(A)

MAT 2384 A
DIFFERENTIAL EQUATIONS AND NUMERICAL METHODS
TEST #1

October 14, 2016

Instructor: Dr. Steve Desjardins

Duration: 80 minutes

Name: $\boxed{\text{Solutions}}$

Student Number:____________________________

Instructions:
- Print your name and student number on this page.
- Verify that your copy of the exam has all 5 pages.
- You must answer all questions.
- Write your answers in the spaces below the questions. You may use the backs of the pages if necessary.
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Signature:____

A

Question 1 (5 marks) Find the general solutions of the differential equations:

(a) $y' = (1+y^2)\,e^x$

separable

$$\frac{dy}{dx} = (1+y^2)e^x$$

so $\dfrac{dy}{1+y^2} = e^x dx$

thus $\displaystyle\int \frac{dy}{1+y^2} = \int e^x dx + C$

which gives $\arctan y = e^x + C$

and so $\boxed{y = \tan(e^x + C)}$

(b) $y' - \dfrac{2}{x}y = x^4$

linear $f(x) = -\dfrac{2}{x}$, $r(x) = x^4$

$$\mu(x) = e^{\int -\frac{2}{x}dx} = e^{-2\ln x} = x^{-2}$$

so $\displaystyle y = \frac{1}{\mu(x)}\left[\int \mu(x) r(x)\,dx + C\right]$

$$= x^2\left(\int (x^{-2})(x^4)\,dx + C\right)$$

$$= x^2\left(\int x^2 dx + C\right)$$

$$= x^2\left(\frac{1}{3}x^3 + C\right)$$

$$= \boxed{\frac{1}{3}x^5 + Cx^2}$$

A

Question 2 (5 marks) Find the general solution:

$$(21y^2 + 16x\cos y)\,dx + (14xy - 4x^2\sin y)\,dy = 0$$

$$M(x,y) = 21y^2 + 16x\cos y \Rightarrow M_y = 42y - 16x\sin y$$
$$N(x,y) = 14xy - 4x^2\sin y \Rightarrow N_x = 14y - 8x\sin y$$
$M_y \neq N_x$, DE not exact.

$$\frac{M_y - N_x}{N} = \frac{28y - 8x\sin y}{14xy - 4x^2\sin y} = \frac{2}{x} \quad (\text{function of } x \text{ only})$$

then $\mu(x) = e^{\int \frac{2}{x}dx} = x^2$ and the DE becomes

$$(21x^2y^2 + 16x^3\cos y)\,dx + (14x^3y - 4x^4\sin y)\,dy = 0$$

$$M^*(x,y) = 21x^2y^2 + 16x^3\cos y \Rightarrow M^*_y = 42x^2y - 16x^3\sin y$$
$$N^*(x,y) = 14x^3y - 4x^4\sin y \Rightarrow N^*_x = 42x^2y - 16x^3\sin y$$
$M^*_y = N^*_x$, DE exact

$$F(x,y) = \int M^*(x,y)\,dx + g(y) \quad \left(\text{OR } \int N^*(x,y)\,dy + g(x)\right)$$
$$= \int (21x^2y^2 + 16x^3\cos y)\,dx + g(y)$$
$$= 7x^3y^2 + 4x^4\cos y + g(y)$$

then $\frac{\partial F}{\partial y} = 14x^3y - 4x^4\sin y + g'(y) = N^*(x,y)$
$= 14x^3y - 4x^4\sin y$

so $g'(y) = 0 \Rightarrow g(y) = \text{constant}$, so take $g(y) = 0$

then $F(x,y) = 7x^3y^2 + 4x^4\cos y$

$\therefore$ the general solution is $\boxed{7x^3y^2 + 4x^4\cos y = C}$

(A)

Question 3 (5 marks) Solve the initial value problems:

(a) $y'' + 2\pi y' + \pi^2 y = 0, \quad y(0) = 0, \quad y'(0) = 1$

the char. eq. is $\lambda^2 + 2\pi\lambda + \pi^2 = (\lambda+\pi)^2 = 0$

so $\lambda_1 = \lambda_2 = -\pi$ and the general

solution is $y(x) = C_1 e^{-\pi x} + C_2 x e^{-\pi x}$

$y(0) = 0 \Rightarrow 0 = C_1 e^0 + C_2 (0) e^0 \Rightarrow C_1 = 0$

$y'(x) = -\pi C_1 e^{-\pi x} + C_2 e^{-\pi x} - \pi C_2 x e^{-\pi x}$

$y'(0) = 1 \Rightarrow 1 = -\pi C_1 e^0 + C_2 e^0 - \pi C_2 (0) e^0 \Rightarrow C_2 = 1$

$\therefore$ the unique solution is $\boxed{y(x) = x e^{-\pi x}}$

(b) $x^2 y'' - 3xy' + 8y = 0, \quad x > 0, \quad y(1) = 2, \quad y'(1) = 8$

the char. eq. is $m(m-1) - 3m + 8 = 0$

or $m^2 - 4m + 8 = 0$

so $m_{1,2} = \dfrac{4 \pm \sqrt{(-4)^2 - 4(1)(8)}}{2} = \dfrac{4 \pm \sqrt{-16}}{2} = 2 \pm 2i$

The general solution is

$y(x) = C_1 x^2 \cos(2\ln x) + C_2 x^2 \sin(2\ln x)$

$y(1) = 2 \Rightarrow 2 = C_1 (1)^2 \cos(0) + C_2 (1)^2 \sin(0) \Rightarrow C_1 = 2$

$y'(x) = 2C_1 x \cos(2\ln x) - 2C_1 x \sin(2\ln x) + 2C_2 x \sin(2\ln x)$
$+ 2C_2 x \cos(2\ln x)$

$y'(1) = 8 \Rightarrow 8 = 2C_1(1)\cos(0) - 2C_1(1)\sin(0) + 2C_2(1)\sin(0)$
$+ 2C_2(1)\cos(0)$

or $2C_1 + 2C_2 = 8 \Rightarrow C_2 = 2$

$\therefore$ unique solution is $\boxed{y(x) = 2x^2\cos(2\ln x) + 2x^2 \sin(2\ln x)}$

A

Question 4 (5 marks)

Consider the data points ((x_j, f_j), where $f_j = f(x_j)$): $(1.0, 2.5)$, $(1.6, 3.8)$ and $(2.1, 6.9)$. Find $p_2(x)$ (with coefficients to 4 decimal places) via Lagrangian Interpolation. Interpolate a value at $x = 1.75$. Given that $1 \le f'''(x) \le 3$ on $[0, 3]$, estimate the error bounds.

$$p_n(x) = \sum_{j=0}^{n} L_j(x)\, f_j\,, \qquad L_k(x) = \prod_{j=0,\, j\neq k}^{n} \frac{(x - x_j)}{(x_k - x_j)}$$

$$|\epsilon_n(x)| = |(x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{(n+1)}(t)}{(n+1)!}|$$

$$p_2(x) = L_0(x) f_0 + L_1(x) f_1 + L_2(x) f_2$$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} f_0 + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} f_1 + \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} f_2$$

$$= \frac{(x-1.6)(x-2.1)}{(1-1.6)(1-2.1)}(2.5) + \frac{(x-1)(x-2.1)}{(1.6-1)(1.6-2.1)}(3.8) + \frac{(x-1)(x-1.6)}{(2.1-1)(2.1-1.6)}(6.9)$$

$$= \left(\frac{x^2 - 3.7x + 3.36}{0.66}\right)(2.5) + \left(\frac{x^2 - 3.1x + 2.1}{-0.3}\right)(3.8) + \left(\frac{x^2 - 2.6x + 1.6}{0.55}\right)(6.9)$$

$$= \boxed{3.6667x^2 - 7.3667x + 6.2000}$$

(Check: $p_2(1) = 2.5000$, $p_2(1.6) = 3.8000$, $p_2(2.1) = 6.9001$ okay!)

then $f(1.75) \simeq p_2(1.75) = \boxed{4.5375}$

$$|\epsilon_2(1.75)| = |(1.75-1)(1.75-1.6)(1.75-2.1)\frac{f'''(t)}{3!}|$$

$$= 0.0066\,|f'''(t)|$$

so $\epsilon_{min} = 0.0066(1) = \boxed{0.0066}$

and $\epsilon_{max} = 0.0066(3) = \boxed{0.0198}$

(B)

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B

Question 1 (5 marks) Find the general solutions of the differential equations:

(a) $yy' = x\,(1+y^2)$ separable

$$y\frac{dy}{dx} = x(1+y^2)$$

so $\frac{y}{1+y^2}\,dy = x\,dx$

thus $\int \frac{y}{1+y^2}\,dy = \int x\,dx + C$

we get $\frac{1}{2}\ln(1+y^2) = \frac{1}{2}x^2 + C$

or $\ln(1+y^2) = x^2 + C$

so $1+y^2 = Ke^{x^2}$

thus $\boxed{y^2 = Ke^{x^2} - 1}$

(b) $y' - \dfrac{3}{x}\,y = x^6$ linear $f(x) = \frac{-3}{x}$, $r(x) = x^6$

$\mu(x) = e^{-\int \frac{3}{x}dx} = x^{-3}$

so $y = x^3\left(\int (x^{-3})(x^6)\,dx + C\right)$

$= x^3\left(\int x^3\,dx + C\right)$

$= x^3\left(\frac{1}{4}x^4 + C\right)$

$= \boxed{\frac{1}{4}x^7 + Cx^3}$

(B)

Question 2 (5 marks) Find the general solution:

$$(24xy^3 - 6\sin y)\,dx + (18x^2y^2 - 2x\cos y)\,dy = 0$$

$M(x,y) = 24xy^3 - 6\sin y \Rightarrow M_y = 72xy^2 - 6\cos y$
$N(x,y) = 18x^2y^2 - 2x\cos y \Rightarrow N_x = 36xy^2 - 2\cos y$
$\Big\}$ $M_y \neq N_x$ DE not exact

$$\frac{M_y - N_x}{N} = \frac{36xy^2 - 4\cos y}{18x^2y^2 - 2x\cos y} = \frac{2}{x} \Rightarrow \mu(x) = x^2 \text{ and DE}$$

becomes $(24x^3y^3 - 6x^2\sin y)\,dx + (18x^4y^2 - 2x^3\cos y)\,dy = 0$

$M^*(x,y) = 24x^3y^3 - 6x^2\sin y \Rightarrow M^*_y = 72x^3y^2 - 6x^2\cos y$
$N^*(x,y) = 18x^4y^2 - 2x^3\cos y \Rightarrow N^*_x = 72x^3y^2 - 6x^2\cos y$
$\Big\}$ $M^*_y = N^*_x$ DE exact

$$F(x,y) = \int (18x^4y^2 - 2x^3\cos y)\,dy + g(x)$$
$$= 6x^4y^3 - 2x^3\sin y + g(x)$$

then $\dfrac{\partial F}{\partial x} = 24x^3y^3 - 6x^2\sin y + g'(x) = M^*(x,y)$
$= 24x^3y^3 - 6x^2\sin y$

$g'(x) = 0 \Rightarrow$ take $g(x) = 0$

so $F(x,y) = 6x^4y^3 - 2x^3\sin y$

$\therefore$ the general solution is $\boxed{6x^4y^3 - 2x^3\sin y = C}$

B

Question 3 (5 marks) Solve the initial value problems:

(a) $y'' - 2\pi y' + \pi^2 y = 0, \quad y(0) = 0, \quad y'(0) = 3$

char. eq. $\lambda^2 - 2\pi\lambda + \pi = (\lambda - \pi)^2 = 0 \Rightarrow \lambda_1 = \lambda_2 = \pi$

general solution $y(x) = c_1 e^{\pi x} + c_2 x e^{\pi x}$

$y(0) = 0 \Rightarrow 0 = c_1 e^0 + c_2 (0) e^0 \Rightarrow c_1 = 0$

$y'(x) = \pi c_1 e^{\pi x} + c_2 e^{\pi x} + \pi c_2 x e^{\pi x}$

$y'(0) = 3 \Rightarrow 3 = \pi c_1 e^0 + c_2 e^0 + \pi c_2 (0) e^0 \Rightarrow c_2 = 3$

$\therefore$ the unique solution is $\boxed{y(x) = 3xe^{\pi x}}$

(b) $x^2y'' + 5xy' + 8y = 0, \quad x > 0, \quad y(1) = 2, \quad y'(1) = 8$

char. eq. $m(m-1) + 5m + 8 = 0$

or $m^2 + 4m + 8 = 0$

So $m_{1,2} = \dfrac{-4 \pm \sqrt{(4)^2 - 4(1)(8)}}{2} = -2 \pm 2i$

general solution $y(x) = c_1 x^{-2} \cos(2\ln x) + c_2 x^{-2} \sin(2\ln x)$

$y(1) = 2 \Rightarrow 2 = c_1 (1)^{-2} \cos(0) + c_2 (1)^{-2} \sin(0) \Rightarrow c_1 = 2$

$y'(x) = -2c_1 x^{-3} \cos(2\ln x) - 2c_1 x^{-3} \sin(2\ln x)$
$\quad - 2c_2 x^{-3} \sin(2\ln x) + 2c_2 x^{-3} \cos(2\ln x)$

$y'(1) = 8 \Rightarrow 8 = -2c_1 (1)^{-3} \cos(0) - 2c_1 (1)^{-3} \sin(0)$
$\quad -2c_2 (1)^{-3} \sin(0) + 2c_2 (1)^{-3} \cos(0)$

or $-2c_1 + 2c_2 = 8 \Rightarrow c_2 = 6$

$\therefore$ unique solution is $\boxed{y(x) = 2x^{-2}\cos(2\ln x) + 6x^{-2}\sin(2\ln x)}$

(B)

Question 4 (5 marks)

Consider the data points ((x_j, f_j), where $f_j = f(x_j)$): $(1.0, 2.1)$, $(1.5, 3.7)$ and $(2.3, 7.1)$. Find $p_2(x)$ (with coefficients to 4 decimal places) via Lagrangian Interpolation. Interpolate a value at $x = 1.75$. Given that $2 \le f'''(x) \le 4$ on $[0, 3]$, estimate the error bounds.

$$p_n(x) = \sum_{j=0}^{n} L_j(x)\, f_j\,, \qquad L_k(x) = \prod_{j=0,\, j\neq k}^{n} \frac{(x - x_j)}{(x_k - x_j)}$$

$$|\epsilon_n(x)| = |(x - x_0)(x - x_1)\cdots(x - x_n)\frac{f^{(n+1)}(t)}{(n+1)!}|$$

$$p_2(x) = \frac{(x-1.5)(x-2.3)}{(1-1.5)(1-2.3)}(2.1) + \frac{(x-1)(x-2.3)}{(1.5-1)(1.5-2.3)}(3.7) + \frac{(x-1)(x-1.5)}{(2.3-1)(2.3-1.5)}(7.1)$$

$$= \left(\frac{x^2 - 3.8x + 3.45}{0.65}\right)(2.1) + \left(\frac{x^2 - 3.3x + 2.3}{-0.4}\right)(3.7) + \left(\frac{x^2 - 2.5x + 1.5}{1.04}\right)(7.1)$$

$$= \boxed{0.8077x^2 + 1.1808\,x + 0.1115}$$

(check: $p_2(1) = 2.1000$, $p_2(1.5) = 3.7000$, $p_2(2.3) = 7.1001$ okay!)

$$f(1.75) \approx p_2(1.75) = \boxed{4.6515}$$

$$|\epsilon_2(1.75)| = |(1.75-1)(1.75-1.5)(1.75-2.3)\, f'''(t)/6|$$

$$= 0.0172\,|f'''(t)|$$

So $\quad \epsilon_{min} = 0.0172(2) = \boxed{0.0344}$

$\quad \epsilon_{max} = 0.0172(4) = \boxed{0.0688}$