x_n + h, y_n + k_3)$
  - Only the improved Euler's method and the fourth-order Runge-Katta method formulas are on the exam's formula sheet; we have to know how to do Euler's method.

Formula sheet

August 5, 2015 9:01 PM

Formulas

$f(t)$	$F(s)=\mathcal{L}\{f(t)\}$	
t^n	$n!/s^{n+1}$	; $n=0,1,2,\ldots$ and $s>0$
e^{at}	$1/(s-a)$	; $s>a$
$\sin(kt)$	$k/(s^2+k^2)$	; $s>0$
$\cos(kt)$	$s/(s^2+k^2)$	; $s>0$
$\sinh(kt)$	$k/(s^2-k^2)$	; $s>k$
$\cosh(kt)$	$s/(s^2-k^2)$	; $s>k$
$\delta(t-a)$	e^{-as}	; $s>0$
$u(t-a)$	$\frac{e^{-as}}{s}$	; $s>0$

$$
\begin{aligned}
\mathcal{L}\{f(t)\}(s) &= \int_0^\infty e^{-st}f(t)dt \\
\mathcal{L}\{e^{at}f(t)\} &= F(s-a) \\
\mathcal{L}\{u(t-a)f(t-a)\} &= e^{-as}F(s) \\
\mathcal{L}\{t^n f(t)\} &= (-1)^n\frac{d^n}{ds^n}F(s) \\
\mathcal{L}\left\{\int_0^t f(x)dx\right\} &= \frac{1}{s}F(s) \\
\mathcal{L}\left\{\frac{f(t)}{t}\right\} &= \int_s^\infty F(x)dx \\
\mathcal{L}\left\{\frac{d^n}{dt^n}f(t)\right\} &= s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)-\ldots-f^{(n-1)}(0) \\
(f*g)(t) &= \int_0^t f(x)g(t-x)dx \\
\mathcal{L}\{(f*g)(t)\} &= F(s)\,G(s)
\end{aligned}
$$

$$
\begin{aligned}
\int_a^b f(x)dx &= h\sum_{j=1}^{n} f(x_j^*)\,, \qquad |\epsilon| \le \frac{1}{24}M\,(b-a)\,h^2\,, \qquad M = max_{a\le x\le b}\,|f''(x)| \\
\int_a^b f(x)dx &= \frac{h}{2}\sum_{j=1}^{n}\left(f(x_{j-1})+f(x_j)\right), \qquad |\epsilon| \le \frac{1}{12}M\,(b-a)\,h^2\,, \qquad M = max_{a\le x\le b}\,|f''(x)| \\
\int_a^b f(x)dx &= \frac{h}{3}\sum_{j=0}^{n-1}\left(f(x_{2j})+4f(x_{2j+1})+f(x_{2j+2})\right), \qquad |\epsilon| \le \frac{1}{180}M\,(b-a)\,h^4\,, \\
&\quad M = max_{a\le x\le b}\,|f^{(4)}(x)|
\end{aligned}
$$

$$\begin{aligned} p_n(x) &= \sum_{j=0}^{n} L_j(x)\, f_j\,, \qquad L_k(x) = \prod_{j=0,\, j\neq k}^{n} \frac{(x-x_j)}{(x_k-x_j)} \\ |\epsilon_n(x)| &= |(x-x_0)(x-x_1)\cdots(x-x_n)\frac{f^{(n+1)}(t)}{(n+1)!}| \end{aligned}$$

$$\begin{aligned} y_{n+1}^P &= y_n^C + h\, f(x_n, y_n^C) \\ y_{n+1}^C &= y_n^C + \frac{1}{2} h \left(f(x_n, y_n^C) + f(x_{n+1}, y_{n+1}^P) \right) \end{aligned}$$

$$\begin{aligned} k_1 &= h\, f(x_n, y_n) \\ k_2 &= h\, f(x_n + \frac{1}{2}\, h, y_n + \frac{1}{2}\, k_1) \\ k_3 &= h\, f(x_n + \frac{1}{2}\, h, y_n + \frac{1}{2}\, k_2) \\ k_4 &= h\, f(x_n + h, y_n + k_3) \\ y_{n+1} &= y_n + \frac{1}{6}\,(k_1 + 2\, k_2 + 2\, k_3 + k_4) \end{aligned}$$

$$\int_{-1}^{1} f(t)\, dt \approx \sum_{j=1}^{n} A_j\, f(t_j)\,, \qquad x = \frac{1}{2}\,(a\,(1-t) + b\,(t+1))$$

Order n	Nodes t_j	Coefficients A_j
2	-0.5773502692	1.0
	0.5773502692	1.0
3	-0.7745966692	0.555555556
	0.0	0.888888889
	0.7745966692	0.555555556
4	-0.8611363116	0.3478548451
	-0.3399810436	0.6521451549
	0.3399810436	0.6521451549
	0.8611363116	0.3478548451
5	-0.9061798459	0.2369268850
	-0.5384693101	0.4786286705
	0.0	0.5688888889
	0.5384693101	0.4786286705
	0.9061798459	0.2369268850