⓪

Euclids ~~Postul~~ Postulates

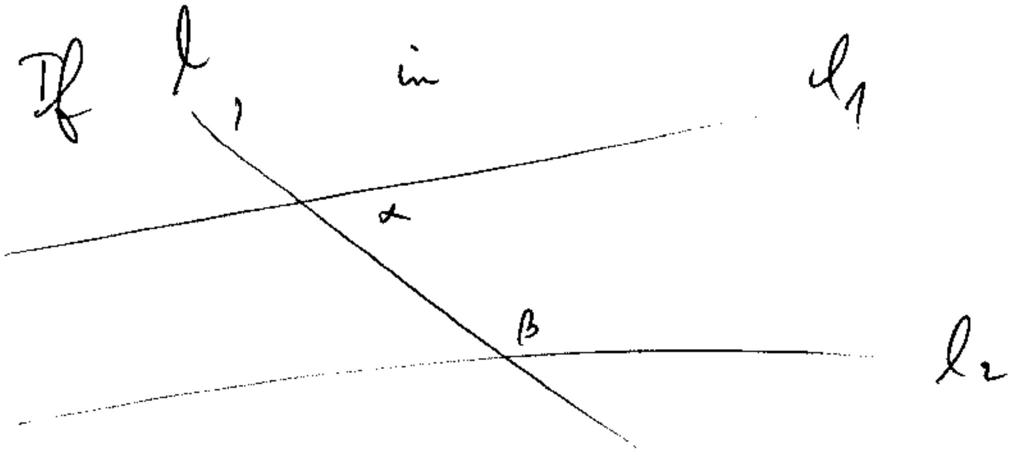
1. A str line can be drawn from any point to any pt

2. A finite str. line can be produced ctsly in a str line

3. A circle may be described with any pt as centre & and any distance as radius

4. All rt angles are equal to each other

5. If l, in

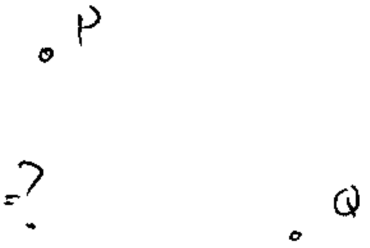


$\alpha + \beta < 2$ rt angles, l_1 & l_2 meet on this side of l If $\alpha + \beta > 2$ rt $\angle$, meet on other side of l

Euclidean Geometry

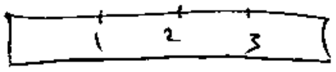
DISTANCE

d(P,Q) = distance between P and Q = ?

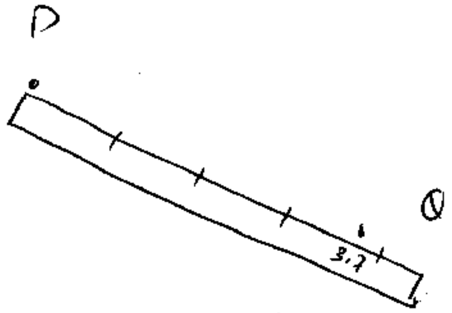


① Make a straight rigid ruler (long enough!)

② decide on a unit; mark it...



③ Move the ruler; "line it up"



$d(P,Q) = 3.7$ (units)

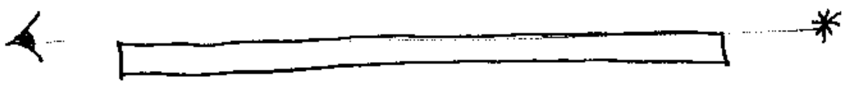
Assumption 1. distance is preserved by "moving" (between the marks)

Come back - same?
copy the ruler.

"rigid" "straight"

"moving the ruler" — translate
— rotate

"straight"



all pts on a straight edge will appear as one

straight line is the path of a light ray!

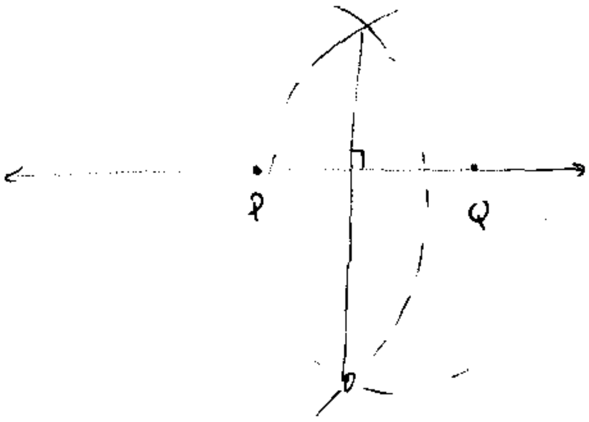
Assumption 2. Straight here is straight there, translated or rotated.

ANGLE

measured by a protractor

angle comes from distance

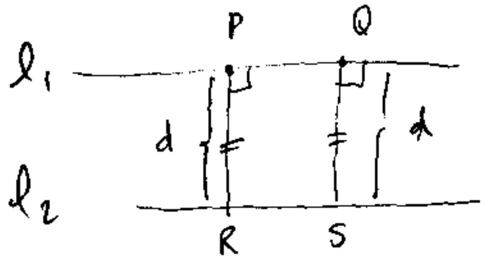
e.g. "right" angle ($\frac{\pi}{2}$, 90°)



rigid rulers (?)

"compass"

e.g. "parallel" lines never intersect
?

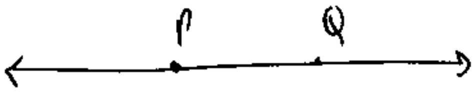


$l_1 \parallel l_2$
$\iff$

$d(P,R) = d(QS)$ ✓

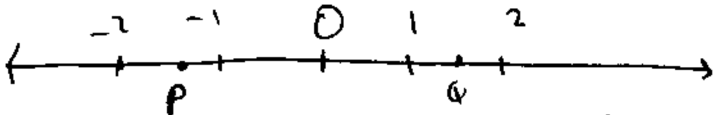
Now we can introduce <u>coordinates</u> into (Euclidean) "space" (Descartes (17 for $\mathbb{R}^2$

E^1



$d(P,Q)$

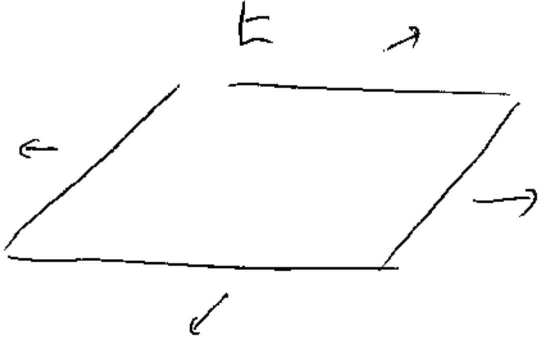
$\mathbb{R}^1$



- origin chosen, unit chosen.
- marked using basic Euclidean ruler

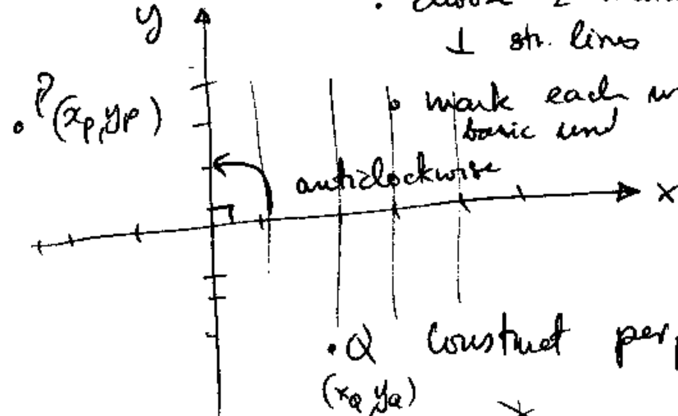
$d(P,Q) = |\text{coord of } P - \text{coord of } Q|$
$= \sqrt{(〃)^2 - (〃)^2}$

E^2



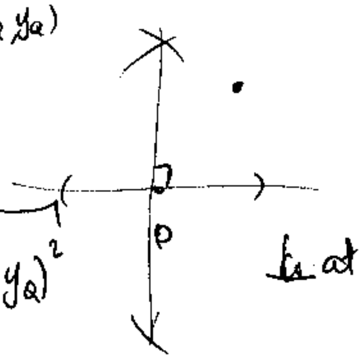
$\mathbb{R}^2$

- choose origin
- choose 2 mutually ⊥ str. lines
- mark each with basic unit



construct perp.

$\mathbb{R}^2 = \{(x,y) \mid x \in \mathbb{R}, y \in \mathbb{R}\}$



⊥ at P.

$d(P,Q) = \sqrt{(x_P - x_Q)^2 + (y_P - y_Q)^2}$

(really?)

E^3 our space (?)

$\mathbb{R}^3 = \{(x,y,3) \mid x, y, z \in \mathbb{R}\}$

- origin, 3 mut. ⊥ axes
- "Right hand" rule

E^n | $\mathbb{R}^n = \{(x_1, \dots x_n) \mid x_i \in \mathbb{R}\}$

We'll study geometry (Euclidean first, then others) starting with the notion of distance, and examine the ways E^n can be "moved around" ~~with~~ while preserving the distance between 2 pts.

Because composition of "Euclidean motions" is useful, instead of moving objects (or pairs of pts.) we'll use the function, map, formula and + think of moving all the pts. Then we can compose the motions, etc.

Note: The geometry of a space can be studied fruitfully by looking at the set of motions of the space that preserve the distance. (like the "symmetries" of a cube ...)

More on assignment # 1. $(\mathcal{M}(E^1))$

E^n Isom(E^n) = {all motions that preserve distance}

1.2 Isometries and Congruence

If $u = (x_1, \dots x_n)$
$v = (y_1 \dots y_n)$

We now use $\mathbb{R}^n$ with its Euclidean distance : $d(u,v) =$

$$d((x_1,\dots,x_n),(y_1,\dots,y_n)) = \|u-v\| = \sqrt{(u-v).(u-v)}$$
$$= \sqrt{(x_1-y_1)^2+\dots+(x_n-y_n^2)}$$

<u>Defn</u> $f : \mathbb{R}^n \to \mathbb{R}^n$ is an <u>isometry</u> (Euclidean isometry)

if $\|f(u) - f(v)\| = \|u-v\|$ for all $u, v \in \mathbb{R}^n$.

e.g. If $t = (t_1,\dots t_n)$ $f : \mathbb{R}^n \to \mathbb{R}^n$ defined $f(u) = u+t$ is an isometry,

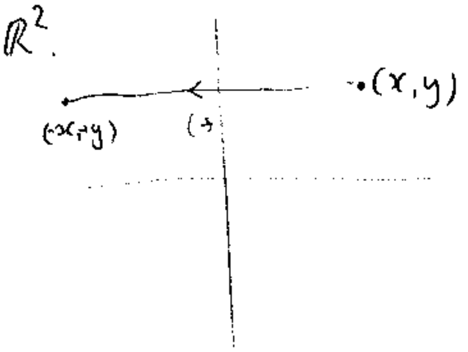
since $\|f(u) - f(v)\| = \|(u+t) - (v+t)\| = \|u-v\|$

for all $u, v \in \mathbb{R}^n$.

e.g. $n=2$ $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by $f(x,y) = (-x,y)$
(reflection in y-axis) is an isometry, since

$$\|f(u) - f(v)\| = \|(-x,y) - (-x',y')\| = \|(-x+x'), y-y'\|$$
$u=(x,y)$ $v=(x',y')$
$$= \sqrt{(x'-x)^2+(y-y')^2}$$
$$= \sqrt{(x-x')^2+(y-y')^2}$$
$$= \|u-v\|$$

holds for all $u, v \in \mathbb{R}^2$.



e.g. $n=2$ $0\leq\theta\leq 2\pi$ define $f:\mathbb{R}^2\to\mathbb{R}^2$ by

$$f_\theta(x,y) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}y\cos\theta - x\sin\theta\\ x\sin\theta + y\sin\theta\end{bmatrix}$$

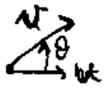
exercise: 1) show directly that f_θ is an isometry.

2) Let $R_\theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$ and $t=(x_0,y_0)\in\mathbb{R}^2$. Define

$f:\mathbb{R}^2\to\mathbb{R}^2$ by $f(v) = R_\theta u + t$.. Show that f is an isometry for any θ or $t\in\mathbb{R}^2$.

We saw before (in our woolly talk..) that anything that preserves distance preserves angles.

Recall: if $u, v\in\mathbb{R}^n$, the angle between u & v



is the unique θ s.t. 1)

$$\cos\theta = \frac{u\cdot v}{\|u\|\,\|v\|}$$

and 2) $0\leq\theta\leq\pi$.

Whoa! Who says $\frac{u\cdot v}{\|u\|\|v\|}$ can be the cosine of an angle?

Schwarz Inequality: If $u, v \in \mathbb{R}^n$, then

$$|u \cdot v| \leq \|u\| \, \|v\|.$$

Proof: Let $x \in \mathbb{R}$. Consider $q(x) = \|u - xv\|^2 = (u - xv) \cdot (u - xv)$

$$= \|u\|^2 - 2x(u \cdot v) + x^2 \|v\|^2$$

Since $q(x) \geq 0$, and q is a quadratic function of x,

"$b^2 - 4ac$" ≤ 0 i.e. $(-2(u \cdot v))^2 - 4 \cdot \|v\|^2 \cdot \|u\|^2 \leq 0$

i.e. $4(u \cdot v)^2 \leq 4 \|u\|^2 \|v\|^2$

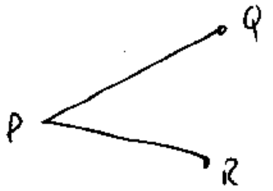
or $(u \cdot v)^2 \leq \|u\|^2 \|v\|^2$

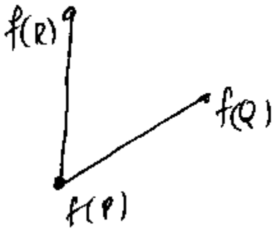
or $|u \cdot v| \leq \|u\| \, \|v\|$. QED ☒

Hence, preserving angles is the same as preserving dot products of vectors. ($\|u\| \|v\|$ - lengths) Q If f is an isometry of $\mathbb{R}^n$, will it preserve dot products?

Well,

Theorem If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, and $P, Q, R \in \mathbb{R}^n$,





$$(f(Q) - f(P)) \cdot (f(R) - f(P)) = (Q - P) \cdot (R - P).$$

Pf. Let $u = Q - P$, $v = R - P$

$u' = f(Q) - f(P)$ $v' = f(R) - f(P)$. 8

Then $\| u' - v' \|^2 = \| f(Q) - f(P) - (f(R) - f(P)) \|^2$

$= \| f(Q) - f(P) \|^2$

$*$ ——— $= \| Q - P \|^2$ (since f is an isom.

$= \| u - v \|^2$

LHS*: $\| u' - v' \|^2 = (u' - v') \cdot (u' - v') = \| u' \|^2 - 2 u' \cdot v' + \| v' \|^2$

But $\| u' \|^2 = \| f(Q) - f(P) \|^2 = \| Q - P \|^2 = \| u \|^2$ (f isom.)

and $\| v' \|^2 = \| f(R) - f(P) \|^2 = \| R - P \|^2 = \| v \|^2$.

RHS*: $\| u - v \|^2 = \| u \|^2 - 2 u \cdot v + \| v \|^2$.

So, LHS = RHS $\Rightarrow -2 u' \cdot v' = -2 u \cdot v$

$\Leftrightarrow u' \cdot v' = u \cdot v$

$\overset{\text{i.e.}}{\Leftrightarrow} (f(Q) - f(P)) \cdot (f(R) - f(P)) = (Q - P) \cdot (R - P)$.

$\square$

e.g. $R_y : \mathbb{R}^2 \to \mathbb{R}^2$ $R_y(x, y) = (-x, y)$.

$(R_y(a, b) - R_y(c, d)) \cdot (R_y(e, f) - R_y(c, d))$

$= (-a, b) - (-c, d) \cdot ((-e, f) - (-c, d))$

$= (c - a, b - d) \cdot (c - e, f - d)$

check : $= [(a, b) - (c, d)] \cdot [(e, f) - (c, d)]$

<u>Remark</u>: We cannot hope that

$$f(u)\cdot f(v) = u\cdot v \quad \text{for } \underline{\text{every}} \text{ isometry,}$$

Since: e.g. Consider $g: \mathbb{R}^2 \to \mathbb{R}^2$ def'd by $g(x,y) = (x,y)+(1,1)$

Then $f(1,0)\cdot f(0,1) = (2,1)\cdot(1,2) = 4$

while $(1,0)\cdot(0,1) = 0$!

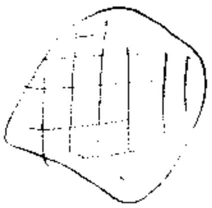
<u>However</u>: (exercise) if $f: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry and $f(0) = 0$, show that $f(u)\cdot f(v) = u\cdot v$ for all $u, v \in \mathbb{R}^n$.

iso = same
metron - Gr. measure

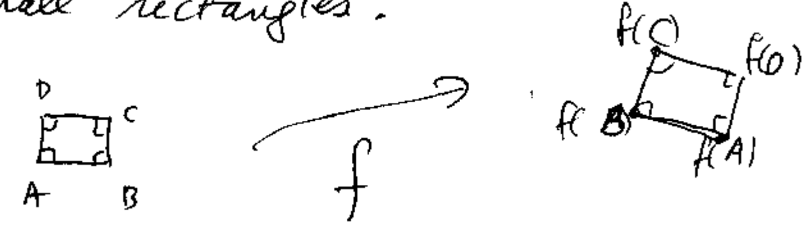
L2
L3

So lengths, angles are preserved by isometries. As well,

- area is preserved by isometries:



the area of any (reasonable) figure can be approximated by adding areas of small rectangles.



R

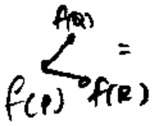
f(R) (still a rectangle!)

area R = $\|A-B\|\,\|A-D\|$

area f(R) = $\|f(A)-f(B)\|\,\|f(A)-f(D)\|$ = area R.

OR:

area = $\|u \times v\|$



= $(f(R)-f(P)) \times f(Q)$

Q: $f(u \times v) = f(u) \times f(v)$? Even if $f(0)=0$?

- volume (hypervolume) is preserved $(u \times v \cdot w) \sim (fu \times fv) \cdot f(w)$ If $f(u) \times f(v) = f(u \times v)$ ... by isometries ... (in the same way)

(ex. show that if $f: \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry, compare $f(u \times v)$ and $f(u) \overset{?}{\times} f(v)$). Find an example $\neq$

- isometries preserve straight lines i.e

if $L \overset{= \{v_0 + s u_0 \mid s \in \mathbb{R}\} \ (u_0 \neq 0)}{\subset} \mathbb{R}^n$ is a line, $f: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, then $f(L) \overset{= \{f(p) \mid p \in \mathbb{R}^n\} \text{ "image of } L \text{ under } f\text{"}}{}$ is also a line!

?? This will follow from:

<u>Theorem</u> If $f: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, and $f(0)=0$, then

1) $f(u+v) = f(u) + f(v)$ for any, $u, v \in \mathbb{R}^n$
2) $f(ku) = k.f(u)$ $k \in \mathbb{R}$.

(i.e. f is a <u>linear transformation</u>).

Pf. It is enough to show that $\forall u, v \in \mathbb{R}, \forall k \in \mathbb{R}$,

$$\| f(ku+v) - (k.f(u) + f(v)) \| = 0 .$$

We compute: $\| f(ku+v) - (k f(u) + f(v)) \|^2$

$$= \| f(ku+v) - k f(u) + f(v)) \| \cdot (f(ku+v) - k f(u) + f(v)))$$

$$= \underset{①}{\| f(ku+v) \|^2} - 2 \underset{②}{f(ku+v) \cdot (k f(u) + f(v))} + \underset{③}{\| k f(u) + f(v) \|^2}$$

Now, ~~since~~ $f(0) = 0$, and f is an isometry ~~of~~ if $w \in \mathbb{R}^n$ then

$$\|f(w)\|^2 = \|f(w) - f(0)\|^2 = \|w - 0\|^2 = \|w\|^2.$$

N.B.
$\|f(w)\|^2 = \|w\|^2$
(f isom, $f(0)=0$)

So, ① $= \|ku + v\|^2 = k^2\|u\|^2 + 2k\, u \cdot v + \|v\|^2.$

② $= -2k f(ku+v) \cdot f(u) - 2 f(ku+v) \cdot f(v)$

$= -2k\,(ku+v) \cdot u - 2\,(ku+v) \cdot v$ (Since $w_1 \cdot w_2 = f(w_1) \cdot f(w_2)$ $\forall\, w_1, w_2 \in \mathbb{R}^n$)

$= -2k^2\|u\|^2 - 2k\, u \cdot v - 2k\, u \cdot v - 2\|v\|^2$

$= -2\,(k^2\|u\|^2 + 2k(u \cdot v) + \|v\|^2)$

While

③ $= k^2\|f(u)\|^2 + 2k\, f(u) \cdot f(v) + \|f(v)\|^2$

$= k^2\|u\|^2 + 2k\, u \cdot v + \|v\|^2$

Clearly ① + ② + ③ $= 0$.

Hence $f(ku+v) = k.f(u) + f(v)$ ☐

Corollary If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry, then f sends lines to lines.

$(f(0) \neq 0)$ nec.

Pf. Set $g(u) = f(u) - f(0)$. Then $g : \mathbb{R}^n \to \mathbb{R}^n$ and

1) $g(0) = 0$

2) $\| g(P) - g(Q) \| = \| f(P) - f(0) - (f(Q) - f(0)) \| = \|f(P) - f(Q)\|$

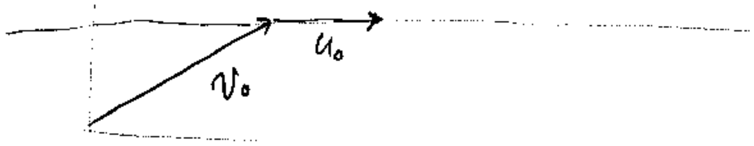
$= \| P - Q \|$ for all $P, Q \in \mathbb{R}^n$,

So g is an isometry with $g(0) = 0$.

So $g(ku + v) = k(g(u) + g(v)$.

Let L be a line in $\mathbb{R}^n$. Then, $L = \{ su_0 + v_0 \mid s \in \mathbb{R} \}$

So $f(L) = \{ f(su_0 + v_0) \mid s \in \mathbb{R} \}$.



Well,

$$f(su_0 + v_0) = f(0) + g(su_0 + v_0)$$

$$= f(0) + s.\, g(u_0) + \underbrace{g(v_0)}_{f(v_0) - f(0)}$$

$$= f(v_0) + s\,(f(u_0) - f(0))$$

let $v_0' = f(v_0)$, $u_0' = f(u_0) - f(0)$

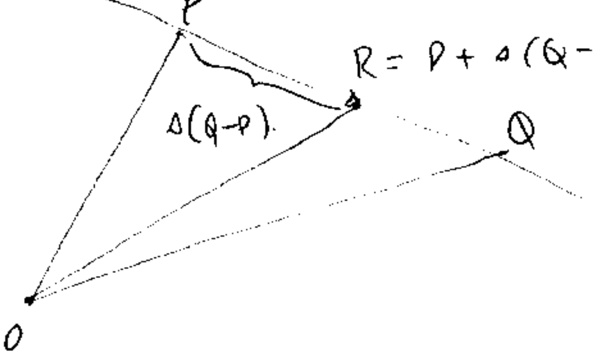
Then, $f(L) = \{ f(su_0 + v_0) \mid s \in \mathbb{R} \}$

$= \{ su_0' + v_0' \mid s \in \mathbb{R} \}$

is again a str. line (through v_0', direction u_0'). □

Remark If $P, Q \in \mathbb{R}^n$, the segment (Def^n) $\overline{PQ} =$

$\{ P + s(Q-P) \mid s \in \mathbb{R} \wedge 0 \le s \le 1 \}$



exercise

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isom, show that

$$f(\overline{PQ}) = \overline{f(P)\, f(Q)}.$$

So FAR

So, we know that an isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ is of the form $f(x) = g(x) + \underbrace{f(0)}_{\text{translation part}}$ where $g : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry which satisfies

1) $g(ku) = k.g(u)$

2) $g(u+v) = g(u) + g(v)$

"linear transformation"

– What do such g's look like?

e.g. in $\mathbb{R}^2$, seen rotations, reflections (not transl in this case)

$R_y(x,y) = (-x,y)$: $\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix} = \begin{bmatrix}-x\\ y\end{bmatrix}$!

$$R_\theta(x,y) = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}x\\ y\end{bmatrix}$$

These isometries are obtained by multiplication by a matrix! Is this always true?

<u>Yes</u>

<u>Thm</u> : If $g : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation, then there is an $n \times n$ matrix A s.t. $g(X) = AX$ (X col.).

<u>Pf.</u> Let $X = (X_1, \ldots, X_n) \in \mathbb{R}^n$. Then if $\{e_1, \ldots, e_n\}$ is the standard bases of $\mathbb{R}^n$,

$$g(X) = g(X_1e_1 + X_2e_2 + \cdots + X_ne_n)$$

$$= X_1 g(e_1) + X_2 g(e_2) + \cdots + X_n g(e_n) \qquad (g \text{ is linear})$$

Now let $A = [g(e_1) \ g(e_2) \ \cdots \ g(e_n)]$ $\qquad g(e_i)$ - ith column of A.

Then $AX = A\begin{bmatrix}X_1\\ \vdots\\ X_n\end{bmatrix} = \underbrace{[g(e_1) \cdots g(e_n)]\begin{bmatrix}X_1\\ \vdots\\ X_n\end{bmatrix}}_{\text{block mult}} = X_1g(e_1) + \cdots + X_ng(e_n)$

$= g(X)$! ▨

Corollary: If $f:\mathbb{R}^n\to\mathbb{R}^n$ is an isometry,

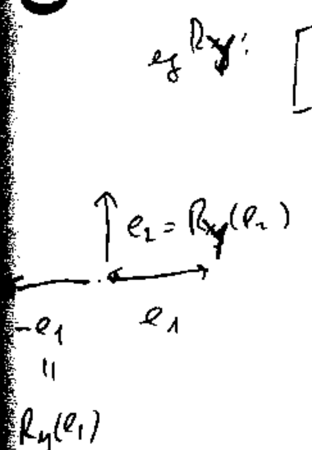
then $$f(x) = \underbrace{Ax}_{\text{linear part}} + \underbrace{b}_{\text{translation}}$$ for some $A\in M_{nn}(\mathbb{R})$, $b\in\mathbb{R}^n$. ($b=f(0)$).

What kind of matrix can A be? Must be special ...

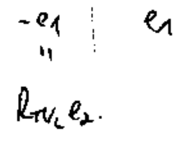
$$\underset{R_y}{\begin{bmatrix}-1&0\\0&1\end{bmatrix}} \quad \underset{I}{\begin{bmatrix}1&0\\0&1\end{bmatrix}} \quad \underset{R_{\pi/2}}{\begin{bmatrix}0&-1\\+1&0\end{bmatrix}} \quad \underset{R_\theta}{\begin{bmatrix}\cos\theta&-\sin\theta\\\sin\theta&\cos\theta\end{bmatrix}} \ldots$$

note: we know now that the columns of these matrices are the image of the standard basis by the isometry.

eg R_y: $\begin{bmatrix}-1&0\\0&1\end{bmatrix} = \begin{bmatrix}R_y(e_1) & R_y(e_2)\end{bmatrix}$



• $R_{\pi/2} = \begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}R_{\pi/2}(e_1) & R_{\pi/2}(e_2)\end{bmatrix}$



Now, $e_1\cdot e_2 = 0$, and the $\overset{\text{linear}}{\text{matrix}}$ part preserves dot products, so of course $R_y(e_1)\cdot R_y(e_2) = 0$
$R_{\pi/2}(e_1)\cdot R_{\pi/2}(e_2) = 0$

note. $R_\theta(e_1) \cdot R_\theta(e_2) = \begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix} \cdot \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix} = -\cos\theta\sin\theta + \sin\theta\cos\theta$
$= 0.$

These preserve the lengths of vectors, too:

$1 = \|e_1\|^2 = e_1 \cdot e_1$; $R_y(e_1) \cdot R(e_1) = \|R_y(e_1)\|^2 = 1$

e_1 ; $R_\theta(e_1) \cdot R_\theta(e_1) = \|(\cos\theta, \sin\theta)\|^2 = \cos^2\theta + \sin^2\theta$
$= 1.$

What have we found? If $g : \mathbb{R}^n \to \mathbb{R}^n$ is a (linear) isometry $^{(g(0)=0)}$, then in the matrix of g, the columns are mutually perpendicular, and each column has length 1 !

$$e_1, \dots, e_n \longmapsto ge_1 \dots ge_n$$
$$[ge_1 \dots ge_n] = A$$

here, $\|e_i\|^2 = 1 \iff \|g(e_i)\|^2 = 1$

$e_i \cdot e_j = 0 \quad (i \neq j) \qquad g(e_i) \cdot g(e_j) = 0 \quad (i \neq j)$

This is a very special kind of matrix.

Def^n A $n \times n$ matrix A is orthogonal if its columns are mutually perpendicular and all have length 1.

Def^n $O(n) = \{A \mid A \text{ is an orthogonal } n \times n \text{ matrix}\}$.

Ex. Show that 1) I_n is orthogonal
2) If $A, B \in O(n)$, then $AB \in O(n)$
3) If $A \in O(n)$ then A^{-1} exists and $A^{-1} \in O(n)$

There is an efficient way to express this using matrix multiplication:

Suppose $A = [v_1 \ \ v_2] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ $\quad v_1$ 1st col of A, $\quad v_2$ 2nd col of A

$$0 = v_1 \cdot v_2 = \begin{bmatrix} a \\ c \end{bmatrix} \underset{\substack{\uparrow \\ \text{dot product}}}{\cdot} \begin{bmatrix} b \\ d \end{bmatrix} = ab + cd$$

$$= \underset{\substack{\uparrow \\ \text{matrix} \\ \text{product}}}{[a \ \ c] \begin{bmatrix} b \\ d \end{bmatrix}} = [ab + cd] \equiv ab + cd$$

$$= v_1^t v_2 \qquad \text{(matrix product)}$$

Look at the matrix product

$$A^t A = [v_1 \ \ v_2]^t \, [v_1 \ \ v_2]$$

$$= \begin{bmatrix} v_1^t \\ v_2^t \end{bmatrix} [v_1 \ \ v_2]$$

$$= \begin{bmatrix} v_1^t v_1 & v_1^t v_2 \\ v_2^t v_1 & v_2^t v_2 \end{bmatrix}$$

$$= \begin{bmatrix} v_1 \cdot v_1 & v_1 \cdot v_2 \\ v_2 \cdot v_1 & v_2 \cdot v_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2!$$

Indeed

Proposition: A matrix is orthogonal if and only if $A^t A = I \quad (= AA^t)$.

This means 1) $A^{-1} = A^t$!

2) A^{-1} is also orthogonal

3) the rows of are mutually perpendicular and all have length 1.

(If $\{v_1, \ldots, v_n\}$ satisfies $v_i \cdot v_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

we call the set orthonormal.

If $A = [v_1 \cdots v_n]$ is orthogonal, then

$\{v_1, \ldots, v_n\}$ is a basis of $\mathbb{R}^n$ (since A is invertible)

It is an orthonormal basis of $\mathbb{R}^n$.

exercise

A linear isometry takes an orthonormal basis of $\mathbb{R}^n$ to another o.n. basis of $\mathbb{R}^n$.

So far: $f: \mathbb{R}^n \to \mathbb{R}^n$ ~~linear~~ isometry
(~~$g(0)=0$ & is an isometry~~)

then $f(x) = Ax + b$ A $n \times n$ matrix
$b \in \mathbb{R}^n$

Consider $f(x) - f(0) =: g(x)$. Then $g(x) = Ax$, $A = [ge_1 \cdots ge_n]$.

(Saw in $\mathbb{R}^2$, matrix A had orthogonal column, ~~each~~ of length 1.)

Suppose $\{e_1, \dots, e_n\}$ is the standard ordered basis of $\mathbb{R}^n$.

Cons $g(e_1), g(e_2), \dots, g(e_n)$

Then, we know that: $g(e_i) \cdot g(e_j) = e_i \cdot e_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$

i.e. the ~~column~~ vectors $ge_1 \cdots ge_n$ are mutually perpendicular and each has length 1. (i.e. $\{ge_1, \dots ge_n\}$ is an orthonormal set in $\mathbb{R}^n$.

If $A = [ge_1 \cdots ge_n]$, we saw that this is the same as saying $A^tA = I$.

Had: Defn An $n \times n$ matrix A is orthogonal if cols are an orthonormal set in $\mathbb{R}^n$.

Propn A is orthogonal $\Leftrightarrow A^tA = I = AA^t$.

Remark: Since $A^t = A^{-1}$, A is invertible. So cols are an orthonormal basis of $\mathbb{R}^n$.

e.g. $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ is an orthogonal matrix. $\begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ is orthogonal.

<u>Def^n</u> $O(n) = \{A \mid A$ is an orthogonal $n \times n$ matrix $\}$

<u>Exercise</u>: Show that 1) $I_n \in O(n)$ 2) If $A, B \in O(n)$, then $AB \in O(n)$ 3) If $A \in O(n)$, then $A^{-1} \in O(n)$.

e.g. $O(1) = \{ a \mid a.a = 1 \} = \{1, -1\}$.

<u>ex</u>. find all isometries of $\mathbb{R}^1$ and describe them geometrically (transl rotn? reflⁿ? ??)

e.g. $O(2) = \left\{ A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; AA^t = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

What kind of matrices are these? Know of examples...

Recall: $A = [Ae_1 \; Ae_2]$; know $\|Ae_1\| = \|Ae_2\| = 1$
$Ae_1 \cdot Ae_2 = 0$.

Since $\|Ae_1\| = 1$, $\|Ae_2\| = 1$
Let $Ae_1 = (\cos\theta, \sin\theta)$ $\quad Ae_2 = (\cos\varphi, \sin\varphi)$ $\quad 0 \leq \theta, \varphi < 2\pi$.

$$0 = Ae_1 \cdot Ae_2 = \cos\theta\cos\varphi + \sin\theta\sin\varphi = \cos(\theta - \varphi)$$

$\therefore \theta - \varphi = \pi/2, \; 3\pi/2$.

<u>2 cases</u> I. $\varphi = \theta - \pi/2$ or
II. $\varphi = \theta - 3\pi/2$.

So I. $\cos\varphi = \cos(\theta - \pi/2) = \sin\theta$
$(\sin\varphi = \sin(\theta - \pi/2) = -\cos\theta$

II $\cos\varphi = \cos(\theta - 3\pi/2)$
$= \cos(\theta + \pi/2) = -\sin\theta$
$\sin\varphi = \sin(\theta + \pi/2) = \cos\theta$

$\therefore$ either $A = \begin{bmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{bmatrix}$ or $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$

I. $\qquad$ II.

rotation $\circlearrowleft\theta$

$$\overset{\shortparallel}{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

itself is a reflection! $\rightarrow$ $\qquad \uparrow \circlearrowleft\theta \qquad R_x$

We've discovered a great deal!

◡ Theorem: If $f : \mathbb{R}^n \to \mathbb{R}^n$ is an isometry,

then $f(x) = Ax + f(0)$, where A is

the orthogonal $\underset{\longrightarrow}{\text{matrix}}$ $A = [\,\underset{\substack{| \\ f(0)}}{f(e_1)} \cdots \underset{\substack{| \\ f(0)}}{f(e_n)}\,]$.

We've seen examples for $n = 2$.

$\underline{n=1}$? What is a 1×1 orthogonal matrix?
exercise previously given

$[a]^t[a] = (1) \iff a^2 = 1 \iff a = \pm 1$.

◡ So any isometry of $\mathbb{R}^1$ is of the form

$f(x) = ax + b$, where $|a| = 1$
$t \in \mathbb{R}$!

compare with $\underline{M(E^1)}$. Note: $f(x) = -x$ is the reflection in 0.

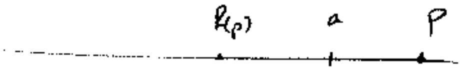
exercise: show that $f(x) = -x + b$ is also a reflection .....

$\underline{n=3}$ .. $\begin{pmatrix} 1.4 \\ 1.5 \\ 1.6 \end{pmatrix}$ Reflections, Rotations, Translations
≥ 3

Reflections in $\mathbb{R}^3$:

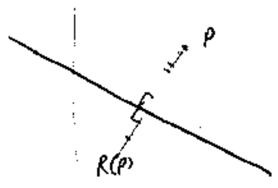
1.3, 1.4 Reflections in $\mathbb{R}^n$

In $\mathbb{R}^1$



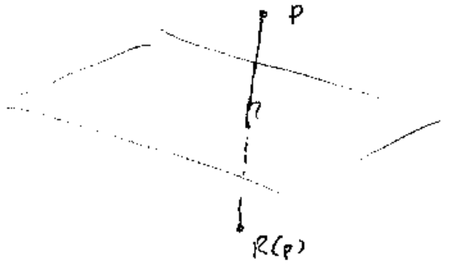
reflection in a point

$\mathbb{R}^2$



reflection in a line

$\mathbb{R}^3$



reflection in a plane

⋮

A reflection is always made w.r.t. a hyperplane

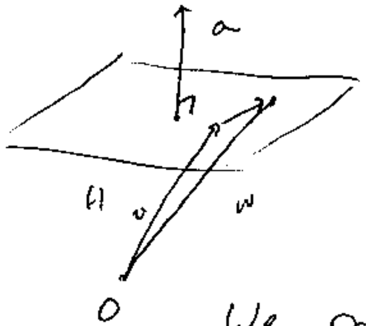
i.e. $H = H_0 + t$ where • H_0 is a subspace of $\mathbb{R}^n$ of dimension $n-1$

("translate" of a subspace of dimension $n-1$ "codimension 1") • $t \in \mathbb{R}^n$

Fix $a \overset{\neq 0}{=} (a_1, \dots, a_n) \in \mathbb{R}^n$, $b \in \mathbb{R}^n$

$$H = \{ (x_1, \dots, x_n) \mid a_1 x_1 + \dots + a_n x_n = b \} \qquad (n=1,\ H = \{\tfrac{b}{a_1}\})$$
$$= \{ v \in \mathbb{R}^n \mid a \cdot v = b \}$$

($H_0 = \{ v \in \mathbb{R}^n \mid a \cdot v = 0 \}$ is a subspace: all vectors in $\mathbb{R}^n$ $\perp$ to a!



Note: if $v, w \in H$, then

$$a \cdot (v - w) = a \cdot v - a \cdot w = b - b = 0$$

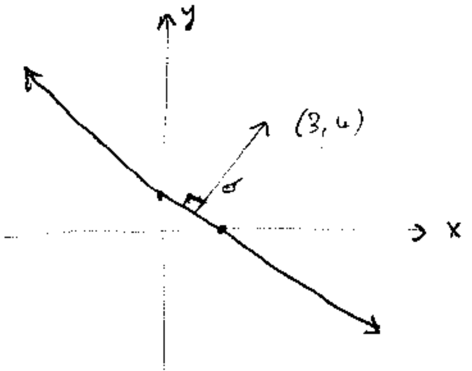
So $a \perp v - w$.

We say that a is a normal to the hyperplane H

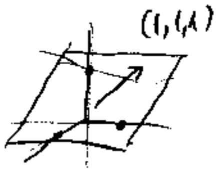
e.g. $n = 3$ $\quad a_1 x_1 + a_2 x_2 + a_3 x_3 = b$ is eqn of a plane with normal (a_1, a_2, a_3)

$n = 2$ $\quad a_1 x + a_2 y = b$ is eqn of line in $\mathbb{R}^2$ with normal (a_1, a_2)

e.g. $3x + 4y = 6$



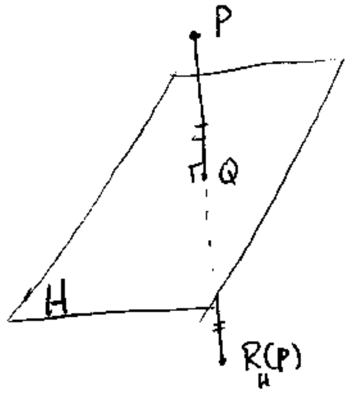
e.g. $x+y+z=1$



19/9/02

<u>Reflection in</u> $H = \{v \mid v\cdot a = b\}$ $(a \neq 0)$.

Clearly, $\vec{PQ} = ta$, $(t \in \mathbb{R})$ and



$$R_H(P) = P + 2\vec{PQ}$$
$$= P + 2ta.$$

To find t, we note that $Q \in H$,

so $Q = P + \vec{PQ} = P + ta$ satisfies

$$Q \cdot a = b$$

Solving for t, we find: $(P + ta)\cdot a = b \iff t\|a\|^2 = b - a\cdot P$
$\iff t = \dfrac{b - a\cdot P}{\|a\|^2}$

Hence $$\boxed{R_H(P) = P + 2\frac{(b - a\cdot P)}{\|a\|^2}.\, a}$$

$R_H(0) = 0 + 2\frac{b}{\|a\|}.a$

$\therefore R_H'(P) = R_H(P) - R_H(0)$
$= P - \dfrac{2(a\cdot P)a}{\|a\|^2}$

<u>Remark</u> $\underline{R_H(0) = 0 \iff b = 0 \iff H \text{ is a subspace} \iff R_H \text{ is linear}}$ $R_{H_0}(P)$ · (linear in P)

$R_H^2 = Id.$

e.g. ($n = 1$) $H = \{\frac{b}{a}\} = \{x \mid ax = b\}$ $(a \neq 0)$; $x_0 = b/a$ $R_{x_0}(x) = x + \dfrac{2(b - ax)}{a\,a^2}.a$

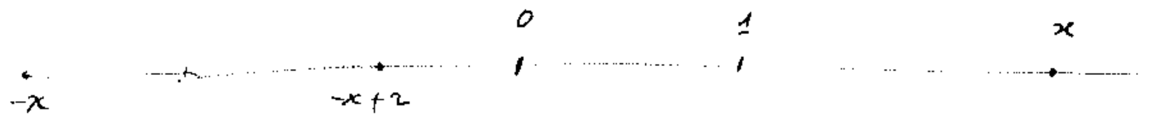
(reflection in a point :

exercise

$= x + 2\frac{b}{a} - 2x$

$= -x + 2b/a$

If $x_0 = b/a$ $R_{x_0}(x) = -x + 2x_0.$



e.g $x_0 = 1$

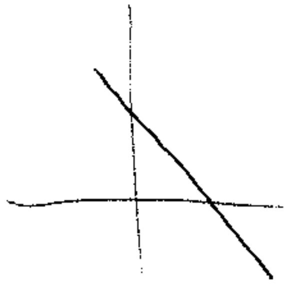
$R_1(x) = -x + 2$

assignment # 1

e.g. $\underline{n=2}$ $H = \{ (x,y) \mid a_1 x + a_2 y = b \}$ $(a_1, a_2) \neq 0$

e.g. $x+y=1$

$$R_H(x,y) = (x,y) + \frac{2(1-x-y)}{2} . (1,1)$$

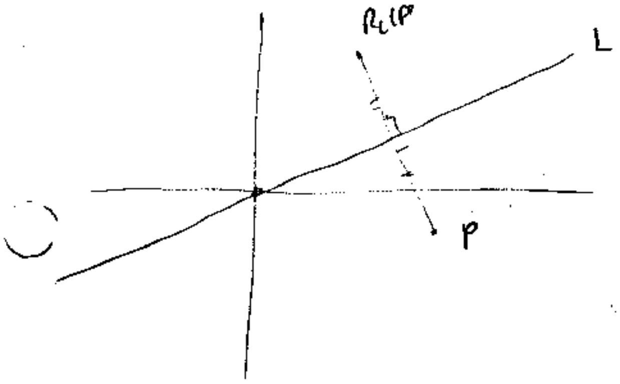


$$= (x,y) + (1-x-y, 1-x-y)$$

$$= (1-y, 1-x) = (1,1) + (-y,-x)$$

Column notation: $R_H(x,y) = \begin{bmatrix}1\\1\end{bmatrix} + \overbrace{\begin{bmatrix}0&-1\\-1&0\end{bmatrix}}^{A}\begin{bmatrix}x\\y\end{bmatrix}$; $AA^t = I_2$ det $A = -1$.

→ e.g. L: $x - 2y = 0$ (line through origin : subspace)



$$R_L(x,y) = (x,y) - \frac{2(x-2y)}{5} . (1,-2)$$

$$= \left(\tfrac{3}{5}x + \tfrac{4}{5}y, \tfrac{4}{5}x - \tfrac{3}{5}y\right)$$

$$= \underbrace{\begin{bmatrix}\frac{3}{5} & \frac{4}{5}\\ \frac{4}{5} & -\frac{3}{5}\end{bmatrix}}_{A}\begin{bmatrix}x\\y\end{bmatrix}$$

$\underline{\text{note}}$: $AA^t = \frac{1}{5}\begin{bmatrix}3&4\\4&-3\end{bmatrix}\frac{1}{5}\begin{bmatrix}3&4\\4&-3\end{bmatrix} = \frac{1}{25}\begin{bmatrix}25&0\\0&25\end{bmatrix} = I_2$!

As expected: A is $\underline{\text{orthogonal}}$.

note: det $A = \frac{-9-16}{25} = -1$. (In fact we'll see that A orthog, det $A = -1 \Leftrightarrow$ A generates a reflection through a line in $\mathbb{R}^2$.)

$\underline{\text{Remark}}$: $R_L(x,y) = (x,y) \underset{(x,y)\in L!}{\Leftrightarrow} x - 2y = 0 \Leftrightarrow (x,y) \in L$. We can $\underline{\text{recover}}$ L from R_L : $R_L(x,y) = A\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix} \Leftrightarrow \begin{bmatrix}x\\y\end{bmatrix}$ is an eigenvector of A with eval 1 !

$R_H(P) = P + 2\frac{(b - a.P)}{\|a\|^2}\, a$

e.g. <u>Reflections in $\mathbb{R}^3$</u>

$H = \{(x, y, z) \mid x + y + z = 0\}$ (Subspace plane through origin)

$$R_H(x,y,z) = (x, y, z) - \frac{2(x+y+z)}{3}.(1,1,1)$$

$$= \left(-\frac{x}{3} - \frac{2}{3}y - \frac{2}{3}z\ ,\ -\frac{2}{3}x + \frac{y}{3} - \frac{2}{3}z\ ,\ -\frac{2}{3}x - \frac{2}{3}y + \frac{z}{3}\right)$$

∴ Colo

$$= \underbrace{-\frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix}}_{A}\begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

check: $AA^t = I$
($\det A = -1.$)

Again, $H = \{v \mid R_H(v) = v\}$ (eigenspace corres. to $\lambda = 1$).

$\{v \mid Av = v\} = \{v \mid (A - I)v = 0\}$; $\left[\begin{array}{ccc|c} -2/3 & -2/3 & -2/3 & 0 \\ -2/3 & -2/3 & -2/3 & 0 \\ -2/3 & -2/3 & -2/3 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} & s & t & \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \{(x,y,z) \mid x+y+z=0\}$! above

<u>Remark</u>: The matrix of R_H is symmetric, so (recall!) it is diagonalizable! What will the diagonal form be? What are the possible eigenvalues of an orthogonal matrix?

Suppose $AA^t = A^tA = I$, and $Av = \lambda v$ $(v \neq 0)$.

Then 1) $v^tA^t = \lambda v^t$

and 2) $v = A^t(Av) = \lambda A^t v$, so $\lambda \neq 0$ and $A^t v = \frac{1}{\lambda} v$

Consider $v^tA^tv = (v^tA^t)v = \lambda(v^t)v = \lambda \|v\|^2$

$= v^t(A^tv) = v^t\left(\frac{1}{\lambda}v\right) = \frac{1}{\lambda}\|v\|^2$

∴ $\lambda^2 = 1$, so $\lambda = \pm 1$.

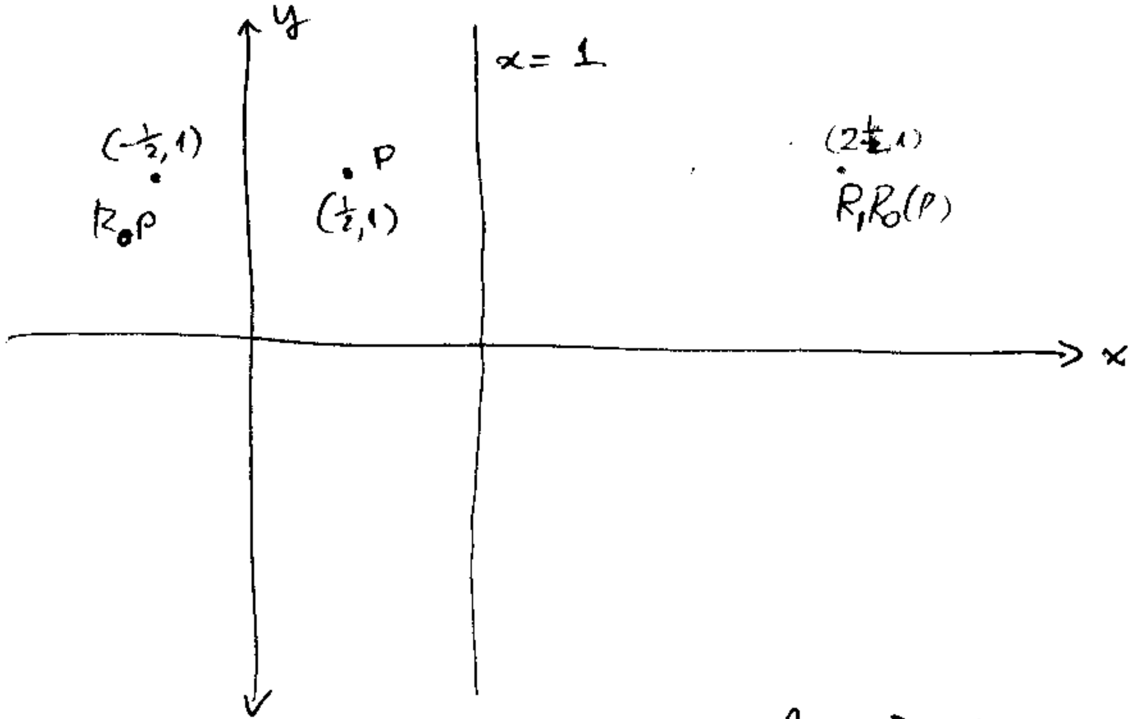
$$\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1 \end{bmatrix} \quad \overbrace{\begin{bmatrix} -1 & & \\ & 1 & \\ & & 1 \end{bmatrix}}^{\text{reflection}} \quad \overset{\text{rotation}}{\begin{bmatrix} -1 & & \\ & -1 & \\ & & 1 \end{bmatrix}} \quad \begin{bmatrix} -1 & & \\ & -1 & \\ & & -1 \end{bmatrix}$$

Translations

$v_0 \in \mathbb{R}^n$, fixed. $T_{v_0}: \mathbb{R}^n \to \mathbb{R}^n$ def'd by $T_{v_0}(v) = v_0 + v$

is the translation by v_0; we already know it is an isometry

eg. $T_{(2,0)}: \mathbb{R}^2 \to \mathbb{R}^2$; Let R_0 = refl^n in y-axis
R_1 = " $x=1$.



Consider $f: \mathbb{R}^2 \to \mathbb{R}^2$ def'd by $f(v) = R_1 R_0(v)$

$R_0 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x \\ y \end{bmatrix}$; $R_1 \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -x+2 \\ y \end{bmatrix}$ (check)

$R_1 R_0 \begin{pmatrix} x \\ y \end{pmatrix} = R_1 \begin{bmatrix} -x \\ y \end{bmatrix} = \begin{bmatrix} x+2 \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \end{bmatrix} = T_{(2,0)}\left(\begin{bmatrix} x \\ y \end{bmatrix}\right)$!

Prop^n Every translation of $\mathbb{R}^n$ is a composition ("product") of 2 reflections

Pf. Let $a \in \mathbb{R}^n$, $a \neq 0$, and $T_a: \mathbb{R}^n \to \mathbb{R}^n$ be $T_a(v) = v + a$.
Define $H_0 = \{v \in \mathbb{R}^n \mid a \cdot v = 0\}$
$H_1 = \{v \in \mathbb{R}^n \mid a \cdot v = \frac{\|a\|^2}{2}\}$.

Rough check: $d(H_0, H_1) = \frac{\|a\|}{2}$

claim : $T_a = R_{H_1}R_{H_0}$

$R_H(p) = p + 2\frac{(b - a\cdot p)}{\|a\|^2} a$

Well, $R_{H_0}(v) = v - 2\frac{(a_0\cdot v)\, a_0}{\|a_0\|^2}$

$$R_{H_1}(v) = v + 2\frac{\left(\frac{\|a_0\|^2}{2} - a_0\cdot v\right)}{\|a_0\|^2} a_0$$

$$= v + a_0 - 2\frac{(a_0\cdot v)\, a}{\|a_0\|^2}$$

$$= R_{H_0}v + a \quad !$$

Hence, $R_{H_1}(R_{H_0}v) = R_{H_0}(R_{H_0}v) + a_0$

$= v + a$!

$= T_a(v)$

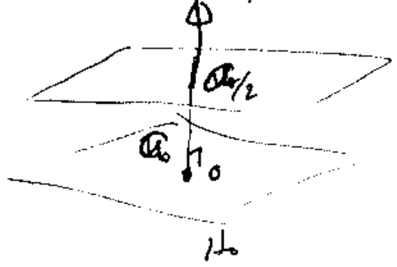
i.e. $R_{H_1}R_{H_0} = T_a$

$R_{H_0}v = R_{H_1}v - a$

So

$R_{H_0}R_{H_1}(v) = R_{H_1}(R_{H_1}(v)) - a$
$= v - a$!

Rmk: 1) $H_0 \parallel H_1$
2) The normals are (equal!) parallel to the translation
3) The distance between H_0 & H_1 is $\frac{\|a_0\|}{2}$,



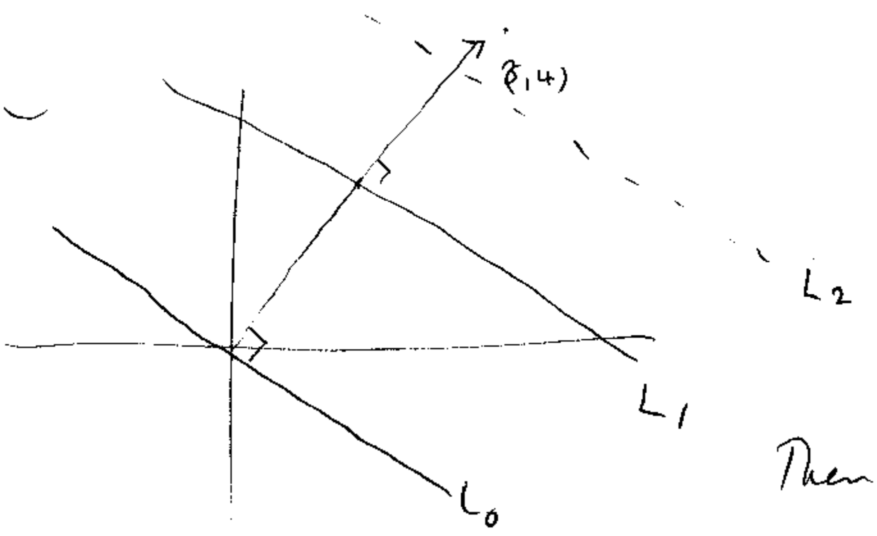
Since 1) $a \perp H_0$
2) $\frac{a}{2} \in H_1$

$\therefore \|\frac{a}{2}\|$ is distance

exercise: If H and K are parallel hyperplanes, ~~with~~ unit normal $a \neq 0$, then $R_K R_H = T_{\pm da}$, where

d = distance from H to K
$+$ if a points from H to K
$-$ " K to H.

e.g. $T_{(3,4)} : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$; Let $L_0 = \{v \in \mathbb{R}^2 \mid (3,4)\cdot v = 0\}$



$= \{(x,y) \mid 3x+4y=0\}$

$L_1 = \{v \in \mathbb{R}^2 \mid (3,4)\cdot v = \frac{25}{2}\}$

$= \{(x,y) \mid 3x+4y = \frac{25}{2}\}$

Then $R_{L_1}R_{L_0} = T_{(3,4)}$

Let $L_2 = \{v \in \mathbb{R}^2 \mid (3,4)\cdot v = 25 \ (=\|(3,4)\|^2)\}$.

Then $R_{L_1}(v) = v + 2\,\frac{(\frac{25}{2} - (3,4)\cdot v)}{25}\,(3,4)$

$= (3,4) + R_{L_0}(v)$

$R_{L_2}(v) = v + 2\left(\frac{25 - (3,4)\cdot v}{25}\right)(3,4)$

$= 2(3.4) + R_{L_0}(v) = (3,4) + R_{L_1}(v)$

$\therefore \quad R_{L_2}R_{L_1}(v) = (3,4) + R_{L_1}(R_{L_1}v) = (3,4) + v.$

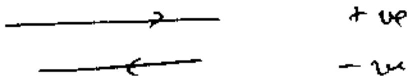
Note: $R_{L_1}R_{L_2}(v) = R_{L_2}(R_{L_2}(v)) - (3,4)$

$= v - (3,4)$!

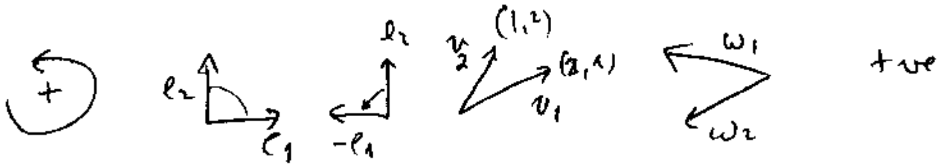
ex. Express $T_{(1,2,2)} : \mathbb{R}^3 \to \mathbb{R}^3$ as a product of 2 reflections "before" rotation.

Orientation.

$\mathbb{R}$



$\mathbb{R}^2$



$\{e_1, e_2\}, \{e_2, -e_1\}, \{v_1, v_2\}, \{w_1, w_2\},$

Sensitive to order

$\{-e_1, e_2\}$ $\{(1,1), e_1\}$

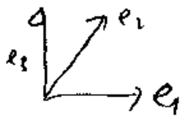
note $\det\begin{bmatrix} e_1 & e_2 \end{bmatrix} = \det\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$ $\det\begin{bmatrix} e_2 & -e_1 \end{bmatrix} = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$

write as cols

$\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ all +ve

but $\begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix} = -1 < 0$ $\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 < 0$ all -ve

$\mathbb{R}^3$



$\{e_1, e_2, e_3\}$ $\det|e_1\ e_2\ e_3| = e_1 \times e_2 \cdot e_3 = 1$

Right hand rule determines +ve direction of third.

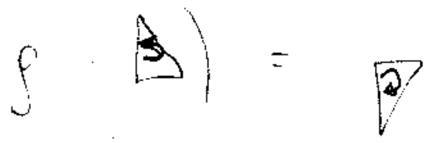
Def^n An ordered basis $\{v_1, \dots, v_n\}$ of $\mathbb{R}^n$ is +vely oriented (oriented +vely, has +ve orientation ...) if $\det\ [v_1 \cdots v_n] > 0$.

Effect of isometries on orientation

e.g. $\underline{n=2}$ $\quad R_x \begin{bmatrix} x \\ y \end{bmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$

$\{e_1, e_2\} \longmapsto \{e_1, -e_2\}$

+ve $\qquad$ − ve



Let $\{v_1, v_2\}$ be any ordered basis of $\mathbb{R}^2$. Is the orientation of $\{R_x v_1, R_x v_2\}$ opposite to that of $\{v_1, v_2\}$? Recall $R_x v = Av$

$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \in O(2)$

Well, $\det [R_x v_1 \; R_x v_2] = \det [Av_1 \; Av_2]$ $\qquad$ block mult^n

$= \det (A [v_1 \; v_2])$

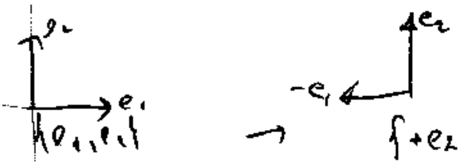
$= \det A \, . \, \det [v_1 \; v_2]$

$= -1 \, . \, \det [v_1 \; v_2]$

So yes! R_x reverses all orientations.

L6/L7 2015

e.g. ($\frac{\pi}{2}$) $R_{\frac{\pi}{2}}$ $\rho : \mathbb{R}^2 \to \mathbb{R}^2$ $\qquad \rho \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$



$\{e_1, e_2\} \rightarrow \{+e_2, -e_1\}$

$\det [\rho v_1 \; \rho v_2] = \det \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} [v_1 \; v_2] \right)$

$= 1 \, . \, \det [v_1 \; v_2]$

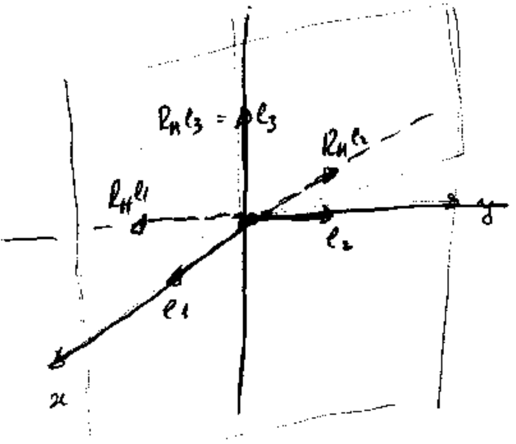
preserves orientation

$R_H(P) = P + 2\frac{(b - a\cdot P)}{\|a\|^2} a = P - 2a\cdot P \frac{a}{\|a\|^2}$

e.g. Reflection in the plane $H = \{(x,y,z) \mid x+y = 0\}$

$$R_H(x,y,z) = (x,y,z) - \frac{2(x+y)}{2}(1,1,0) = (x,y,z) - (x+y, x+y, 0)$$

$$= (-y,-x,z) = \underbrace{\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

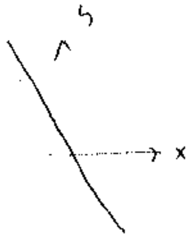


$R_H e_1 = R_H(1,0,0) = (0,-1,0) = -e_2$

$R_H e_2 = R_H(0,1,0) = (-1,0,0) = -e_1$

$R_H e_3 = R_H(0,0,1) = (0,0,1) = e_3$

(looks like a reversal).



$\{e_1, e_2, e_3\}$ +ve ✓

$\{R_H e_1\ R_H e_2\ R_H e_3\}$: $R_H e_1 \times R_H e_2 \cdot R_H e_3$

$$= \det \begin{bmatrix} R_H e_1 & R_H e_2 & R_H e_3 \end{bmatrix}$$

$$= \det \begin{bmatrix} \overset{+}{0} & \overset{-}{-1} & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad = \overset{-}{(-1)}\overset{(1,2)\text{ entry}}{(-1)} \begin{vmatrix} -1 & 0 \\ 0 & 1 \end{vmatrix}$$

$$= -1.$$

So R_H reverses the orientation in this case.

Suppose $\{v_1, v_2, v_3\}$ is any basis of $\mathbb{R}^3$. Does R_H reverse its orientation too?

Again,

$$[R_H v_1 \ R_H v_2 \ R_H v_3] = \overbrace{\begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{A} [v_1 \ v_2 \ v_3)$$

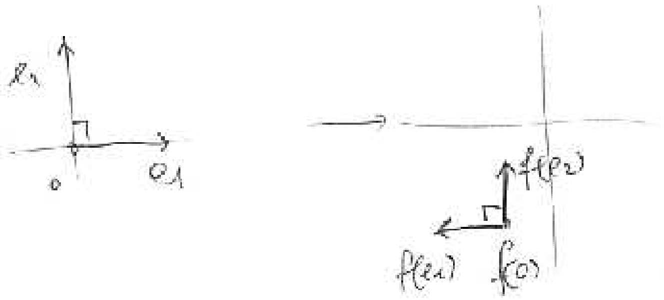
$$\text{So } \det(R_H v_1 \ R_H v_2 \ R_H v_3) = \underbrace{\det A}_{} \cdot \det [v_1, v_2, v_3)$$

29/9/03 8/3/07

Remark: All previous examples have fixed origin. What if not?

eg $f : \mathbb{R}^2 \to \mathbb{R}^2$ $f(x,y) = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} -2 \\ -4 \end{bmatrix}$; $f(0) = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$

$f(e_1) = \begin{bmatrix} -3 \\ -4 \end{bmatrix}$



appears to reverse orientation

$f(e_2) = \begin{bmatrix} -2 \\ -3 \end{bmatrix}$

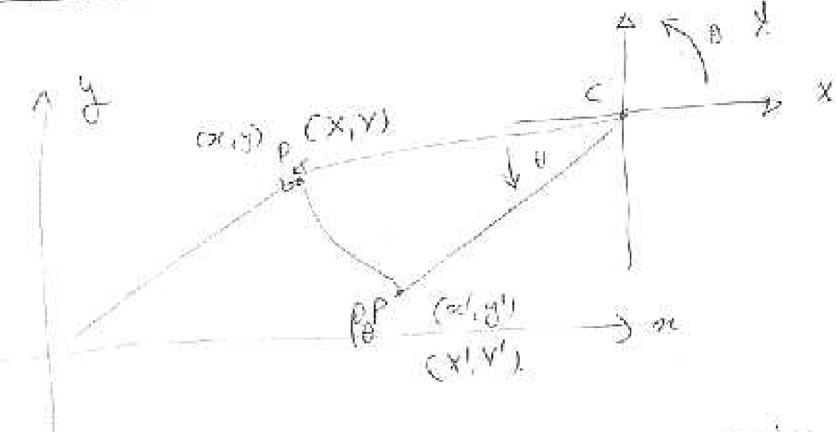
Should check $\det [f(e_1) - f(0) \quad f(e_2) - f(0)) = \det \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$
$= -1$.

not $\det [f(e_1) \quad f(e_1)] = \det \begin{bmatrix} -3 & -2 \\ -4 & -3 \end{bmatrix} = +1$)

Defⁿ An isometry $f : \mathbb{R}^n \to \mathbb{R}^n$ is orientation preserving if $\det [f(e_1) - f(0) \ \dots \ f(e_n) - f(0)] > 0$. ("reversing" if < 0)

Rmk: ① If $f(x) = Ax + b$ $A \in O(n)$, $b \in \mathbb{R}^n$, f or pres $\Leftrightarrow$ $\det A = 1$; Remark ② f, g isom. fg preserve orient?

Rotations in $\mathbb{R}^2$ about $c = (c_1, c_2)$



New Coord

$$X = x - c_1 \qquad ; \begin{bmatrix}X\\Y\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix} - c$$
$$Y = y - c_2$$

$$A_\theta = \begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}$$

Rotn in (X, Y) coords is $\begin{bmatrix}X'\\Y'\end{bmatrix} = A_\theta . \begin{bmatrix}X\\Y\end{bmatrix}$

$$\therefore \begin{bmatrix}x'\\y'\end{bmatrix} = \begin{bmatrix}X'\\Y'\end{bmatrix} + c = A_\theta\begin{bmatrix}X\\Y\end{bmatrix} + c = A_\theta\left(\begin{bmatrix}x\\y\end{bmatrix} - c\right) + c$$

$$\therefore \rho_\theta\begin{bmatrix}x\\y\end{bmatrix} = \begin{bmatrix}x'\\y'\end{bmatrix} = A_\theta\left(\begin{bmatrix}x\\y\end{bmatrix} - c\right) + c$$

$$\rho_{\theta, c}(v) = A_\theta(v - c) + c \qquad \left(= A_\theta v + (I - A_\theta)c \right)$$

eg rotn by $\pi/3$ about $(1, 1)$ $\qquad \rho_{\pi/3} = \begin{bmatrix}\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \sqrt{3}/2 & \frac{1}{2}\end{bmatrix}$

$$\therefore \rho_c\begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + \frac{1}{2}\begin{bmatrix}\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\sqrt{3}/2 & \frac{1}{2}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1 & -\sqrt{3}\\ \sqrt{3} & 1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} + \frac{1}{2}\begin{bmatrix}1+\sqrt{3}\\ 1-\sqrt{3}\end{bmatrix}$$

Rmk. Every orientation preserving isometry of $\mathbb{R}^2$ is of the form $f(v) = A_\theta v + b$ for some θ, some $b \in \mathbb{R}^2$, $A_\theta = \begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$

Thm Every orientation preserving isometry of $\mathbb{R}^2$ is either a translation or a rotation.

Pf. We know $f(v) = A_\theta v + b$; if $A_\theta = I_2$, $f(v) = v + b$ is a translation, so assume $A_\theta \neq I_2$.

To see that f is a rotation, it suffices to find $c \in \mathbb{R}^2$ s.t.

$$A_\theta v + b = A_\theta(v - c) + c \quad \text{for all } v \in \mathbb{R}^2$$

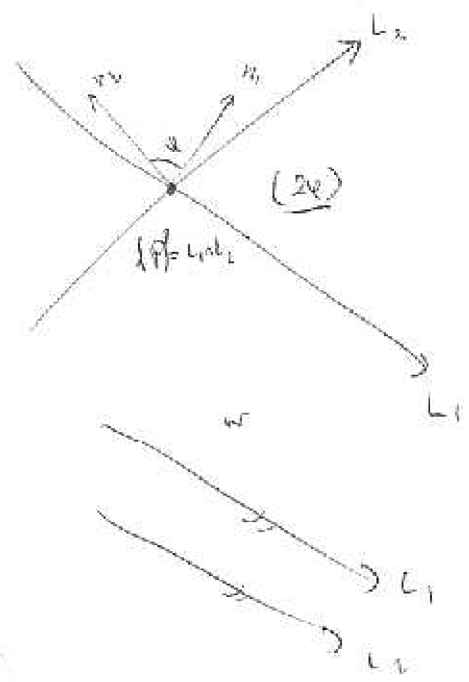
$$\text{i.e.} \quad b = \underline{(I - A_\theta)c} \quad *$$

Now $\det(I - A_\theta) = \begin{vmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & 1-\cos\theta \end{vmatrix} = 1 - 2\cos\theta + \cos^2\theta + \sin^2\theta$
$= 2(1-\cos\theta)$
$= 0 \iff \cos\theta = 1 \iff \begin{matrix}\cos\theta = 1\\ \sin\theta = 0\end{matrix}$
$\iff A_\theta = I_2$.

If $A_\theta \neq I_2$
$\therefore \det(I - A_\theta) \neq 0$ ∴

Hence $I - A_\theta$ is invertible; so we can solve $*$ for c. So we can write
$f(v) = A_\theta(v - c) + c$, how f is a rotn.
$\forall v \in \mathbb{R}^2$ ☒

<u>Corollary</u> If L_1 and L_2 are lines in $\mathbb{R}^2$, then the isometry $f = R_{L_2}R_{L_1}$ is either a translation or a rotn.



Pf. f preserves orientation, since R_{L_1} reverses it, as does R_{L_2}. (Why? : matrix of R_L is always $\begin{pmatrix}\cos\theta & \sin\theta \\ \sin\theta & -\cos\theta\end{pmatrix}$! det $= -1$.)

Hence f is either a translation or a rotation. ☒

<u>Rmk</u>. • If $L_1 \parallel L_2$ (have parallel normals) $R_{L_2}R_{L_1}$ is a translation (in dirn of common normal by twice dist. betw L_1 & L_2).

<u>NEW</u> • If $L_1 \nparallel L_2$, $R_{L_2}R_{L_1}$ is a rotn an ~~[illegible]~~ <u>exercise</u> about $L_1 \cap L_2$ by <u>twice</u> acute angle between L_1 & L_2; orientation of rotation is $\{n_1, n_2\}$ (normals of $n_1 \cdot n_2 \geq 0$.

pblm 15

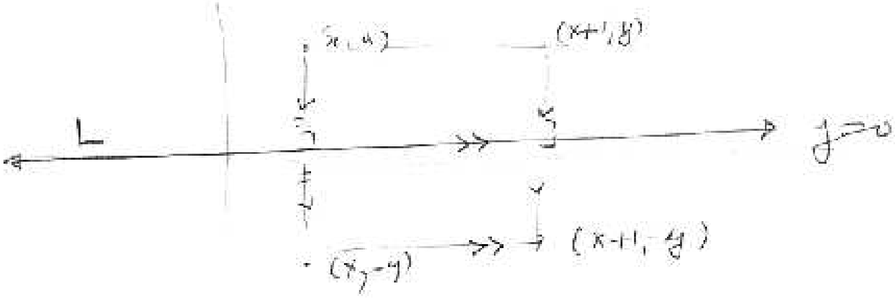
Orientation reversing isometries : $f(v) = Av + b$; $A \in O(2)$, $\det A = -1$.

These include reflections

e.g. $f\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} x \\ -y+2 \end{bmatrix}$ is reflection in line $y = 1$. ($h(x,y) = (x,-y)$ refl^n in $y=0$). (Note : $\underline{\underline{\begin{bmatrix} 0 \\ 2 \end{bmatrix} \parallel L : y = 1}}$)

but not always e.g. $g(v) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} v + \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is <u>not</u> a reflection,

since $g(x,y) = (x,y) \iff (x+1,-y) = (x,y)$, which has no sol^{ns}.



g is a glide <u>reflection</u>, since $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is <u>parallel</u> to the line L of reflection.

<u>Def^n</u> A <u>glide reflection</u> is an isometry which is reflection in a line K, followed by translation by a vector <u>parallel to K</u>. N.B. "glide" before as well.... i.e.

exercise: show $g(v) = R_L\left(v - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$

<u>Remark</u>: ~~Exercise~~ If $g(v) = Av + b$; $A \in O(2)$, $\det A = -1$

then g is either a glide reflection or a reflection.

(hint : A is the matrix of ~~the~~ reflection in the line L (say), through 0, with normal n. (say). Write $b = \text{~~for~~}\ \underbrace{\text{proj}_n b}_{b_\perp} + \underbrace{(b - \text{proj}_n b)}_{b_\parallel}$

If $b_\perp = 0$, g is a reflection (in a line $\parallel$ to L)

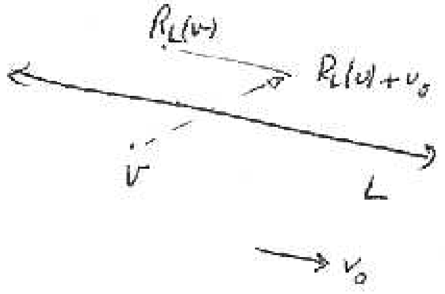
$b_\perp \neq 0$ g is a glide reflection! (line $\parallel$ to L, $b_\perp$ as glide vector.

/11

Def^n Let v_0, n be non-zero, perpendicular in $\mathbb{R}^2$.
$L = \{v \mid v \cdot n = b\}$ a line in $\mathbb{R}^2$ $\parallel$ to v_0
($\perp$ to n)

- The <u>glide reflection</u> associated to L and v_0 is the isometry

$$f(v) = R_L(v) + v_0$$



An isometry g <u>is</u> a glide reflection if g ~~[illegible]~~ is the glide reflection associated to some line L and v_0, $v_0 \parallel L$.

ex.

e.g. $g(v) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} v + \underset{v_0}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$; L is $y=0$

<u>Exercise</u>. If $0 \neq v_0 \parallel L$, $R_L(v) + v_0 = R_L(v+v_0)$, $\forall v \in \mathbb{R}^2$.

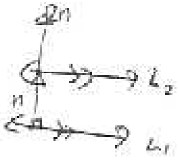
Rmk. A glide refl^n reverses orientation.

<u>Theorem</u> (3 reflection thm). Every isometry of $\mathbb{R}^3$ is ~~either~~ a product of 1, 2 or 3 reflection. <u>Cor</u> Every isometry of $\mathbb{R}^3$ is either a rot^n, transl, refl^n, or a glide refl^n.

Pf - in a minute.

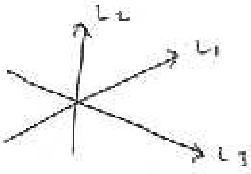
I.  L R_L

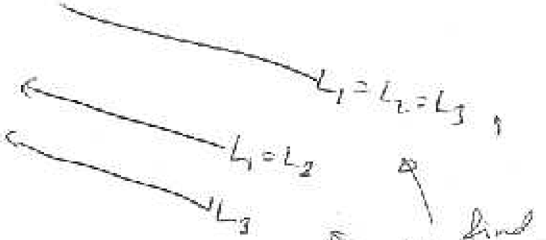
II (i). 

$R_{L_2}R_{L_1}$ rot^n about P by $\overset{\curvearrowleft}{2\alpha}$ (ii) 

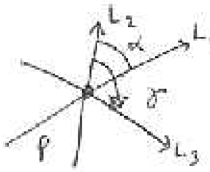
$R_{L_2}R_{L_1}$ is transl^n by $2n$.

III 3 lines. $\exists$ degenerate cases



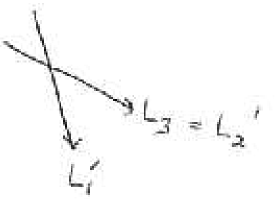


ex. find $R_{L_3}R_{L_2}R_{L_1}$

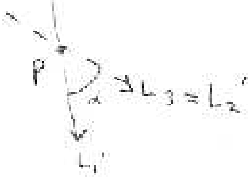


$R_{L_3}R_{L_2}R_{L_1}$

rotate L_1, L_2, about P, leave L_3 fixed, by γ until get



P



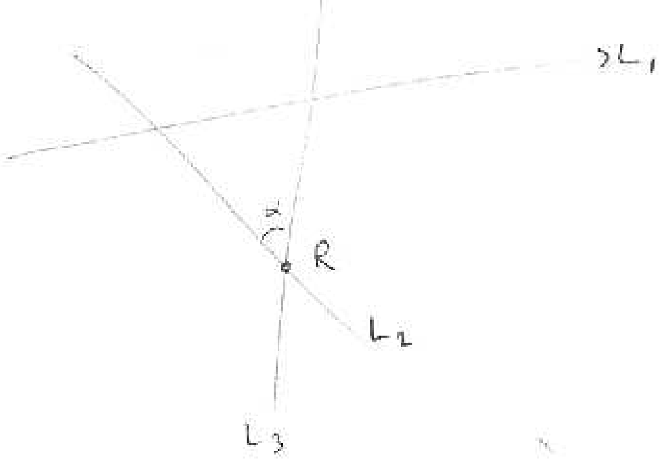
Note: $R_{L_2'}R_{L_1'} = R_{L_2}R_{L_1}$

$$\therefore R_{L_3}R_{L_2}R_{L_1} = R_{L_3}R_{L_2'}R_{L_1'} = R_{L_3}R_{L_3}R_{L_1'} = R_{L_1'}$$

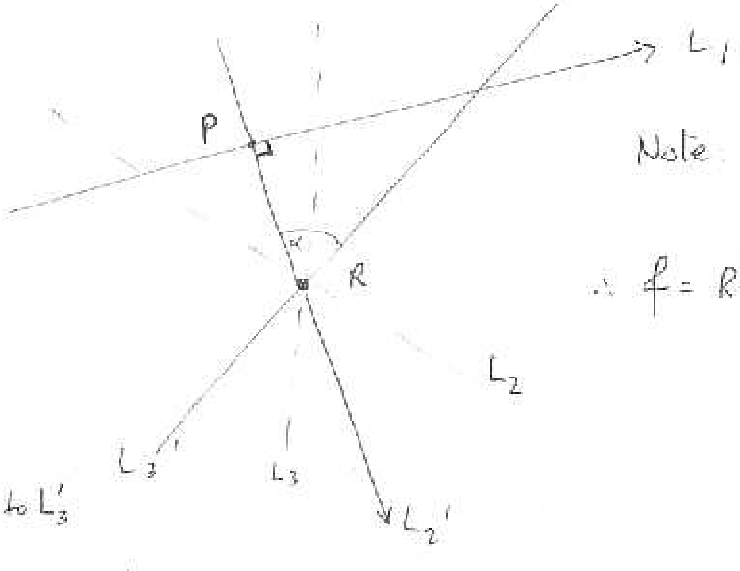
Hence in this case, $R_{L_3}R_{L_2}R_{L_1}$ is a reflection (in L_1').

<u>Generic case</u>

$$f = R_{L_3}R_{L_2}R_{L_1}$$



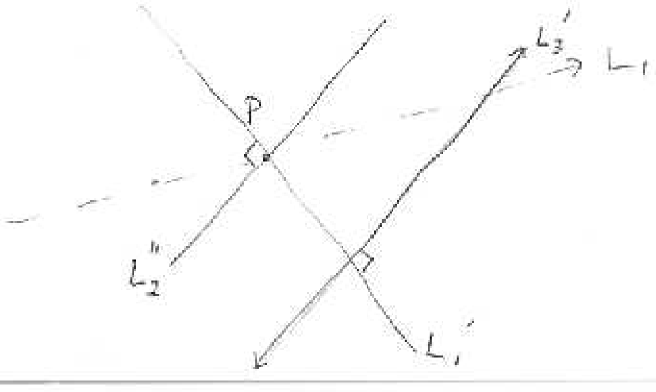
Ⓘ Rotate L_2, L_3 about R by same angle to make $L_2' \perp L_1$.



Note $R_{L_3}R_{L_2} = R_{L_3'}R_{L_2'}$

$$\therefore f = R_{L_3'}R_{L_2'}R_{L_1}$$

Ⓘⓘ Rotate L_1 & L_2 about P by same angle to make L_2'' ∥ to L_3'



Note: $R_{L_2''}R_{L_1'} = R_{L_2'}R_{L_1}$

$$\therefore f = \underbrace{R_{L_3'}R_{L_2''}}_{\substack{\uparrow \\ \text{translation} \\ \text{by a vector} \parallel \\ \text{to } L_1'}} \underset{\substack{\uparrow \\ \text{refl}^n \text{ in } L_1'}}{R_{L_1'}}$$

$\therefore f$ is a glide reflection

Theorem (3 reflections theorem)

Every isometry of $\mathbb{R}^2$ is the product of 1, 2 or 3 reflections.

Pf. First ~~note~~ we show that every isometry of $\mathbb{R}^2$ is uniquely determined by what it does to $0=(0,0)$, $e_1=(1,0)$ & $e_2=(0,1)$ (or indeed, any 3 non-collinear pts): For suppose $f, g : \mathbb{R}^2 \to \mathbb{R}^2$ are isometries with $f(0)=g(0)$, $f(e_1)=g(e_1)$ & $f(e_2)=g(e_2)$.

Recall: $f(v) = Av + f(0)$, $g(v) = Bv + g(0)$ for some $A, B \in O(2)$

Now $f(0)=g(0)$ so

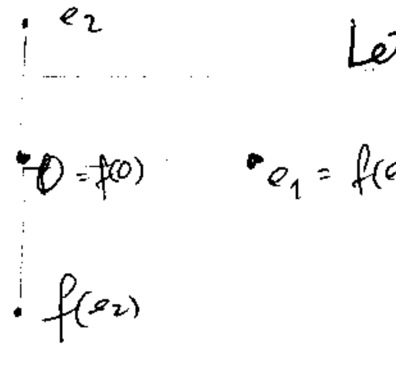
$$\left.\begin{array}{ll} f(e_1)=g(e_1) \Rightarrow & Ae_1 = Be_1 \\ \& \ f(e_2)=g(e_2) \Rightarrow & Ae_2 = Be_2 \end{array}\right\} \quad A = B.$$

Hence $f=g$. Now suppose $f : \mathbb{R}^2 \to \mathbb{R}^2$ is an isometry.

Case I $f(0)=0$, $f(e_1)=e_1$, $f(e_2)=e_2$.

By our remark above, f agrees with id on these 3 pts, and so $f = id$, which is the product $R_L R_L$ for any line L.

Case II $f(0)=0$, $f(e_1)=e_1$, $f(e_2) \neq e_2$.



Let L be the line $L = \{P \in \mathbb{R}^2 \mid \|P-e_2\| = \|P-f(e_2)\|\}$.

Claim: $0 \in L$ and $e_1 \in L$:

Well, $\|0-e_2\| = \|f(0)-f(e_2)\| = \|0-f(e_2)\|$ so $0 \in L$

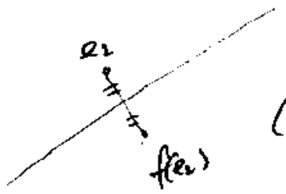
Similarly $\|e_1-e_2\| = \|f(e_1)-f(e_2)\| = \|e_1-f(e_2)\|$ so $e_1 \in L$.

Claim: $f = R_L$. So consider the isometry

$R_L f$: $R_L f(0) = R_L(0) = 0$ (Since $0 \in L$)

R $R_L f(e_1) = R_L(e_1) = e_1$ (Since $e_1 \in L$

$R_L f(e_2) = e_2$ by ass#3, Q4



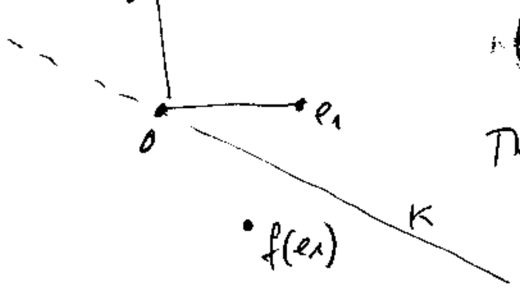
(i.e. R_L interchanges e_2 & $f(e_2)$)

Hence, $R_L f = id$, so $R_L(R_L f) = R_L$

i.e. $f = R_L$.

Hence, in this case f is a single reflection.

Case $\underline{III}$ $f(0) = 0$, $f(e_1) \neq e_1$ $f(e_2) \neq e_2$ ($f(e_2) = e_2$ done in similar manner to Case II)



Let K be the line equidistant fr e_1 & $f(e_1)$
$K = \{P \in \mathbb{R}^2 \mid \|P - e_1\| = \|P - f(e_1)\|$

Then, $\|0 - e_1\| = \|f(0) - f(e_1)\| = \|0 - f(e_1)\|$

so $0 \in K$.

Consider the isometry $R_K f$. Then, $R_K f(0) = R_K(0) = 0$ $(0 \in K)$

& $R_K f(e_1) = e_1$

(refl^n in K interchanges e_1, $f(e_1)$.). Now, either $R_K f(e_2) = e_2$ or not. If $R_K f(e_2) = e_2$, then $R_K f = id$ so $\underline{f = R_K}$.

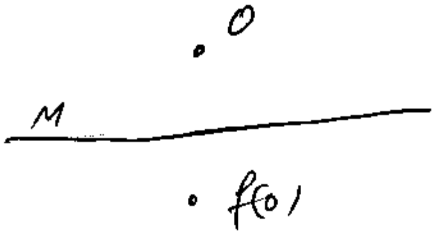
If $R_K f(e_2) \neq e_2$, apply Case II to $g = R_K f$ to conclude that $g = R_K f = R_L$ for some line L (equidist e_2, $R_K f(e_2)$)

Then $\underbrace{R_K R_K}_{id} f = R_K R_L$ so $f = R_K R_L$ is the product of 2 reflection.

Case IV $f(0)$ moves each of $0, e_1$ & e_2.

Let M be the line equidistant from 0 & $f(0)$

$$M = \{ P \in \mathbb{R}^2 / \|P - 0\| = \|P - f(0)\| .$$



Then $R_M f(0) = 0$. (by constr.)

Hence, the isometry $h = R_M f$ fixes the origin. • If h fixes e_1 & e_2, apply Case I to conclude $h = R_M f = id$, so again, $f = R_M$ is a single reflection. • If h fixes one of e_1 & e_2 (but not both) apply case II to conclude $h = R_M f = R_L$ so $f = R_M R_L$ is a product of 2 refl^{ns}.

• If h fixes only 0, apply case III to conclude that $h = R_K$ or $h = R_K R_L$

so $R_M f = R_K$ or $R_M f = R_K R_L$

so $f = R_M R_K$ or $f = R_M R_K R_L$.

Hence, in all possible cases, f is a product of at most 3 reflections!

$\square$

Corollary ~~#~~ Every isom of $\mathbb{R}^2$ is either a transl^n, a rot^n, a refl^n or a glide reflection.

(pf: use 3 lines and argue as on P. 37 of notes)

e.g. $f\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = \underbrace{\begin{bmatrix}0&-1\\-1&0\end{bmatrix}}_{A}\begin{bmatrix}x\\y\end{bmatrix} + \underbrace{\begin{bmatrix}0\\2\end{bmatrix}}_{v_1}$; a) since $A^tA = I_2$, f is an isom

b) $\det A = -1$, so f is orient^n reversing. Hence it is a refl^n or a glide refl^n.

c) which? and what are the details?

METHOD: (idea: f is a refl$^n \Leftrightarrow f(f(v)) = v$ why? f refl$^n \Rightarrow f(f(v)) = v$.

f glide refl$^n \Rightarrow f = R_L(v) + v_0 = R_L(v+v_0)$ ($v_0 \parallel$ to L, $v_0 \neq 0$) so $f(f(v)) =$

$$= R_L(\underbrace{R_L(v) + v_0}_{w}) + v_0$$; $R_L(v) = v + \frac{2(b - a\cdot v)\,a}{\|a\|^2}$

P

$$= R_L(w) + v_0 + v_0$$

$$= R_L(R_L(v)) + 2v_0$$

$= v + 2v_0$; hence if $v_0 \neq 0$, $f(f(v)) \neq v$.)

Compute $f(f(v)) = A(Av + v_1) + v_1 = A^2v + Av_1 + v_1$

$$= v + \begin{bmatrix}0&-1\\-1&0\end{bmatrix}\begin{bmatrix}0\\2\end{bmatrix} + \begin{bmatrix}0\\2\end{bmatrix}$$

$$= v_1 + \begin{bmatrix}-2\\0\end{bmatrix} + \begin{bmatrix}0\\2\end{bmatrix} = v + \begin{bmatrix}-2\\2\end{bmatrix} \neq v!$$

Hence f is a glide refl^n: moreover, the glide vector v_0 is $\frac{1}{2}\begin{bmatrix}-2\\2\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix}$.

So $f(v) = R_L(v) + \begin{bmatrix}-1\\1\end{bmatrix}$ for some line L. What's L?

Well, $g(v) = f(v) - \begin{bmatrix}-1\\1\end{bmatrix} = R_L(v)$ & $L = \{v \in \mathbb{R}^2 \mid g(v) = v\}$!

Compute: $gv = Av + \begin{bmatrix}1\\1\end{bmatrix}$ $\therefore gv = v \Leftrightarrow \left[\begin{array}{cc|c}-1&-1&-1\\-1&-1&-1\end{array}\right] \sim \left[\begin{array}{cc|c}1&1&1\\0&0&0\end{array}\right]$

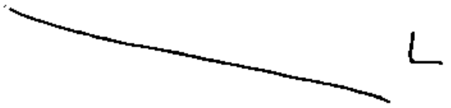
i.e. $L = \{(x,y) \mid x+y = 1\}$!

Hence f is the glide refl^n associated to the line $x+y=1$ and the glide vector $v_0 = \begin{bmatrix}-1\\1\end{bmatrix}$!

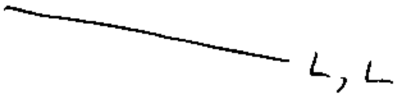
Check Rmk: $\begin{bmatrix}-1\\1\end{bmatrix} \parallel L$!

Remarks

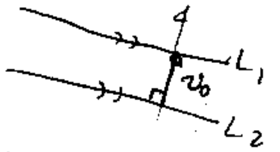
1 reflection



2 reflections



or

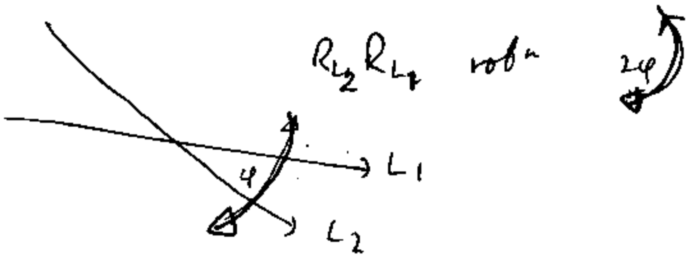


$R_L R_L = id$

$R_{L_2}R_{L_1}$ trans^l $\underset{2v_0}{\perp}$ L_1, L_2

(note: $L_1 \parallel L_2 \Rightarrow$
$R_{L_1}R_{L_2} = -R_{L_2}R_{L_1}$

or

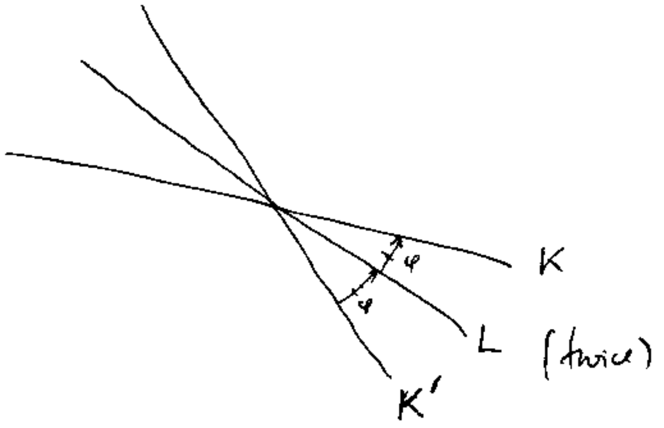


$R_{L_2}R_{L_1}$ rot^n 2φ

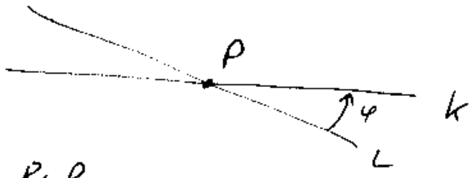
3 reflections $L_1 = L_2 = L_3 = L$; $R_L R_L R_L = R_L$ reflection

$L_1 = L_2 = L \neq K$ $R_L R_L R_K = R_K$ reflect

$L \nparallel K$? $R_L R_K R_L$



note:



$R_L R_K$

$R_{K'}R_L$

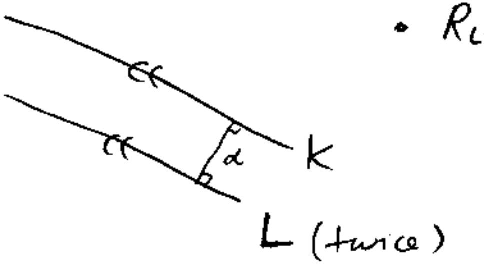
So $R_L R_K R_L = R_{K'} R_L R_L$

$= R_{K'}$!

3 reflections

$L_1 = L_2 = L \neq K$

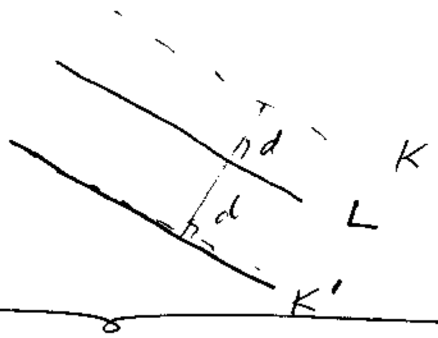
$L \parallel K$



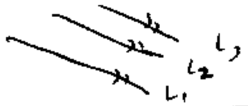
• $R_L R_L R_K = R_K$

• for $\underbrace{R_L R_K} R_L = R_L \underbrace{R_L R_{K'}}$

$= R_{K'}$

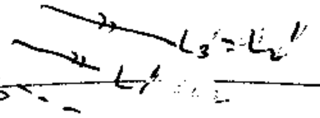


• $R_{L_3} R_{L_2} R_{L_1}$



shift

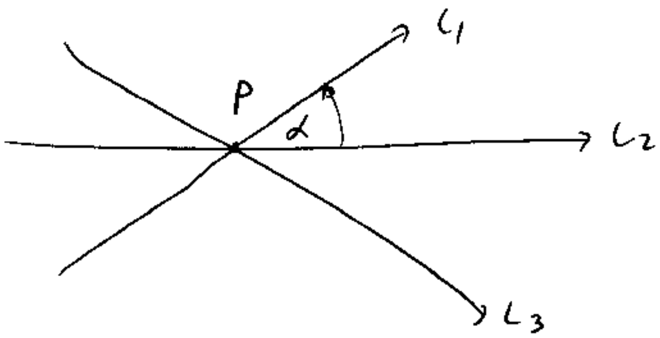
make 2 coincide



$321 = 3'2'1' = 1'$

as above

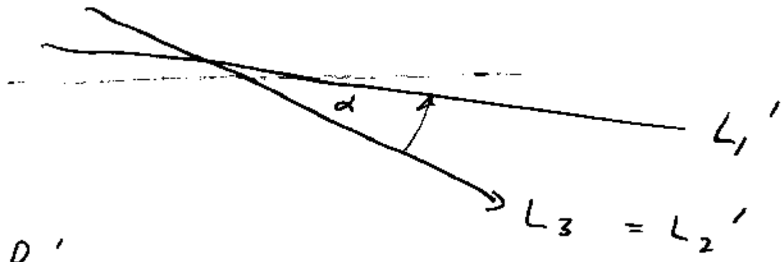
3 reflections



$R_{L_2} R_{L_1} = R_{L_2'} R_{L_1'}$

$R_{L_3} R_{L_2} R_{L_1}$

$=$

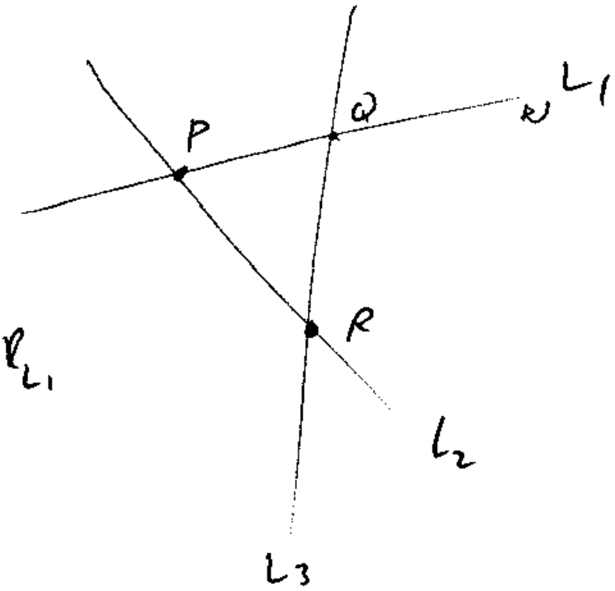


$R_{L_3} R_{L_2'} R_{L_1'}$

$= R_{L_3} R_{L_3} R_{L_1'} = R_{L_1'}$

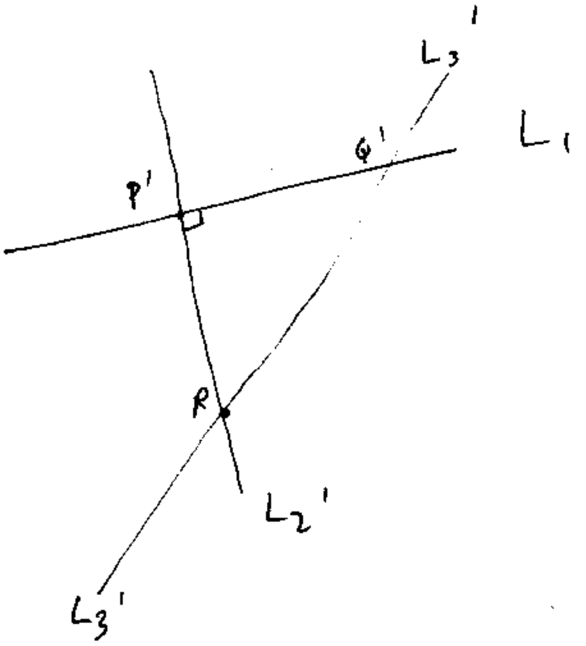
(other orders treated similarly) ex:

Product of 3 reflections in generic case:



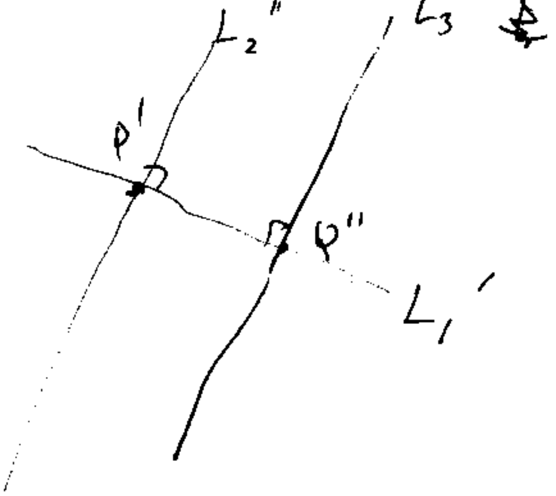
$f = \underbrace{\rho_{L_3}\rho_{L_2}}_{}\rho_{L_1}$

about R to make $L_2' \perp L_1$

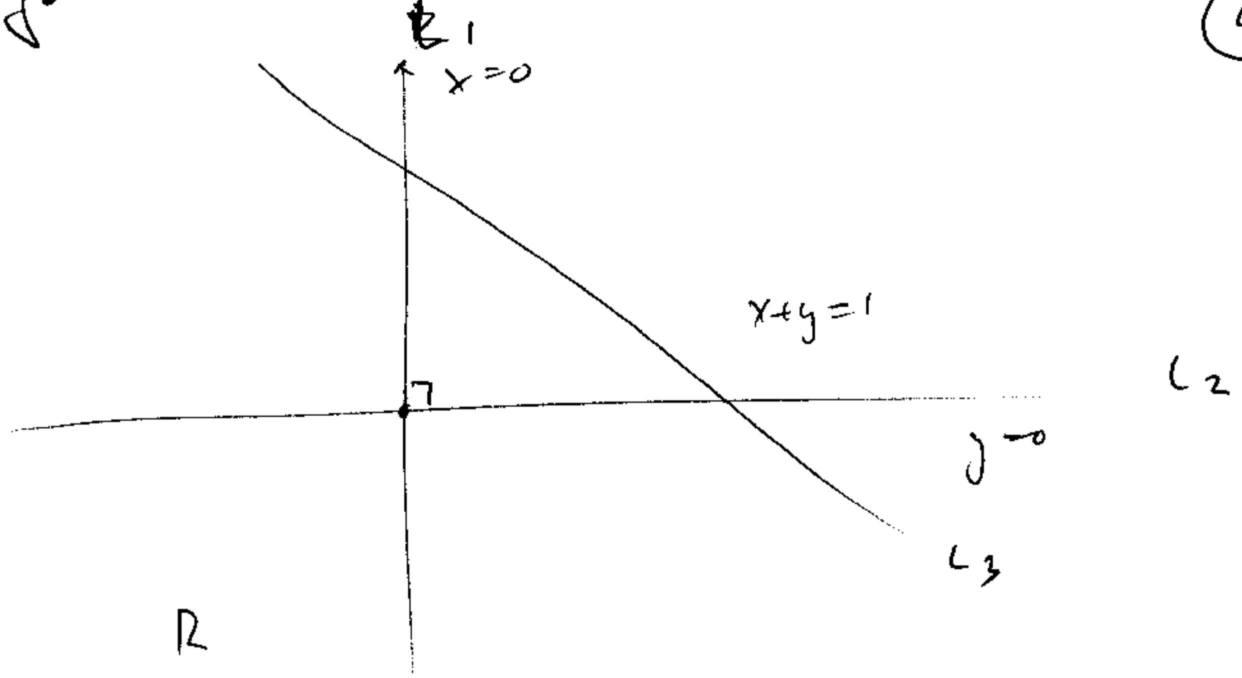


about P'

to make $L_2'' \parallel$ to L_3'

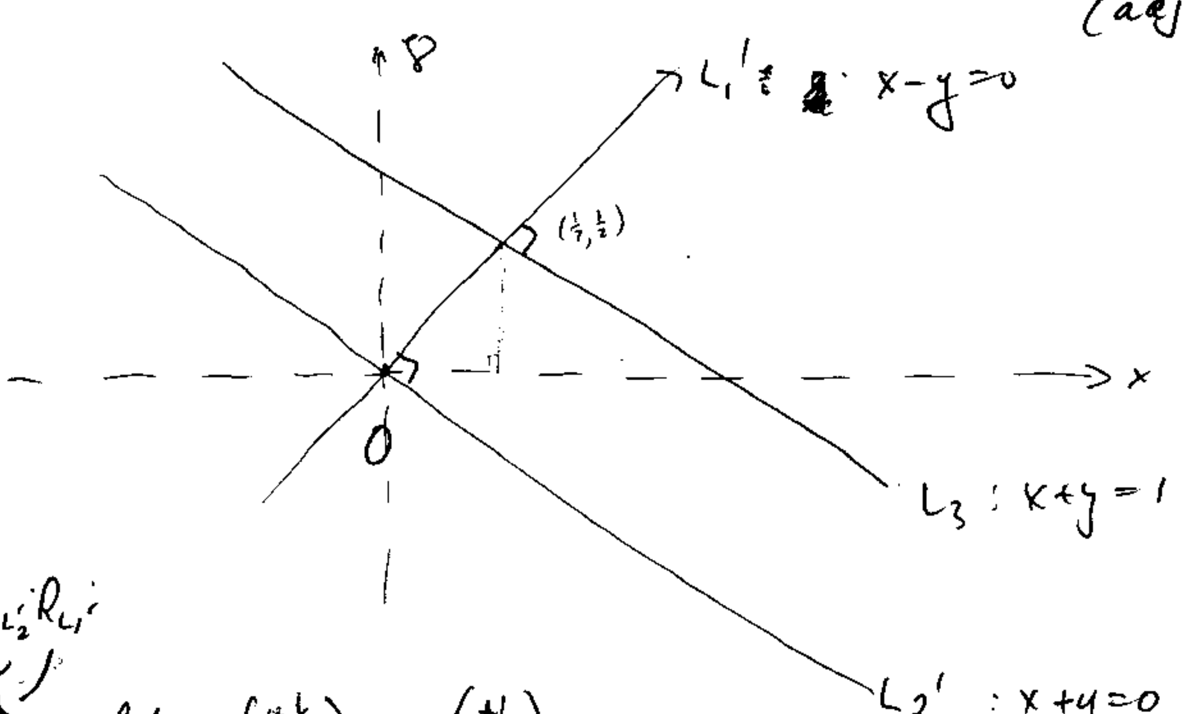


e.g.



$R_{L_3}R_{L_2}R_{L_1}$: note $L_2 \perp L_1$ already!

rotate about origin to make $L_2' \parallel L_3$ (adjacent in formula)



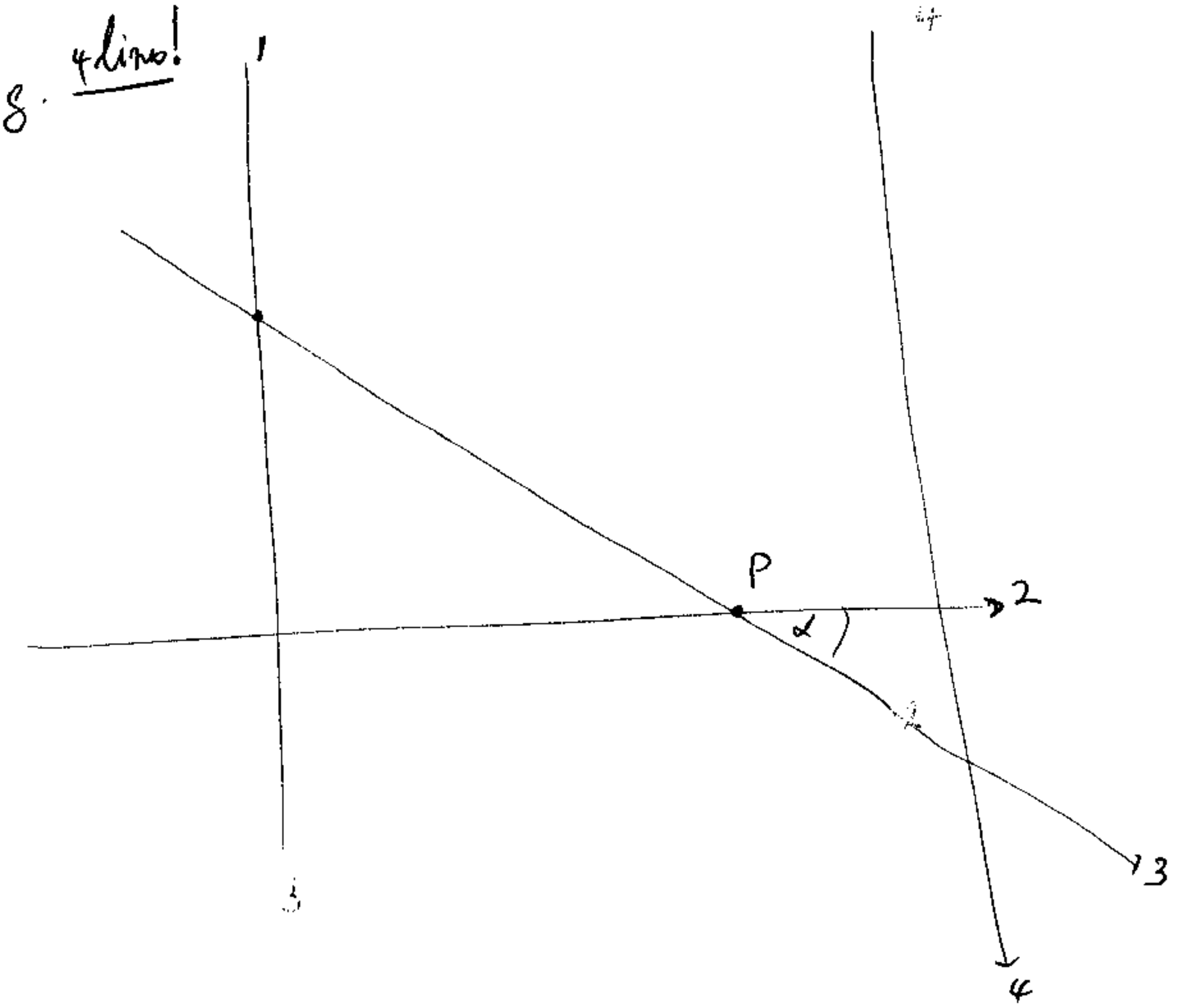
$= \underbrace{R_{L_3}R_{L_2'}}R_{L_1'}$

transl by $2\begin{pmatrix}+1/2\\+1/2\end{pmatrix} = \begin{pmatrix}+1\\+1\end{pmatrix}$

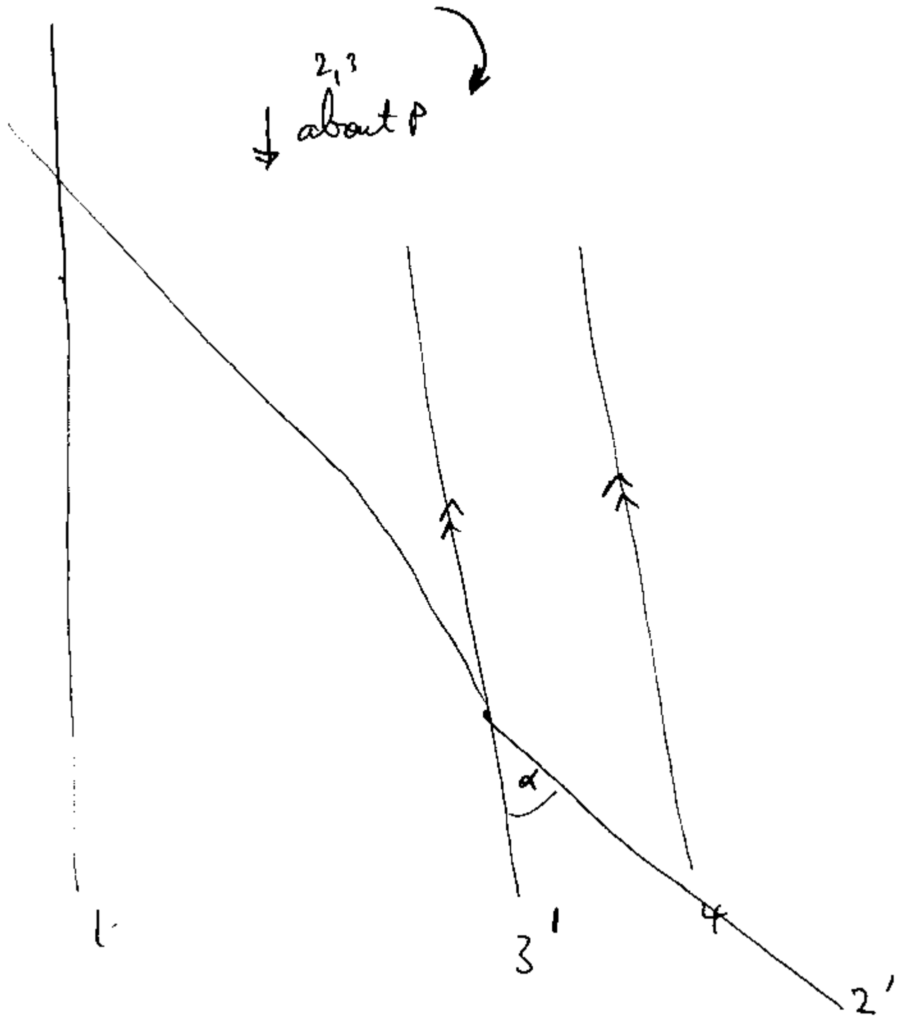
Refl: $y=x$ $\begin{bmatrix}0&1\\1&0\end{bmatrix}$

$\therefore R_{L_3}R_{L_2}R_{L_1}(v) = \begin{bmatrix}0&1\\1&0\end{bmatrix}v + \begin{pmatrix}+1\\+1\end{pmatrix}$ check!

e.g. 4 lines!



$4\ \underbrace{3\ 2}\ 1$



||

$4\ 3'\ 2'\ 1$

move 3', 4 //

So 1, 2', 3'' intersect in a pt.

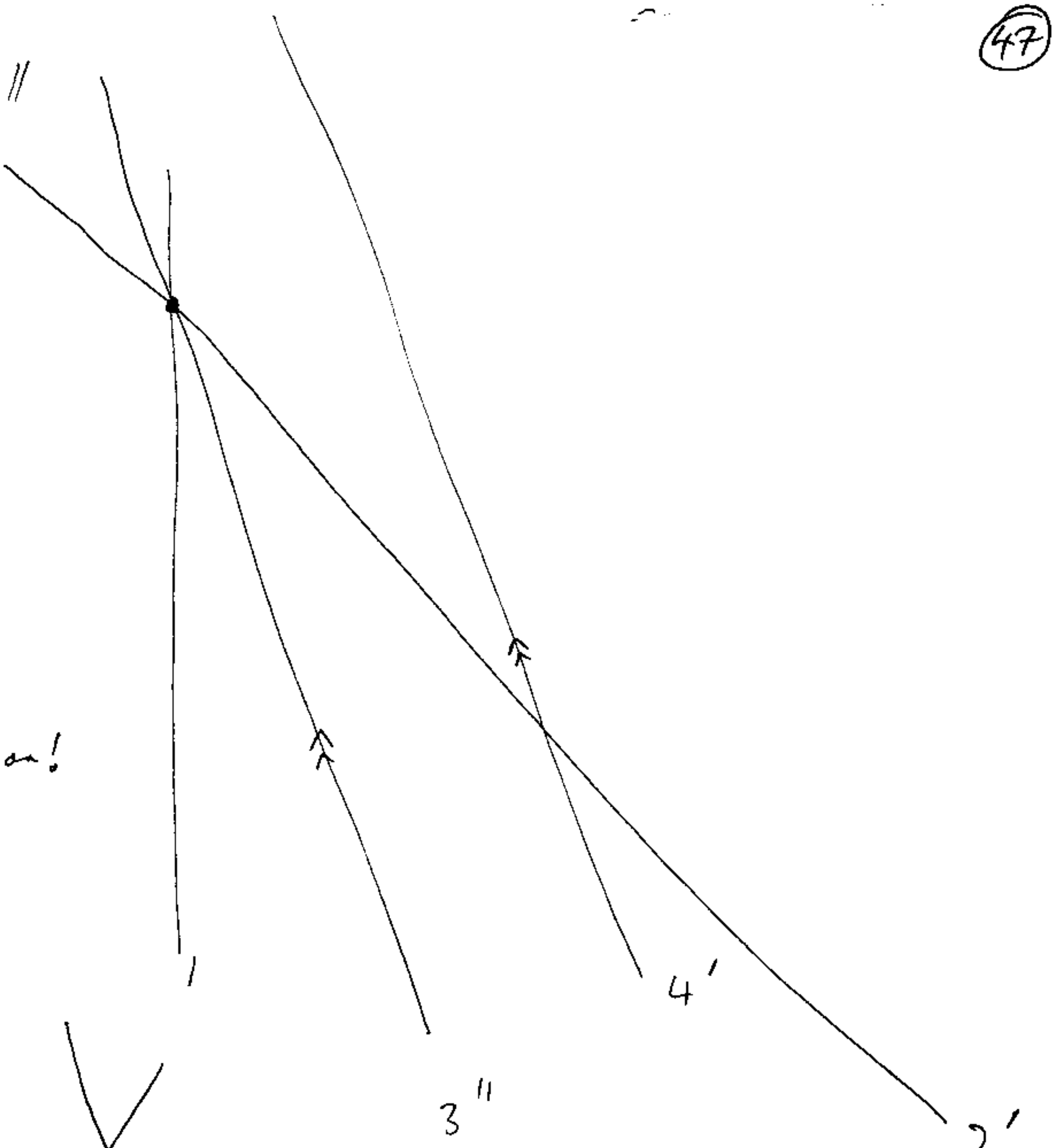
$4\,3'\,2'\,1$

||

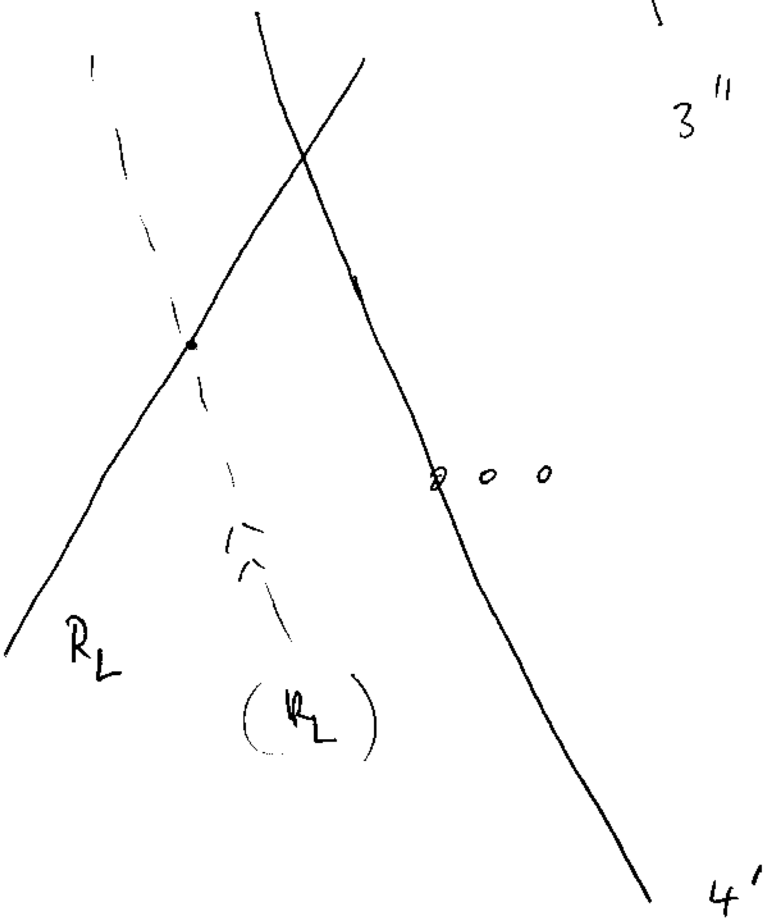
$4'\,\underbrace{3''\,2}\,1$

single reflection!

R_L



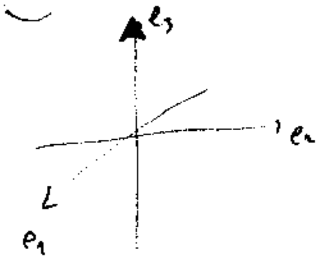
=



transl rot^n.

$\frac{L10}{L11}$ <u>Rot^n in $\mathbb{R}^3$</u> Easy examples $R_{a,\theta}$  RH rule

<u>About z-axis</u>: $a = e_3$; $\rho = R_{e_3,\theta}$. N.B. <u>ρ fixes origin</u>, ρ linear



In x-y plane, $\rho(x,y,0) = \left(\begin{bmatrix}\cos\theta & -\sin\theta \\ \sin\theta & \cos\theta\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix}, 0\right)$

$\rho(0,0,z) = z$

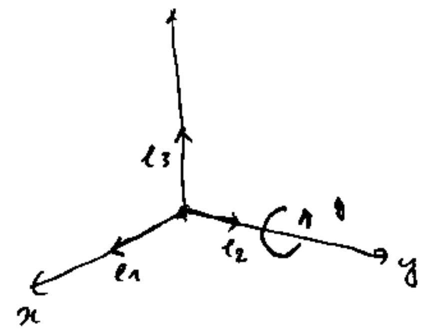
i.e. $\rho e_1 = \cos\theta e_1 + \sin\theta e_2$
$\rho e_2 = -\sin\theta e_1 + \cos\theta e_2$
$\rho e_3 = e_3$

* Note: $\{\underbrace{e_1, e_2}_{\text{plane rot-}}, \overset{\text{axis}}{\overset{\downarrow}{e_3}}\}$ +ve orient^n

$\{e_3, e_1, e_2\}$

$$\therefore \rho(x,y,z) = \begin{bmatrix}\cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}$$

<u>About y-axis</u> $\rho = R_{e_2,\theta}$



Here, to imitate the above,

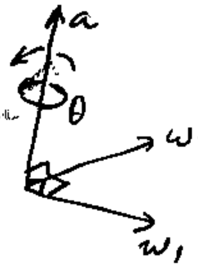
$\{e_3, e_1\}$ is the +ve orient^n (in the "plane of rot^n")

i.e. $\{e_3, e_1, e_2\}$ has +ve orient

So $\rho e_3 = \cos\theta e_3 + \sin\theta e_1$
$\rho e_1 = -\sin\theta e_3 + \cos\theta e_1$
$\rho e_2 = e_2$

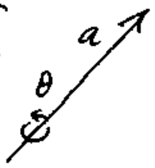
$$\therefore \rho(x,y,z) = \begin{bmatrix}\cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta\end{bmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}$$

<u>IDEA</u>
Given $a \neq 0$



Need <u>orthogonal</u> basis, $\|w_1\| = \|w_2\|$ $\{w_1, w_2\} \perp$ to a so that $\{w_1, w_2, a\}$ is +vely oriented. Then use * on w_1, w_2.

General Rot^n about $\overset{\text{directed line through}}{O}$, dirn $\underline{a}$



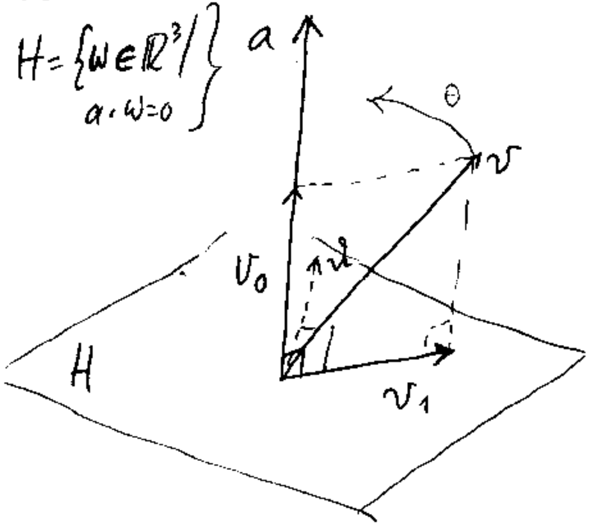
(49)

$R_{a,\theta} = \rho$; an isometry.

- Let $a \neq 0$, $\theta \in \mathbb{R}$. Suppose $\|a\| = 1$ (convenience).

<u>Remarks:</u> Since $\rho(0) = 0$, ρ will be linear

Let
$H = \{w \in \mathbb{R}^3 / a \cdot w = 0\}$



Write $v = v_0 + v_1$

$$= \operatorname{proj}_a v + (v - \operatorname{proj}_a v)$$
$$(\| a) \qquad (\perp a)$$
$$= (v \cdot a) a + (v - (v \cdot a) a)$$

Then $\rho(v) = \rho(v_0 + v_1) = \rho(v_0) + \rho(v_1)$
$$= v_0 + \rho(v_1)$$
$$= (v \cdot a) a + \rho(v_1)$$

- We need an orthogonal basis $\{v_1, v_2\}$ in H with $\|v_1\| = \|v_2\|$. If $v_1 = 0$, then $\rho(v) = v$ and we're done. Otherwise, if $v_1 \neq 0$, let $v_2 = a \times v_1$. Then

① $\|v_2\| = \|a\| \|v_1\| \sin\varphi$, $(\varphi = \frac{\pi}{2})$ $= \|v_1\|$

② $v_1 \perp v_2$;

③ $v_2 \cdot a = 0$, so $v_2 \in H$. Moreover, $\{v_1, v_2, a\}$ is +vely oriented since

$\det [v_1, v_2\ a] = \det [a\ v_1\ v_2]$, $\det [\overset{u}{a}\ \overset{v}{v_1}\ \overset{w}{a \times v_1}]$ $(= (u \times v) \cdot w)$
$= (a \times v_1) \cdot (a \times v_1) = \| a \times v_1\|^2 > 0.$

With respect to the $\overset{\text{ordered}}{\text{basis}}$ $\{v_1, v_2\}$ of H, we have

$$\rho v_1 = \cos\theta v_1 + \sin\theta v_2 = \cos\theta v_1 + \sin\theta (a \times v_1)$$
$$(\rho v_2 = -\sin\theta v_1 + \cos\theta v_2)$$

$$\therefore \rho(v) = (v \cdot a) a + \cos\theta (v - (v \cdot a) a) + \sin\theta (a \times (v - (v \cdot a) a))$$

$a \times a = 0$

$$\underline{\rho(v) = (v\cdot a)(1-\cos\theta)a + (\cos\theta)v + (\sin\theta)(a\times v)}$$

N.B. ($\|a\|=1$)

<u>Rmk.</u> ① If $v \parallel a$ so that $v = (v\cdot a)a$, then $\rho(v) = v$, as we wished.

② ρ is an isometry, since $\|\rho(v)\|^2 = \|\rho(v_0)+\rho(v_1)\|^2$

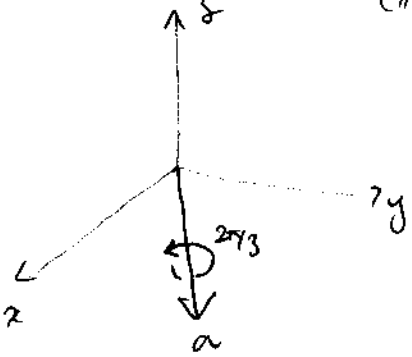
$= \|\rho(v_0)\|^2 + \|\rho(v_1)\|^2$ (because $\rho(v_0) \perp \rho(v_1)$) ex. (use $v_1 \perp v_2$, $\|v_1\| = \|v_2\|$)

$= \|v_0\|^2 + \|v_1\|^2 = \|v_0+v_1\|^2 = \|v\|^2$.

---

e.g. $a = \frac{\sqrt{3}}{3}(1,1,-1)$; $\theta = 2\pi/3$; $\cos\theta = -\frac{1}{2}$, $\sin\theta = \frac{\sqrt{3}}{2}$

($\|a\|=1$)



$v = (x,y,z)$

$v\cdot a = \frac{\sqrt{3}}{3}(x+y-z)$

$(1-\cos\theta) = \frac{3}{2}$

$$(v\cdot a)(1-\cos\theta)a = \frac{\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{3}\cdot\frac{3}{2}(x+y-z)(1,1,-1)$$

$$= \frac{(x+y-z)}{2}(1,1,-1)$$

$$(\cos\theta)v = -\frac{1}{2}(x,y,z)$$

$$(\sin\theta)(a\times v) = \frac{\sqrt{3}}{3}\cdot\frac{\sqrt{3}}{2}\begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & 1 & -1 \\ x & y & z \end{vmatrix} = \frac{1}{2}(y+z, -z-x, y-x)$$

$$\therefore R_{a\theta}(v) = \frac{x+y-z}{2}(1,\ 1,-1)$$

$$- \frac{1}{2}(x, y, z)$$

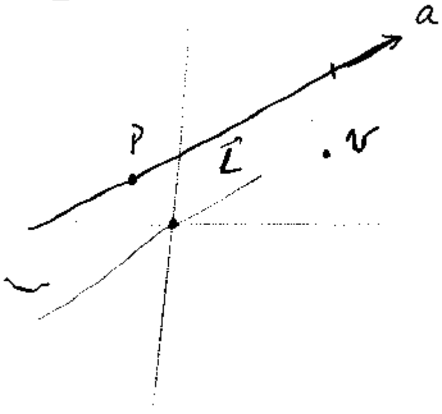
$$+ \frac{1}{2}(z+y,\ -z-x,\ y-x)$$

$$= (y, -z, -x)$$

$$\therefore R_{a\theta}(v) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

([illegible])

(2) Rotation about any (directed line) axis, not necessarily through the origin.



As before: pick $P \in L$

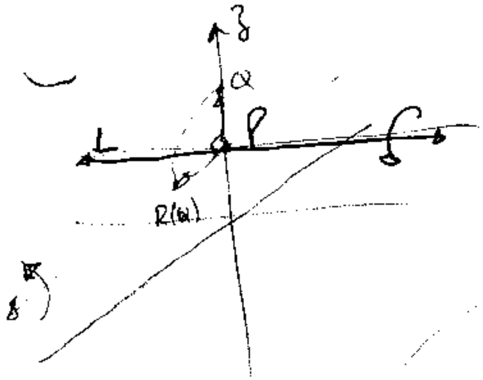
Rotate $v-P$ using the above, and then translate back:

$$R_{L\theta}(v) = \underline{R_{(a,\theta)}(v-P) + P.} = R_{(a,\theta)}(v) - R_{a\theta}(P) + P.$$

[illegible] $)((1-\cos\theta)a$ [illegible] $)$

$+ \sin\theta$ [illegible] $-P))$ $\quad \underline{+\ P}$

eg. $L = \{ e_3 + t e_2 \mid t \in \mathbb{R} \}$, $\theta = \pi/2$; $a = e_2$, $P = e_3$.



$$R_{a,\theta} = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \longrightarrow \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$$ as we saw

$\{e_3, e_1, e_2\}$ +ve or.

$$\begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\therefore R_{L,\theta}(x,y,z) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z-1 \end{bmatrix} + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$R_{a,\theta}(v-P)+P$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} + \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} z-1 \\ y \\ -x+1 \end{bmatrix}$$

$Q = (0,0,2)$ $\quad R_{L,\theta}(Q) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$$\left(\left(1 - \overset{0}{\cos\pi/2}\right)\left((x,y,z)\cdot e_2\right)e_1 + \overset{\cos\frac{\pi}{2}}{0}(v) + \left(e_2 \times (x,y,z)\right) \right)$$

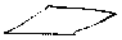
$$= (0,y,0) + (z,0,-x)$$

$$= (z,y,-x).$$

$$= \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

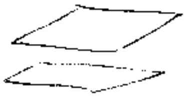
Isometries of $\mathbb{R}^3$ (1–4 reflections)

1 reflection ✓

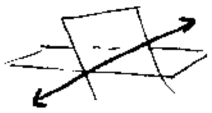


$A \sim \begin{matrix} -1 & & \\ & +1 & \\ & & +1 \end{matrix}$

2 reflections



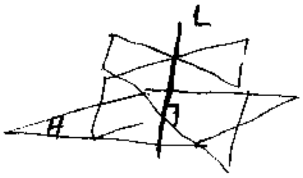
translation



rotation about line of intersection

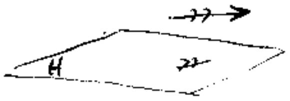
$Iv + b$, $\begin{bmatrix} 1 & & \\ & c & -s \\ & s & c \end{bmatrix}$ $\begin{matrix} & & \\ & e^{i\theta} & \\ & & \bar{e}^{i\theta} \end{matrix}$

3 reflections



rotn + refln H
$L \perp H$
(see later)

$\begin{matrix} -1 & & \\ & e^{i\theta} & \\ & & e^{-i\theta} \end{matrix}$



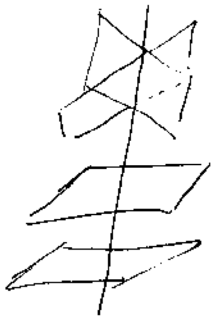
reflh + transl ∥ to H

$\begin{pmatrix} -1 & & \\ & +1 & \\ & & +1 \end{pmatrix} + b \parallel$ to H

L11
L12

\- - - - -

4 reflections



rotn about L
trans ∥ to L

$\begin{pmatrix} 1 & & \\ & e^{i\theta} & \\ & & e^{-i\theta} \end{pmatrix} v + b \parallel L$

$A \in O(3)$ λ evel of A $\Rightarrow |\lambda| = 1$

$\lambda_1 \; \lambda_2 \; \lambda_3$

all real $\begin{matrix} 1 & & \\ & 1 & \\ & & 1 \end{matrix}$, $\begin{matrix} -1 & & \\ & -1 & \\ & & -1 \end{matrix}$ $\begin{matrix} -1 & & \\ & -1 & \\ & & 1 \end{matrix}$ $\begin{matrix} -1 & & \\ & -1 & \\ & & -1 \end{matrix}$

1 real
2 complx conjugats $\begin{matrix} 1 & & \\ & e^{i\theta} & \\ & & e^{-i\theta} \end{matrix}$

High School Geom revisited

<u>Defⁿ</u> 2 subsets $A, B \subset \mathbb{R}^n$ are <u>congruent</u> $(A \cong B)$ if there is an isometry f of $\mathbb{R}^n$ st $f(A) = B$

<u>Rmk</u> $A \cong A$, $A \cong B \Rightarrow B \cong A$, $\underset{f}{A \cong B}$, $\underset{g}{B \cong C} \Rightarrow A \cong C$

$f(A) = B$ $g(B) = C$ $\Rightarrow (g f)(A) = C$.

e.g. <u>Any 2 pts of $\mathbb{R}^n$ are congruent</u> ; If $P = Q$, $f = id$.

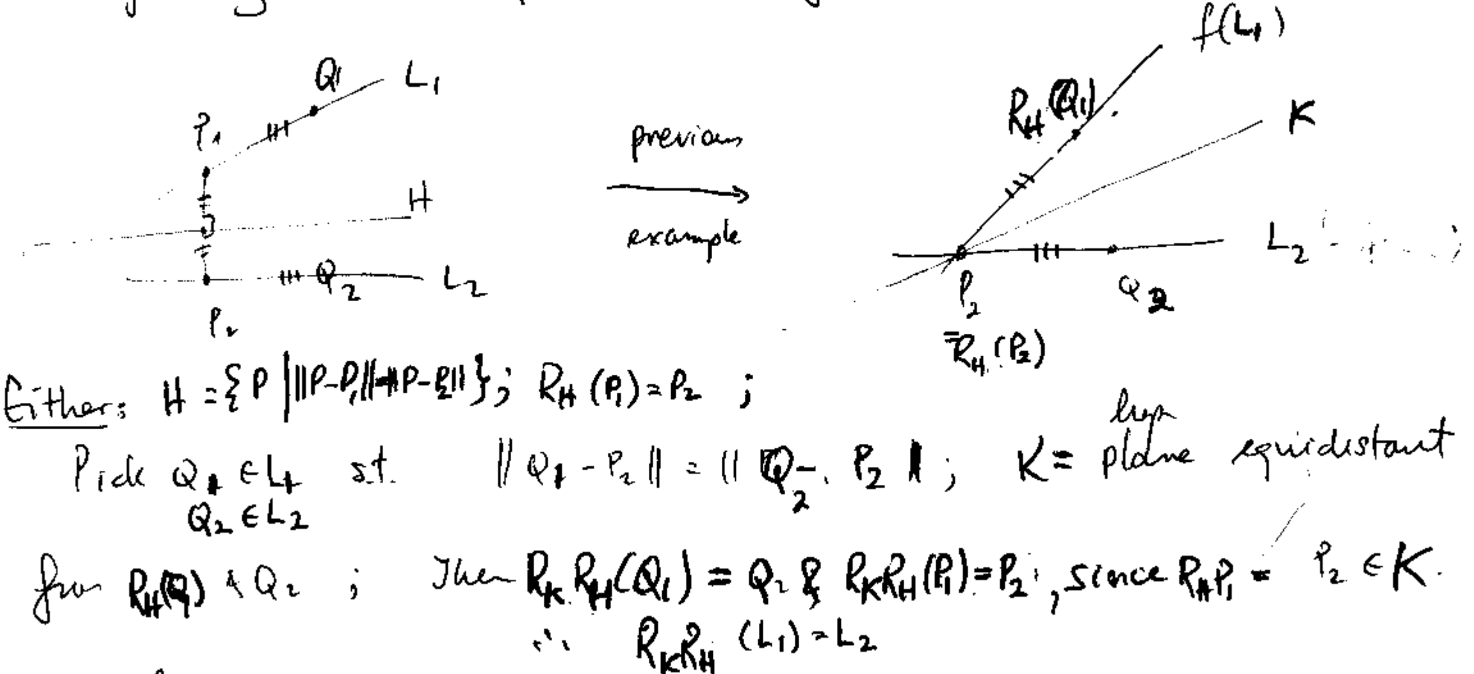
If $P \neq Q$, either $f(v) = v + (Q - P)$

or if $H = \{ v \in \mathbb{R}^n \mid \|v - P\| = \|v - Q\| \}$ then

$R_H(P) = Q$

So $R_H(\{P\}) = R_H(\{Q\})$

e.g. Any 2 lines of $\mathbb{R}^n$ are congruent. (by a rotⁿ!)



<u>Either</u>: $H = \{ P \mid \|P - P_1\| = \|P - P_2\| \}$; $R_H(P_1) = P_2$;

Pick $Q_1 \in L_1$ st. $\|Q_1 - P_2\| = \|Q_2 - P_2\|$; K = $\overset{hyp}{\text{plane}}$ equidistant
$Q_2 \in L_2$

from $R_H(Q)$ & Q_2 ; Then $R_K R_H(Q_1) = Q_2$ & $R_K R_H(P_1) = P_2$, since $R_H P_1 = P_2 \in K$.

$\therefore R_K R_H (L_1) = L_2$

<u>OR</u>: (in $\mathbb{R}^3, \mathbb{R}^2$): use a rotⁿ ... Then $R_H f(L_1) = L_2$.

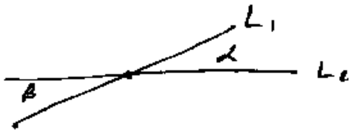
(using fact that if $P, Q \in L_1$ & $g(P), g(Q) \in L_2$

then $g(L_1) = L_2$.)

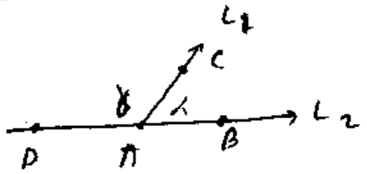
x: Show that any 2 segments of equal (finite) length are congruent

Theorem:

e.g. Opposite angles are equal.



Pf① lemma

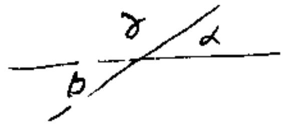


A, B, C, D as shown. Then $\angle BAC + \angle CAD = \pi$.

$\alpha + \beta = \pi$

pf of lemma: ass #2? Millen, exercise

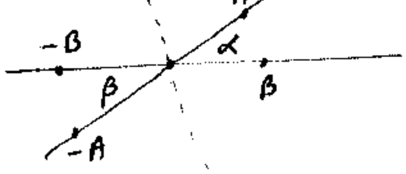
Pf of Thm:



$\alpha + \beta = \pi = \gamma + \beta$ so $\alpha = \gamma$.

Pf② wlog assume L_1, L_2 intersect at origin. (translations take lines to lines, preserve angles)

Pick A, B, −A, −B as shown.



Then $\cos\alpha = \frac{A \cdot B}{\|A\|\|B\|}$

$\cos\beta = \frac{(-A)\cdot(-B)}{\|-A\|\|-B\|} = \cos\alpha$.

Since $0 \le \alpha, \beta \le \pi$, this implies $\alpha = \beta$.

pf③ diagram as in pf②, chosen A, B st. $\|A\| = \|B\|$. Let H be the ~~hyperplane~~ line equidistant from 0 & A, B. Then $R_H(A) = -B$, $R_H B = -A$, $R_H 0 = 0$ so R_H takes α to β. Since R_H is an isometry & isometries preserve angles, $\alpha = \beta$.

pf④ Use a rotation by π about pt of intersection (You must prove it works, though!)

Def

Defn 2 lines in $\mathbb{R}^3$ are $\parallel$ if 1) they lie in the same plane and 2) do not intersect.

- Exercise $L_1 \parallel L_2 \Leftrightarrow L_1 = f(L_2)$, where f is a translation.

(see notes on the web.)

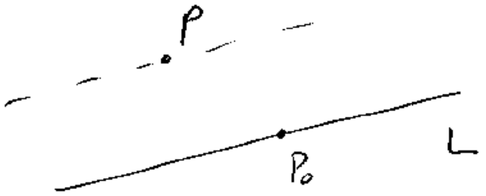
$f(v) = v_0 + (Q - P)$

~~[illegible]~~ for any $Q \in L_2$, $P \in L_1$

15/10/03

Thm Euclid's Parallel Postulate: Given a line L and $P \notin L$, there is a unique line through P, $\parallel$ to L.

Pf.

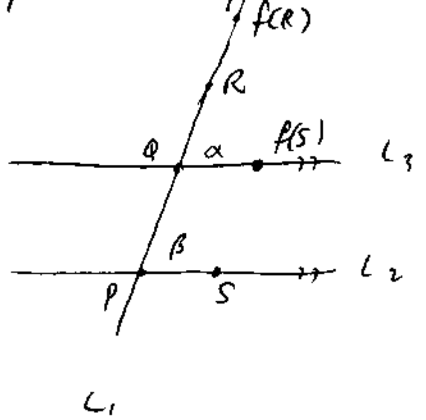


Choose $P_0 \in L$, $v_0 = (-P_0 + P)$.

Then ~~let~~ if $f(v) = v + v_0$ ~~show that~~ $f(L)$ is the desired line.

→ exercise: prove uniqueness! (hint: use previous exercise)

Thm



If $L_2 \parallel L_3$, $\alpha = \beta$

Pf. Let $f(v) = v + (Q-P)$ so $f(L_2) = L_3$, and $f(L_1) = L_1$.

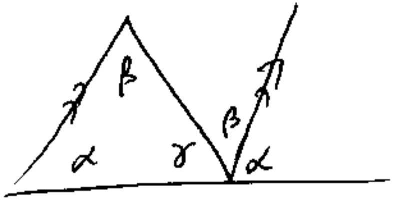
Choose R, S as shown, then $\alpha = \angle f(S) \overset{f(P)}{Q} f(R)$

$= \angle SPR$ (since f is an isometry)

$= \beta$,

Thm 1.8.1 Sum of angle in a Euclidean Δ is π

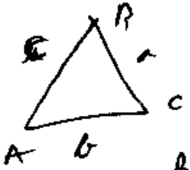
— Pf:



1.8 – 1.9 Congruence thm's for Δ's

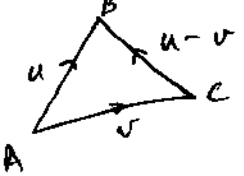
1.8.6 Cosine law:

If



then $a^2 = b^2 + c^2 - 2bc\cos A$

pf:

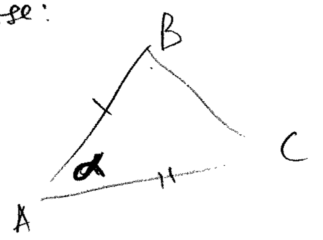


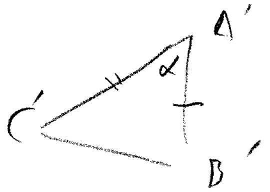
$$a^2 = \|u - v\|^2 = (u - v)\cdot(u - v)$$
$$= \|u\|^2 - 2u\cdot v + \|v\|^2 \; ; \; u\cdot v = \cos A\,\|u\|\,\|v\| = bc\cos A$$
$$= c^2 + b^2 - 2bc\cos A$$

∎

Prop^n 1.9.1 "SAS"

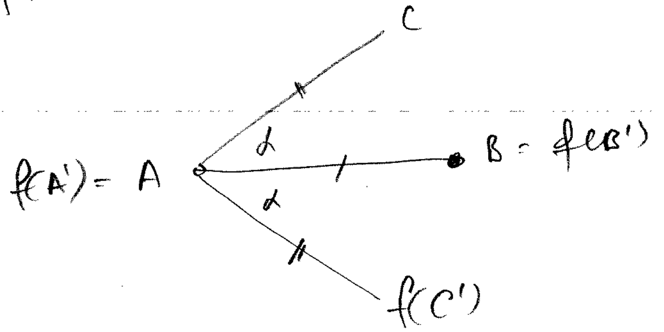
Suppose:





Then $\triangle ABC \cong \triangle A'B'C'$

Pf. Let f be an isometry taking $\overline{A'B'}$ to $\overline{AB}$.



Case (I) $C = f(C')$. Done.

(II) $C \neq f(C')$.

Let $L = \{v \mid \|v - C\| = \|v - f(C')\|.\}$

Then $A \in L$ since $\|A - C\| = \|A' - C'\|$
$= \|f(A') - f(C')\|$
$= \|A - f(C')\|$.

$B = f(B') \in L$: By the cosine rule, $\|B - C\| = \|B' - C'\|$

$\|B - C\| = \|B' - C'\| = \|f(B') - f(C')\| = \|B - f(C')\|$.

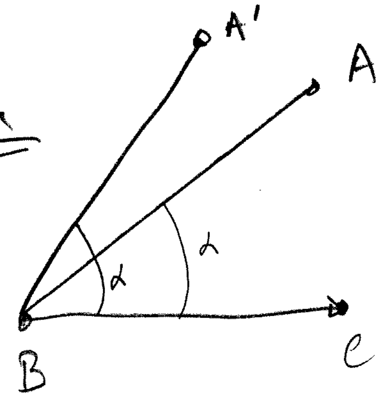
$\therefore$ $R_L(A) = A$, $R_L(B) = B$

and $R_L(f(C')) = C$.

Hence $g = R_L f$ satisfies

$g(A') = R_L f(A') = R_L(A) = A$
$B' = R_L f(B') = R_L(B) = B$
$g(C') = R_L(f(C')) = C$ ∎

Lemma



($\alpha \neq 0$)
($\alpha = 0$: ex.)

and $\|A'-B\| \neq 0 \neq \|A-B\|$

and $\{C-B, A-B\}$
same orient^n as $\{C-B, A'-B\}$.

$\Rightarrow \quad (A-B) = \lambda\,(A'-B)$ for $\lambda > 0$..

Pf. Wlog assume $\overleftrightarrow{BC}$ is the x-axis, $B=0$ and $C=(a,0)$ for $a>0$. Let $A=(x,y)$, $A'=(x',y')$. Then (

$$\frac{x}{\|(x,y)\|} = \frac{(x,y)\cdot(a,0)}{\|(x,y)\|a} = \frac{(x',y')\cdot(a,0)}{\|(x',y')\|a} = \frac{x'}{\|(x',y')\|}$$

So x, x' have same sign.

Moreover $\frac{x^2}{x^2+y^2} = \frac{x'^2}{x'^2+y'^2} \Rightarrow x'^2y^2 = x^2y'^2$

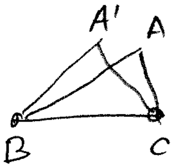
Now $x \neq 0 \Leftrightarrow x' \neq 0$, and if $x=x'=0$, then $(0,y) = \frac{y}{y'}(0,y')$

If $x, x' \neq 0$, $\frac{y^2}{x^2} = \frac{y'^2}{x'^2}$ so $y = \pm\frac{x}{x'}\cdot y'$. But

$\begin{vmatrix} a & x \\ 0 & y \end{vmatrix}$ and $\begin{vmatrix} a & x' \\ 0 & y' \end{vmatrix}$ have same sign so y, y' have same sign. $\therefore$ $y = \frac{x}{x'}\cdot y'$. i.e. $(x,y) = \frac{x}{x'}(x',y')$. $\boxtimes$

ex. Use this to prove "ASA"

get to:



So $A' \in \overleftrightarrow{AB}$
$A \in \overleftrightarrow{AB}$ ...

15/10 FEB

(60-61)

Similarity

Def^n ① $f : \mathbb{R}^n \to \mathbb{R}^n$ is a <u>similarity (transformation)</u> (with constant k)
if there is $k > 0$ s.t. $\forall P, Q \in \mathbb{R}^n$

$$\| f(P) - f(Q) \| = k \, \| P - Q \|.$$

② Two subsets $A, B \subseteq \mathbb{R}^n$ are <u>similar</u> if $\exists$ similarity f st $f(A) = B$. We write $A \sim B$.

ex. ① $A \sim A$, $A \sim B \Rightarrow B \sim A$, $A \sim B$ & $B \sim C \Rightarrow A \sim C$

<u>Thm</u> A map f is a similarity w constant k iff $\exists A \in O(n)$ and $b \in \mathbb{R}^n$ s.t. $f(v) = kAv + b$, $\forall v \in \mathbb{R}^n$.

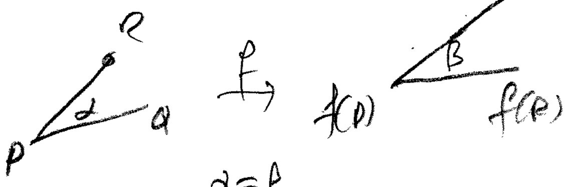
Pf (outline) Simply Note that $g : \mathbb{R}^n \to \mathbb{R}^n$ def'd by $g(v) = \frac{1}{k} \cdot f(v)$ will be an isometry of $\mathbb{R}^n$. ⊠

---

ex. ① If f, g are similarities (i) so $f \circ g$;
(ii) f is invertible & f^{-1} is a similarity.

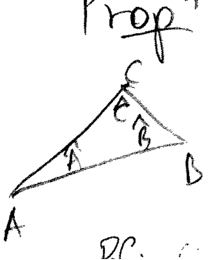
② If f is a similarity then $f(\overline{AB}) = \overline{f(A) f(B)}$

③ Similarities preserve <u>angles</u>.



$\underline{\alpha = \beta}$

Prop^n Suppose 2 triangles $\triangle ABC$, $\triangle A'B'C'$ satisfy



$\hat{A} = \hat{A'}$, $\hat{B} = \hat{B'}$, $\hat{C} = \hat{C'}$. Then

$$\triangle ABC \sim \triangle A'B'C'!$$

Pf - exercise. Hint: Let $k = \frac{\|A'-B'\|}{\|A-B\|}$, $g(v) := kv$ and then show (using ASA) that $\triangle\, g(A)\, g(B)\, g(C) \cong \triangle A'B'C'$.

Corollary If 2 $\triangle$s have equal angles, the lengths of their corresponding sides have a common ratio.

(Pf - ex.)

12 3 06
13 4

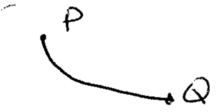
Spherical Geometry

#

● **Def^n** $S^2 = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 = 1\}$; $S^2_R = \{v \in \mathbb{R}^3 \mid \|v\| = R\}$ $R>0$
$= \{v \in \mathbb{R}^3 \mid \|v\| = 1\}$

Geometry on S^2 : points ✓
("2-dim'l") lines ?
isometries ?

"Def^n" If $P, Q \in S^2$, then the (Euclidean) distance $d(P,Q)$ between P and Q is the length of the shortest curve on S^2 between P and Q.
(Latter is called a geodesic on S^2).
"str." lines

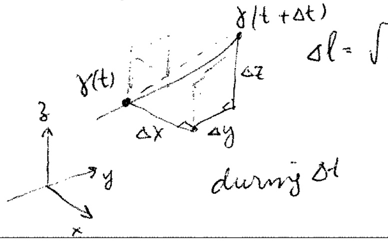


What are they?

Recall: The length of a curve γ in $\mathbb{R}^3$: suppose $\gamma(t) = (x(t), y(t), z(t))$,

● 

$$l(\gamma) : \int_0^1 \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}(t)^2}\,dt = \int_0^1 \|\dot{\gamma}(t)\|\,dt \quad \dot{x} = \frac{dx}{dt}, \text{etc}$$



$\Delta l = \sqrt{\Delta x^2 + \Delta y^2 + \Delta z^2}$; $\Delta x \approx \dot{x}(t)\Delta t$ $\therefore$ $\Delta l = \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)}\ \Delta t$;

---

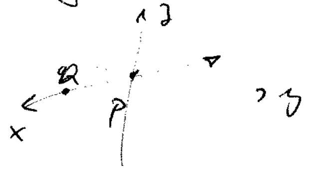
14 07
15

exercise Suppose $f: \mathbb{R}^3 \to \mathbb{R}^3$ is a Euclidean isometry, and $\gamma: [0,1) \to \mathbb{R}^3$ is a smooth curve. Show that $l(f \circ \gamma) = l(\gamma)$ (pblm # 24 on the list)

i.e. if $\gamma(t)$ $(x(t), y(t), z(t))$, where $f \circ \gamma(t) := f(x(t), y(t), z(t))$

1st step: let's reassure ourselves that Euclidean lines are curves of shortest length between 2 pts on them. (shortest distance between 2 point is obtained using a straight line) ∴ Euclidean geodesics (defined analogously) are ordinary (lines), line segments:

So Let $P, Q \in \mathbb{R}^3$ and (do as exercise) wlog assume $P=(0,0,0)$, $Q=(q,0,0)$ for some $q\in\mathbb{R}$, $q>0$



and let $\overset{=(x,y,z)(t)}{\gamma}$ be any curve from P to Q. Then

$$\ell(\gamma) = \int_0^1 \sqrt{\dot x^2+\dot y^2+\dot z}\; dt \;\geq\; \int_0^1 \sqrt{\dot x^2}\,dt = \int_0^1 |\dot x|\,dt \;\geq\; \int_0^1 \dot x\, dt = x(1)-x(0) = q-0=q,$$

which is the length of the segment $\overline{PQ}$!

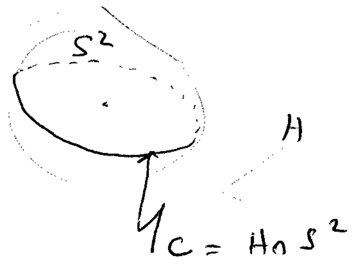
2nd step: candidates for geodesics = str. lines on S^2:

Def^n A great circle on S^2 is a subset of the form

$$C = \{ v \in S^2 \mid a\cdot v = 0\} \quad \text{for some } a\in\mathbb{R}^3,\ a\neq 0.$$

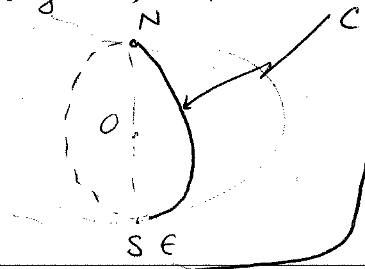
i.e. $C = S^2 \cap H$ for some plane $H \ni 0$.

is the intersection of S^2 with some plane through the origin.



e.g. (plane $z=0$) $\cap S^2$ = equator

e.g. plane through O, (0,0,1) (=N, north pole) $\cap S^2$ is a "line of longitude" (defined later)



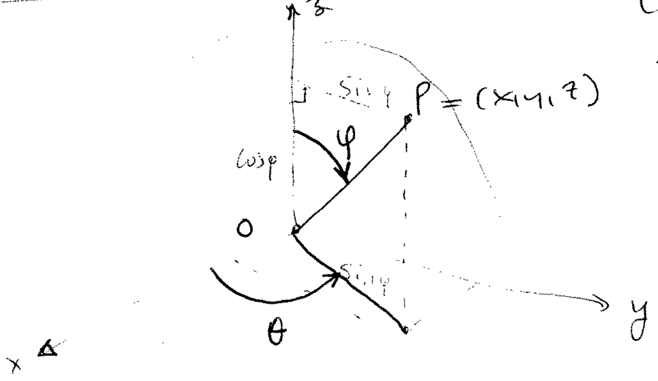
e.g.

There is (at least one) great circle containing any 2 pts $P\neq Q$ in S^2. Suppose $Q\neq -P$. Then Let $H = \{ v\in\mathbb{R}^3 \mid (P\times Q)\cdot v = 0\}$, and then $H\cap S^2$ contains P, Q. (If $Q=-P$, choose any plane K containing $\overline{PQ}$.)

3rd step How to identify great circles e.g. lines of longitude,

Use spherical polar coordinates: (θ,φ) def'd as follows: $0\leq\theta<2\pi$, $0\leq\varphi\leq\pi$.

05 L14/L15



Can see that

$x = \cos\theta\sin\varphi$

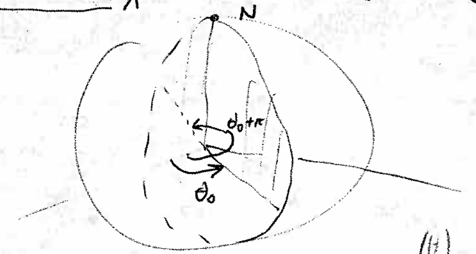
$y = \sin\theta\sin\varphi$

$z = \cos\varphi$

All OK except at $\pm(0,0,1)$, where θ is not defined...

Let $\theta_0 \in [0,\pi]$. $\theta_L = \theta_0$ " $L_{\theta_0} = \{P \mid \theta(P)=\theta_0 \text{ or } \theta(P)=\theta_0+\pi, \text{ or } P = \pm N\}$.

A <u>line of longitude</u>, is curve on S^2 with $\theta(P)=\theta_0$ or $\theta(P)=\theta_0+\pi$, for all $P \in L$:

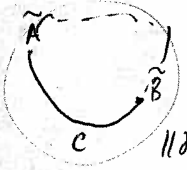


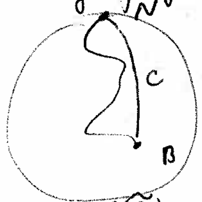
<u>exercise</u>(ass #4 5?) A line of longitude (θ_0) IS a great circle. consequence

<u>exercise</u> (ass #4 5?) A Euclidean isometry fixing O takes a great circle to another great circle

13/4 06 14/4/15 <u>Theorem</u> (2.2.1 in suggested text 1) <u>A geodesic on S^2 is part of a great circle</u>

Pf. Let $\tilde{A}, \tilde{B} \in S^2$ and $\tilde{\gamma}$ any curve $\tilde{\gamma}:[0,1) \to S^2$ with $\tilde{\gamma}(0)=\tilde{A}$, $\tilde{\gamma}(1)=\tilde{B}$.
We establish the theorem by showing that $\ell(\tilde{\gamma}) \geq$ the length of the short part of the great circle $\tilde{C}$ containing $\tilde{A}$ and $\tilde{B}$. To simplify matters, choose a Euclidean isometry f of $\mathbb{R}^3$ s.t. $f(0)=0$, $f(\tilde{A}) = N := (0,0,1)$; and let $B := f(\tilde{B})$.





Define $\gamma = f \circ \tilde{\gamma}$. Since f preserves lengths, for $t \in [0,1]$
$\|\gamma(t)\| = \|f(\tilde{\gamma}(t))\| = \|\tilde{\gamma}(t)\| = 1$, so $\gamma(t)$ is a curve on S^2 from N to B. Moreover, previous exercises show that $\ell(\tilde{\gamma}) = \ell(\gamma)$, and $f(\tilde{C}) = C$ (a line of long.!) is also a great circle, so we establish our desired conclusion if we can show that $\ell(\gamma) \geq$ the length of the part of the line of longitude from N to $B = (\cos\theta_0 \sin\varphi_0, \sin\theta_0 \sin\varphi_0, \cos\varphi_0)$.

So write $\gamma(t) = (x(t), y(t), z(t)) = (\cos\theta(t)\sin\varphi(t), \sin\theta(t)\sin\varphi(t), \cos\varphi(t))$,
and let $\cdot \equiv \frac{d}{dt}$; Then $\ell(\gamma) = \int_0^1 \|\dot{\gamma}(t)\| \, dt = \int_0^1 \sqrt{\dot{x}^2(t) + \dot{y}^2(t) + \dot{z}^2(t)} \, dt$

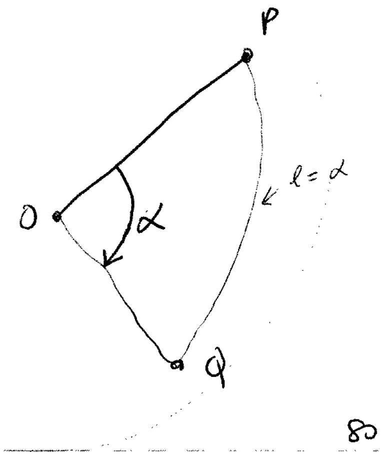
$$\overset{\text{see week 1}}{=} \underset{\text{for details}}{\cdots} = \int_0^1 \sqrt{\dot{\theta}^2(t)\sin^2\varphi(t) + \dot{\varphi}^2(t)} \, dt \geq \int_0^1 \sqrt{\dot{\varphi}^2(t)} \, dt \geq \int_0^1 \dot{\varphi}(t) \, dt = \varphi(1) - \varphi(0) = \varphi_0 - 0 = \varphi_0.$$

Now, set $\sigma(t) = (\cos\theta_0 \sin(\varphi_0 t), \sin\theta_0 \sin(\varphi_0 t), \cos(\varphi_0 t))$. Then $\sigma(0) = N$ and $\sigma(1) = B$, and <u>σ is part of the line of longitude $\theta = \theta_0$!</u> We finish by showing $\ell(\sigma) = \varphi_0$!
Well, $\ell(\sigma) = \int_0^1 \sqrt{\dot{\varphi}^2(t) + \dot{\theta}^2(t)\sin^2\varphi(t)} \, dt = \int_0^1 \varphi_0 \, dt = \varphi_0$, (Since $\frac{d}{dt}(\theta_0) = 0$, $\frac{d}{dt}(\varphi_0 t) = \varphi_0$.)

Hence the curve $\sigma(t)$ attains the shortest length, and $\sigma(t)$ is part of a great circle $\boxed{\equiv}$

N.B. 15/10^{07}

We showed that the spherical distance $d(P,Q)$ between 2 pts (in this case, and so in general)



$d(P,Q) = \alpha$, the angle subtended $= \angle POQ$ $(0 \leq \ \leq \pi)$ at the origin by P, Q

i.e. $\cos\alpha = P \cdot Q$ (since $\underset{=1}{\|P\|} = \|Q\|$)

so $\alpha = \arccos(P \cdot Q)$

$d(P,Q) = \arccos(P \cdot Q)$. Remark: this is always longer than the Euclidean distance between P & Q

We can now see great circles (which are the geodesics! on S^2, by our calculations above: any great circle can be taken to a line of longitude by a suitable Euclidean isometry) as sets of pts equidistant between 2 pts $P \neq Q$ on S^2. Let $P \neq Q$ be points on S^2. Then

$$\begin{aligned} C &= \{ v \in S^2 \mid d(P,v) = d(Q,v) \} \\ &= \{ v \in S^2 \mid \arccos(P \cdot v) = \arccos(Q \cdot v) \} \\ &= \{ v \in S^2 \mid P \cdot v = Q \cdot v \} \\ &= \{ v \in S^2 \mid (P-Q) \cdot v = 0 \} \\ &= \{ \underbrace{v \in \mathbb{R}^3 \mid (P-Q) \cdot v = 0}_{\text{plane through } 0 \text{ w normal } P-Q} \} \cap S^2 \end{aligned}$$

apply cos, which is a bijection $[0,\pi] \to [-1,1]$

i.e. C is a great circle.

Exercise Let $C = H \cap S^2$ be any great circle on S^2. Find $P, Q \in S^2$ s.t. $C = \{ v \in S^2 \mid d(P,v) = d(Q,v) \}$.

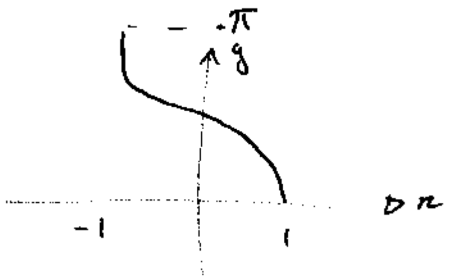
Isometries of S^2

Def^n $f: S^2 \to S^2$ is an isometry if

$$d(f(P), f(Q)) = d(P,Q), \quad \forall P,Q \in S^2$$

i.e. $\arccos(f(P)\cdot f(Q)) = \arccos(P\cdot Q)$

note: $\arccos : [-1,1] \to [0,\pi]$



is a 1-1 function i.e.

$$\arccos x = \arccos y \iff x = y$$

∴ Theorem $f: S^2 \to S^1$ is an isometry of $S^2 \iff$

$$f(P)\cdot f(Q) = P\cdot Q, \quad \forall P,Q \in S^2.$$

Note: (Recall!) isometries of $\mathbb{R}^3$ fixing 0 preserve dot products!

Theorem

Suppose $g: \mathbb{R}^3 \to \mathbb{R}^3$ is an isometry of $\mathbb{R}^3$ s/t $g(0)=0$. Then

1) $g(S^2) \subseteq S^2$

2) g restricts to an isometry of S^2

pf. We know 2) already, if 1) is true! So suppose $P \in S^2$

then $1 = P\cdot P \underset{\substack{\uparrow \\ g \text{ isom } \mathbb{R}^3}}{=} g(P)\cdot g(P) = \|g(P)\|^2 \quad \therefore \quad g(P) \in S^2.$ $\boxtimes$

Hence every isometry of $\mathbb{R}^3$ fixing the origin is isom of S^2

Conversely, As Suppose $f: S^2 \to S^2$ is an isometry of S^2. Define $\tilde{f}: \mathbb{R}^3 \to \mathbb{R}^3$ by $\tilde{f}(x) = \begin{cases} 0 & \text{if } x = 0 \\ \|x\| f(\frac{x}{\|x\|}), & x \neq 0 \end{cases}$

<u>Theorem</u>: If $f: S^2 \to S^2$ is an isometry of S^2, then $\tilde{f}: \mathbb{R}^3 \to \mathbb{R}^3$ is a Euclidean isometry!

pf. Let $x, y \in \mathbb{R}^3$ $\neq 0$. Then

$$\tilde{f}(x) \cdot \tilde{f}(y) = \|x\| f\left(\frac{x}{\|x\|}\right) \cdot \|y\| f\left(\frac{y}{\|y\|}\right)$$

$$= \|x\| \cdot \|y\| \cdot \left(f\left(\tfrac{x}{\|x\|}\right) \cdot f\left(\tfrac{y}{\|y\|}\right) \right)$$

$$= \|x\| \|y\| \left(\frac{x}{\|x\|} \cdot \frac{y}{\|y\|} \right) \quad \text{(since } f \text{ is an isom of } S^2\text{)}$$

$$= x \cdot y.$$

Note: this is true if either of $x, y = 0$ too.

Hence, $\tilde{f}$ preserves inner products, and $\tilde{f}(0) = 0$. Thus, in particular, $\|\tilde{f}(v)\|^2 = \tilde{f}(v) \cdot \tilde{f}(v) = v \cdot v = \|v\|^2$.

Hence, $\|\tilde{f}(x) - \tilde{f}(y)\|^2 = (\tilde{f}(x) - \tilde{f}(y)) \cdot (\tilde{f}(x) - \tilde{f}(y))$

$$= \|\tilde{f}(x)\|^2 - 2\, \tilde{f}(x) \cdot \tilde{f}(y) + \|\tilde{f}(y)\|^2$$

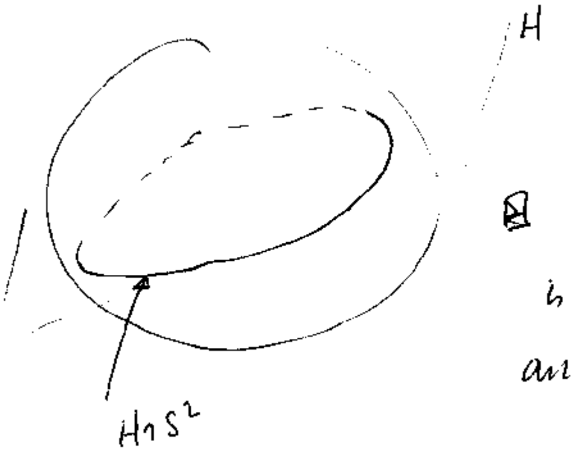
$$\therefore \|\tilde{f}(x) - \tilde{f}(y)\|^2 = \|x\|^2 - 2 x \cdot y + \|y\|^2 = \|x - y\|^2$$

i.e. $\tilde{f}$ preserves Euclidean distances! $\tilde{f}$ is an isometry of $\mathbb{R}^3$. $\boxed{\Xi}$

Hence, any isometry $f: S^2 \to S^2$ can be thought of as the restriction to S^2 of an isometry of $\mathbb{R}^3$ which preserves the origin!

4/11/02 So, we have rotations of S^2 (about, lines, or, thinking of $L \cap S^2 = \{P, -P\}$, about 'points') $R_{P,\theta} v = \cdots$

reflections of S^2 (refl^n in a ~~hyperplane~~, or, thinking intrinsically on S^2, a reflection in $H \cap S^2$ – a "great circle" "lines" of S^2



But, no translations!

☒ Hence, every isometry of S^2 is of the form $f(x) = Ax$ with $A \in O(3)$, and if $A \in O(3)$, then $f(x) = Ax$ gives an isometry of S^2.

Remark: Just as in $\mathbb{R}^3$, not every isometry of S^2 is a rotation or a reflection: $a(x) = -x$ is an isometry which is not a reflection. (It's a (orientation reversing) rotation by $\overset{\curvearrowleft}{\pi}$ in in x-y plan (ie about $N = (0,0,1)$) followed by a reflection in the x-y plane (ie. reflection in great circle x-y plane $\cap S^2$)

Theorem In fact, every isometry of S^2 is either

1) a reflection (in a great circle) (great circle $\{v \in | Av = v\}$ is "plane" ... a 0^2

2) a rotation about $(P_0, -P)$ or ($\{v \in | Av = v\}$ has dim $\underline{1}$, dim 3

3) a rotation about $P_0 \in S^2$, followed by a reflection $\{v | Av = v\} = 0$ or ϕ. in V $H \cap S^2 = \{v \in S^2 / v \cdot P_0 = 0\}$ great circle $\perp P_0$ ie (or equivalently, reflection in $H \cap S^2$, followed by a rotation about P_0)

Pf. Suppose $g: S^2 \to S^2$ is an isometry and let $f: \mathbb{R}^3 \to \mathbb{R}^3$ be the Euclidean isometry extending it. So, $f(0)=0$.

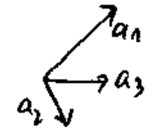
Every isometry of $\mathbb{R}^3$ that fixes a point is the product of 1, 2, or 3 reflections in planes (exactly same kind of argument as for isometries of $\mathbb{R}^2$ shows this - ex.) <u>If f fixes 0, 0 is in each plane.</u>

So further (a) A single reflection

(b) A product of 2 reflections in H_1 & H_2, which is a rotn about $H_1 \cap H_2$ by twice the acute angle between H_1 & H_2

(c) a product of 3 reflections: Let $H_1 = \{v \in \mathbb{R}^3 \mid a_1 \cdot v = 0\}$, $H_2 = \{v \mid a_2 \cdot v = 0\}$ & $H_3 = \{v \mid a_3 \cdot v = 0\}$ be the hyperplanes. Let R_i denote R_{H_i}

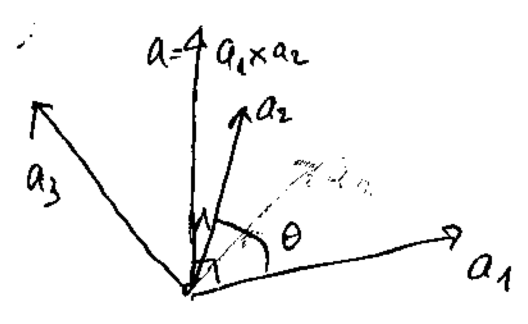
<u>Case I</u> $\{a_1, a_2, a_3\}$ is linearly dependent



(e.g. collinear, coplanar)

<u>Exercise</u>: Show that $R_3 R_2 R_1$ is then a (single) reflection.

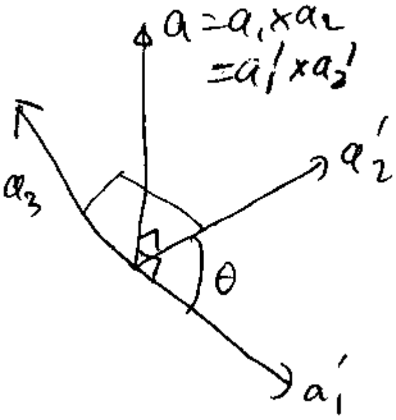
<u>Case II</u> $\{a_1, a_2, a_3\}$ is l.i. Let $a = a_1 \times a_2$, so $L = \text{span}\{a\} = H_1 \cap H_2$. Now $R_2 R_1$ is rotn about a by 2θ;



rotate a_1 & a_2 about L keeping angle between them fixed, until $a_2 \perp a_3$. (This is clearly possible)

when $\text{proj}_{\text{span}\{a_1, a_3\}} a_3$ is $\parallel$ to a_2', angle is $0 < \theta < \pi$, minimum $< \pi/2$, max

So have

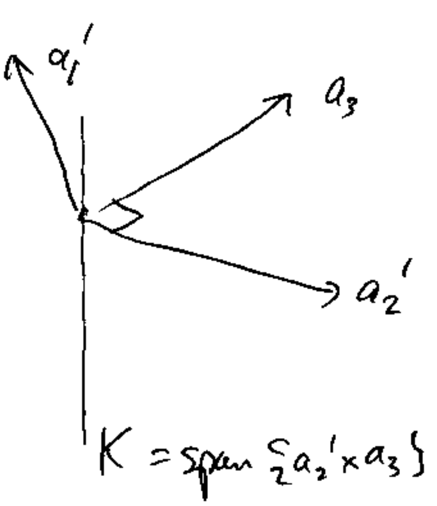


Moreover,

$R_2R_1 = R_2'R_1'$

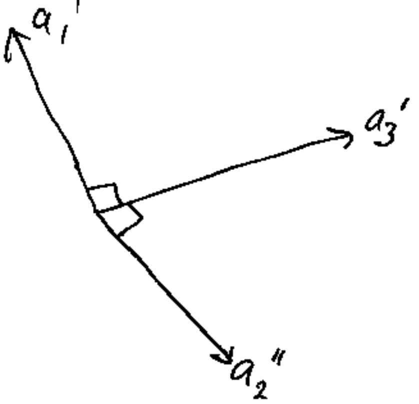
so $R_3R_2R_1 = R_3R_2'R_1'$

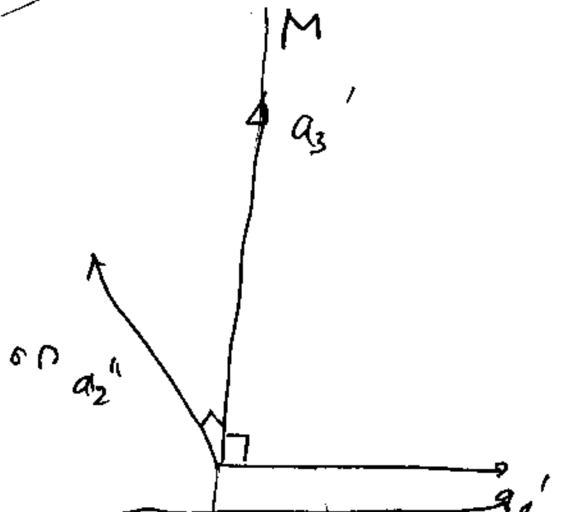
Now consider



Again, rotate a_2', a_3 about $K = \text{span}\{a_2' \times a_3\}$ (keeping $\pi/2$ between them) until $a_1' \perp a_3'$

Then have



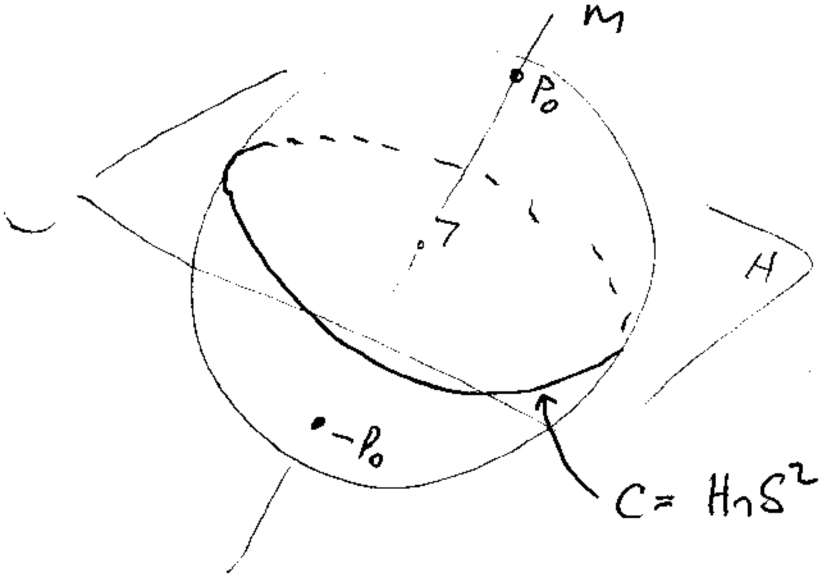


Then $a_3' = \pm a_1' \times a_2'$

and $R_3R_2R_1 = R_3R_2'R_1' = \underline{R_3'}\,\underline{R_2''R_1'}$.

note: $a_1' \times a_2'' \parallel a_3'$

But then $R_3R_2R_1 = R_3'\rho$ is a rotation about a line M followed by reflection in a plane $\perp$ to M.



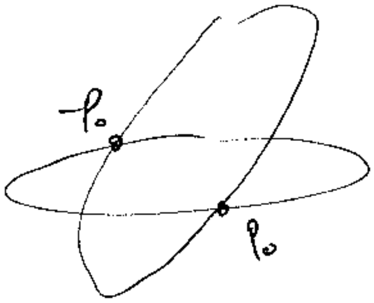
i.e. (in terms of S^2) rotn about P_0 followed by reflection in a great circle ⊥ to P_0.

∎

<u>Corollary</u> If $C_1 \neq C_2$ are great circles and R_{C_1} & R_{C_2} denote reflection of S^2 in them, then $R_{C_2}R_{C_1}$ is a rotation about $P_0 \in S^2$ where $\{P_0, -P_0\} = C_1 \cap C_2$.

(by twice acute angle between C_1 & C_2 i.e. normals to them).

(ex ↑.)



∎

see Comms q on ASS 4

smple:

e.g. Suppose $g: S^2 \to S^2$ is defined by

$$g(v) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix}}_{A} v .$$

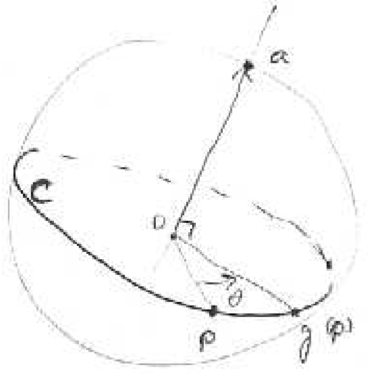
(1) g is an isometry of S^2 because ~~$g(0)=0$ and~~ $A \in O(3)$ (cols. are o.n. basis of $\mathbb{R}^3$; or, $AA^t = I_3$ ← check)

(2) $\det A = -1$, so g is not a rotn.
Hence, g is a reflection or a $\overset{\text{spin}}{}$ reflection.
To see which, we consider the fixed point set of g:
$\text{Fixed}(g) = \{v \in S^2 \mid g(v) = v\}$: $Av = v$; $(A-I)v = 0$ $\left[\begin{array}{ccc|c} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$ so $Av = v$ $\iff v = 0$

$\therefore$ $\text{Fixed}g = \emptyset$ (on S^2). Hence g is a spin reflection.
So it is reflection in a great circle C, followed by a rotation about $a \in S^2$ by θ, followed by reflection in $C = S^2 \cap \{v \mid a \cdot v = 0\}$.

To find a, θ, we proceed as follows. It is clear that g will satisfy $g(a) = -a$, so we solve this eqn: $(A+I)v = 0$ $\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ -1 & 0 & -1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

$\therefore v = (s, -s, s)$, $s \in \mathbb{R}$. Since $\|a\| = 1$, $a = \frac{\sqrt{3}}{3}(1,-1,1)$.



To find θ, choose $p \in C$, e.g. $\frac{\sqrt{2}}{2}(1,1,0) = p$.

Then $g(p) \in C$ (check it) so $\cos\theta = p \cdot g(p)$ (since $\|p\| = \|g(p)\| =$

$$= \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \left(\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right)$$

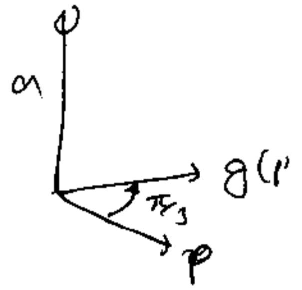
$$= \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{1}{2} \quad \therefore \theta = \pi/3 \text{ or } \underset{(-\pi/3)}{5\pi/3}$$

To see which it is, we consider the orientation of $\{p, g(p), a\}$:

$$\det[p \ g(p) \ a] = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{3}}{3} \begin{vmatrix} 1 & 1 & 1 \\ 1 & 0 & -1 \\ 0 & -1 & 1 \end{vmatrix} = \frac{\sqrt{3}}{6} \begin{vmatrix} 1 & 1 & 1 \\ 0 & -1 & -2 \\ 0 & -1 & 1 \end{vmatrix} = \frac{\sqrt{3}}{6} \begin{vmatrix} -1 & -2 \\ -1 & 1 \end{vmatrix} = \frac{\sqrt{3}}{6}(-1-$$

< 0

$\therefore$ the angle of rotn is $-\pi/3$ or $5\pi/3$. Picture is (



which it is, is determined by the orientation of $\{p, g(p), a\}$: if +ve, $\pi/3$
-ve $-\pi/3$

$$\begin{vmatrix} p \\ g(p) \\ a \end{vmatrix} = \frac{1}{2\sqrt{3}} \begin{vmatrix} 1 & 1 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 1 \end{vmatrix} = \frac{1}{2\sqrt{3}} \begin{vmatrix} 1 & 0 & 0 \\ 1 & -1 & -1 \\ 0 & -2 & 1 \end{vmatrix}$$

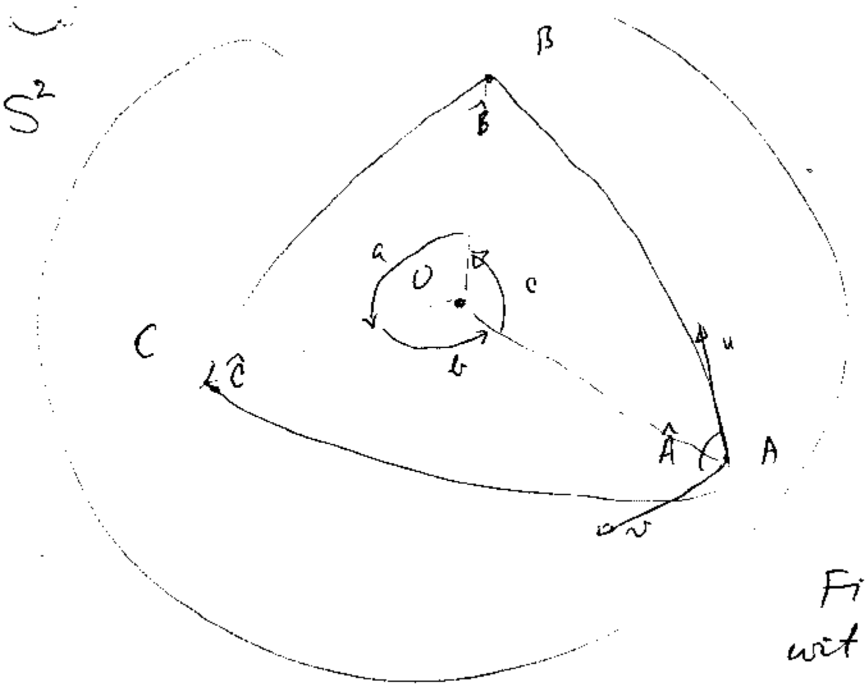
$$= \frac{1}{2\sqrt{3}} \begin{vmatrix} -1 & -1 \\ -2 & 1 \end{vmatrix}$$

$$= \frac{1}{2\sqrt{3}} (-1+2) < 0$$

$\therefore$ the angle of rotn about the oriented line through 0 (orientn a) is $-\pi/3$.

or
($5\pi/3$)

2.3 Spherical Triangles (Goal: Thm 2.3.1)



What is a spherical $\triangle ABC$?

It is the region enclosed by geodesics between A & B, B & C, C & A (and since there are 2, in each case, pick the smaller one*...)

~~* we'll talk briefly about~~

There are formulae for these $\triangle$'s just as in the Euclidean case, though they are different formulae.

First: there are 6 angles associated with a spherical $\triangle$, as illustrated:

$\hat{A}$ is the angle between u & v, tangents to great circles at A which are the sides of the spherical $\triangle$.

Vertex angles are $\hat{A}, \hat{B}, \hat{C}$

arc angles are a, b, c

$$\begin{array}{ccc} a & b & c \\ \| & \| & \| \\ d_{S^2}(B,C) & d_{S}(A,C) & d_{S^2}(A,B) \end{array}$$

Lemma ~~2.3.1~~ In a spherical $\triangle ABC$,

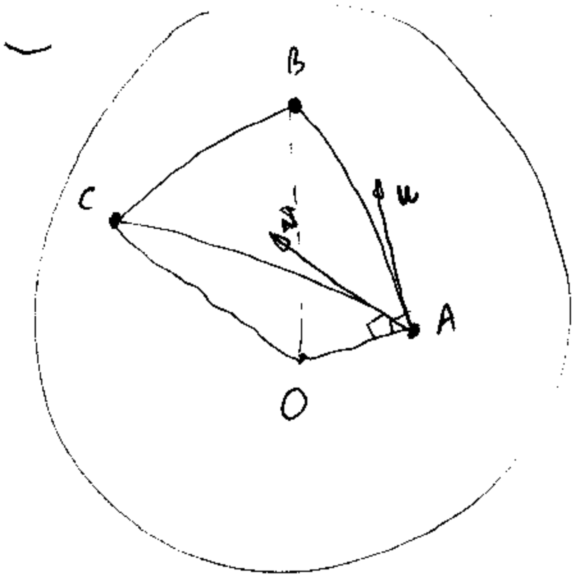
1. $\measuredangle a$ = the angle between B and C $= \angle BOC$
$\measuredangle b$ = " A & C $= \angle COA$
$\measuredangle c$ = " A + B $= \angle AOB$

& 2.
$$\cos \hat{A} = (A \times B) \cdot (A \times C) / \|A \times B\| \, \|A \times C\| = \frac{(A\times B)\cdot(A \times C)}{\sin c \, \sin b}$$
$$\cos \hat{B} = (B \times C) \cdot (B \times A) / \|B \times C\| \, \|B \times A\| = \frac{(B\times C)\cdot(B \times A)}{\sin a \, \sin c}$$
$$\cos \hat{C} = (C \times A) \cdot (C \times B)$$

etc.

Pf 1) is just the defn. 2) Let's do $\cos \hat{A}$

Note: $\cos \hat{A} = \dfrac{u \cdot v}{\|u\| \|v\|}$, where



u is tangent to gt circle from A to B at A $\overrightarrow{\widehat{AB}}$

v " geodesic from A to ~~C~~ at A.

Note: $u \cdot A = 0$ (tangent to S^2 at A) $u \in \operatorname{span}\{A, B\}$,

since gt circle from A to B is $\operatorname{span}\{A,B\} \cap S^2$.

Hence $u = \lambda A + \mu B$. But $u \cdot A = 0 \Rightarrow$

$0 = \lambda + \mu (A \cdot B)$ (since $A \cdot A = \|A\|^2 = 1$)

So $u = \mu_1 (B - (A \cdot B) A)$; Since u points towards B from A, then $u \cdot B \geq 0$

i.e. $\mu_1 (\underbrace{1 - (A \cdot B)^2}_{\geq 0}) \geq 0$, so $\mu_1 > 0$.

Similarly, $v = \mu_2 (C - (A \cdot C) A)$ with $\mu_2 > 0$. ($v \cdot A = 0$, $v \in \operatorname{span}\{A, C\}$)

Consider the rotation ρ on S^2 about A by $\frac{\pi}{2}$ (This is same as rotn about $\overrightarrow{A}$) This has formula $\rho(w) = (w \cdot A) A + A \times w$.

Hence, $\rho(u) = A \times u = \mu_1 (A \times B)$

$\rho(v) = A \times v = \mu_2 (A \times C)$.

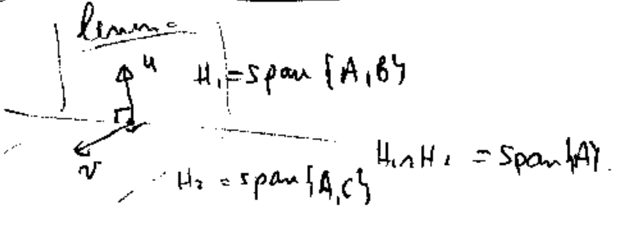
Now, ~~this is an~~ isometry preserving the origin, so

$$\cos \hat{A} = \frac{u \cdot v}{\|u\| \|v\|} = \frac{\rho(u) \cdot \rho(v)}{\|\rho u\| \|\rho v\|} = \frac{\mu_1 \mu_2 (A \times B) \cdot (A \times C)}{\mu_1 \|A \times B\| \mu_2 \|A \times C\|} = \frac{(A \times B) \cdot (A \times C)}{\|A \times B\| \|A \times C\|}.$$

Now $\|A \times B\| = \|A\| \|B\| \sin c = \sin c$ since $\|A\| = \|B\| = 1$.

Siml $\|A \times C\| = \sin b$; This obtain the formula for $\cos \hat{A}$.

lemma



angle between u, v

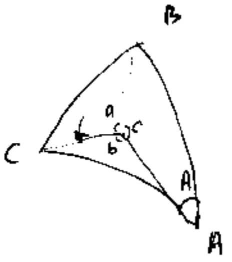
$= \{$ angle between normals H_1, H_2.

" $-\pi$. $\}$.

$A\times u' = -(A\times B)$ & $A\times v' = -A\times C$

∴ The angle between u & v is the same as the angle between $A\times B$ and $A\times C$.

Notation:



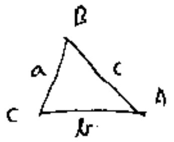
Cor 1. $\cos\hat{A}\sin c\sin b = (A\times B)\cdot(A\times C)$

$\cos B\sin a\sin c = (B\times C)\cdot(B\times A)$

etc.

Corollary (Propn 2.4.1) The law of cosines for spherical triangles: (For Sides) $\cos a = \cos b\cos c + \sin b\sin c\cos\hat{A}$

(cf. $a^2=b^2+c^2-2bc\cos A$)



Rmk: $d_{S^2}(B,C) = a$, not $\cos a$.

$\arccos(\cos a) = a$!

pf. $(A\times B)\cdot(A\times C) = (A\cdot A)(B\cdot C) - (A\cdot B)(A\cdot C)$

$= (A\times B)\times A\cdot C$ $\quad = B\cdot C - (A\cdot B)(A\cdot C)$

$= [-A\times(A\times B)]\cdot C$

$= -[(A\cdot B)(A]\cdot C)$ $\quad = \cos a - \cos c\cos b$

$+(A\cdot A)B]\cdot C$

(check this!)

exercise #5

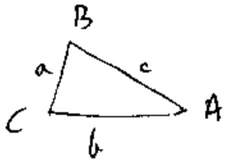
∴ $\cos\hat{A}\sin b\sin c = \cos a - \cos c\cos b$

so result follows.

$(u\times v)\times w$
$= (u\cdot w)v - (v\cdot w)u$

Pause:

Remark How does this compare with the Euclidean formula?



$a^2 = b^2+c^2 - 2bc \cos A$?

Note: when the spherical Δ is small, i.e. a, b, c are small, $\ll 1$

then let's see: First, for $|x| \ll 1$, $\cos x \approx 1 - \frac{x^2}{2} + 0(x^4)$ (Taylor)

$\sin x \approx x. + 0(x^3)$

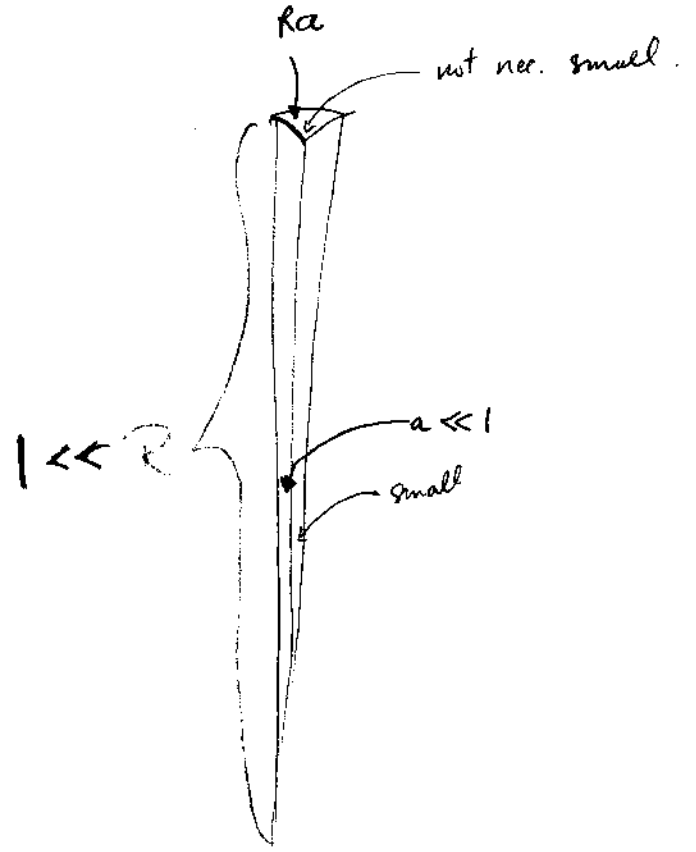
Hence, $\cos a = \cos b \cos c + \sin b \sin c \cos \hat{A}$ yields

to order 2 in a,b,c in each fac.

$1-\frac{a^2}{2} \approx \left(1-\frac{b^2}{2}\right)\left(1-\frac{c^2}{2}\right) + bc \cos \hat{A}$

Terms order 1 : $1 = 1$

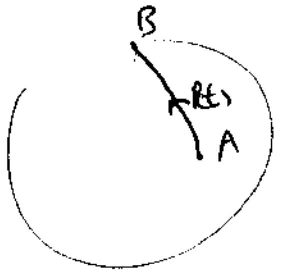
$\underset{2}{0(a^2,b^2,c^2)}$: $-\frac{a^2}{2} = -\frac{b^2}{2} - \frac{c^2}{2} + 2b \cos \underline{A}$ i.e. $a^2 = b^2+c^2 - 2bc \cos \underline{A}$!



Ex. Show that if $A, B \in S^2$ with $d(A,B) < \pi$, then $u \cdot B > 0$ if u is tangent to the ~~pt~~ great circle from A to B $\widehat{AB}$ at B.

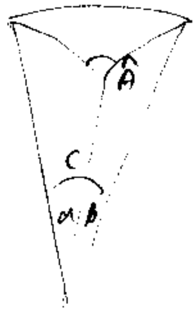
$\frac{\text{or}}{\text{aso}}$ # 5

Hint: choose coords so that $B = (0,0,1)$; find a formula for $P(t)$,



gt. circle for A to P $\widehat{AB}$

e.g. Suppose $a = b = c = \pi/3$. Find ^coss^ vertex angles.



eg. $\cos a = \cos b \cos c + \sin b \sin c \cos \hat{A}$

$$\frac{1}{2} = \frac{1}{4} + \left(\frac{\sqrt{3}}{2}\right)^2 \cos \hat{A}$$

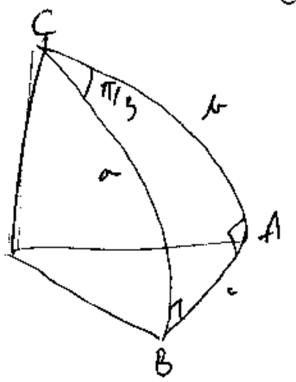
$$\therefore \cos \hat{A} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}.$$

(so $\hat{A} = \arccos(\frac{1}{3}) \approx 1.23$ rad $\approx .4\pi \approx 70.53°$
70° 31′ 44″)

All angles are equal by symmetry

$C \neq A \neq B \neq C$

eg



Suppose $\hat{A} = \hat{B} = \pi/2$, $\hat{C} = \pi/3$. Find interior angles. a, b, c.

$\cos \hat{C} = \frac{1}{2}$

$\cos \hat{A} = \cos \hat{B} = 0$; So

$0 \leq a, b \leq \pi$

$$\left.\begin{array}{l}\cos a = \cos b \cos c \\ \cos b = \cos a \cos c\end{array}\right\} *$$

$$\cos c = \cos a \cos b + \sin a \sin b \cdot \frac{1}{2} \quad **$$

$* \Rightarrow \cos a = \cos a (\cos c)^2 \iff \cos a = 0$ or $\cos c = \pm 1$.

Case I. $\cos a = 0$; $* \Rightarrow \cos b = 0 \Rightarrow \cos c = \frac{1}{2}$ (since $\overset{>0}{\sin a}\ \overset{>0}{\sin b}$ $\Rightarrow = 1$.) $\Rightarrow c = \pi/3$, $a, b = \pi/2$.

~~Case~~ II. $\cos c = 1 \Rightarrow c = 0 \Rightarrow A = B$ impossible ~~$\overset{*,**}{\Rightarrow} 1 = \cos^2 a + \frac{1}{2}\sin^2 a = \cos^2 a + \frac{1}{2}\sin^2 a \Rightarrow \sin a = 0 \Rightarrow$~~

$= -1 \Rightarrow c = \pi \Rightarrow A = -B$; $\cos a = -\cos b$; $\sin a = \sin b$; $\overset{**}{\Rightarrow} -1 = -\cos^2 a + \frac{\sin^2 a}{2} \Rightarrow \ldots \Rightarrow a = \pi \Rightarrow b = 0$ (imposs.

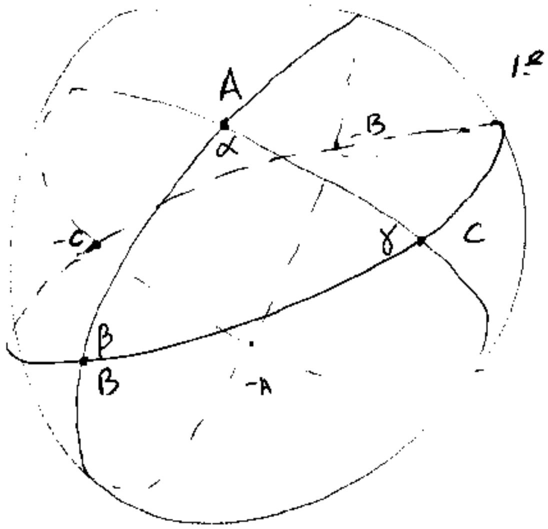
start here
L17↓

! FOR Relationship between angle sum and area!

Theorem In a spherical triangle, ΔABC
(Hariot, 1603!)

$$\hat{A} + \hat{B} + \hat{C} = \pi + \text{area}(\Delta ABC)$$

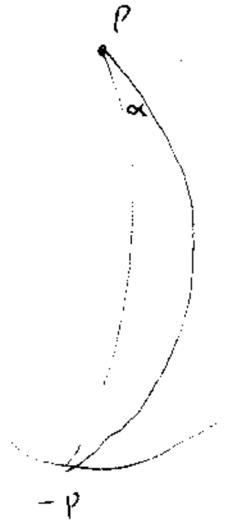
i.e. $\text{area}\,\Delta ABC = \hat{A}+\hat{B}+\hat{C} - \pi$!



Proof: We use several facts about area

(1) area of a sector of angle α is

$$\frac{\alpha}{2\pi} \cdot \underbrace{4\pi}_{\text{area sphere}} = 2\alpha.$$



(2) area is additive (in the obvious sense $\text{area}(X \cup Y) = \text{area}(X) + \text{area}(Y) - \text{area}(X \cap Y)$)

(3) area is invariant under isometries.

Partition S^2 as shown, We know great circles intersect in a pair of antipodal points, as shown. Define triangles

$\Delta = \Delta ABC$

$\Delta_\alpha = \Delta(-A)BC$

$\Delta_\beta = \Delta(-B)AC$

$\Delta_\gamma = \Delta AB(-C)$.

$\Delta_\alpha' = \Delta A(-B)(-C)$

$\Delta_\beta' = \Delta B(-A)(-C)$

$\Delta_\gamma' = \Delta(-A)(-B)C$.

$\Delta' = \Delta(-A)(-B)(-C)$

Note that the isometry $a(v) = -v$ satisfies

$a(\Delta) = \Delta'$

$a(\Delta_x) = \Delta_x'$ for $x = \alpha, \beta, \gamma$

$\therefore$ area $(\Delta) =$ area (Δ')

area $(\Delta_x) =$ area (Δ_x') for $x = \alpha, \beta, \gamma$

Hence area $(\Delta) +$ area $(\Delta_x) =$ area sector $X = 2X$

for $X = \alpha, \beta, \gamma$

$$\therefore \quad \underbrace{3 \text{ area}(\Delta)}_{y} + \underbrace{\sum_{x=\alpha,\beta,\gamma} \text{area } \Delta_x}_{z} = 2(\alpha+\beta+\gamma) = 2 \text{ anglesum}$$

$$\text{Now, area }(S^2) = 4\pi = \overbrace{2 \text{ area}(\Delta) + 2 \sum_{x=\alpha,\beta,\gamma} \text{area } \Delta x}^{\text{Sum of areas of eight } \Delta\text{'s}}$$

$$3y + z = 2(\alpha+\beta+\gamma) \quad ; \; 6y + 2z = 4(\alpha+\beta+\gamma)$$

$$\therefore \quad 2y + 2z = 4\pi$$

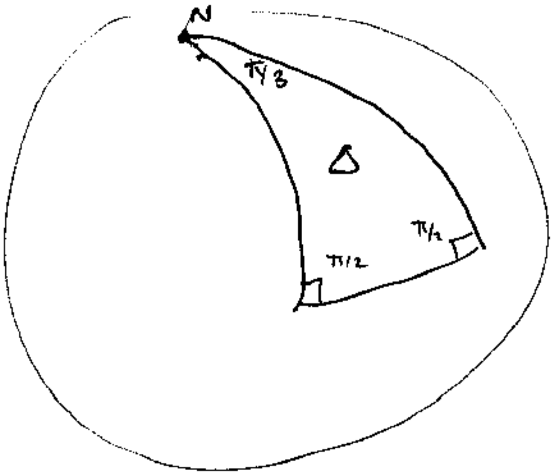
$$\text{Hence} \quad 4y = 4(\alpha+\beta+\gamma) - 4\pi$$

or $\alpha+\beta+\gamma = \pi +$ area (Δ) $\quad (\geq 0!)$ ☒

or area $(\Delta) = \alpha+\beta+\gamma - \pi$

---

eg

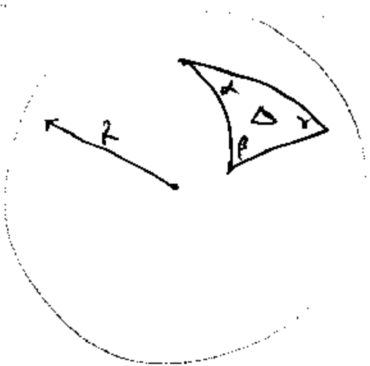


$$\text{area } \Delta = \pi/3 + \pi/2 + \pi/2 - \pi$$

$$= \pi/3$$

Remark①(ex.) On a sphere of radius R, area $S^2_R = 4\pi R^2$,
area sector $\alpha = 2\alpha R^2$

so we obtain



$$\alpha+\beta+\gamma-\pi = \frac{\text{area}\,\Delta}{R^2}$$

② Area of any (non-degenerate) spherical Δ is positive,

so $\alpha+\beta+\gamma > \pi$ in a spherical Δ! (on any sphere)

③ $S^2_R = \{v \in \mathbb{R}^3 \mid \|v\|=R\}$; $d_{S^2_R}(P,Q) = R\arccos(\frac{P\cdot Q}{R^2})$

④ Theorem There is no "isometry" between any spherical triangle and any triangle in $\mathbb{R}^n$.

Defⁿ! If $U \subset S^2_R$, $f: U \to \mathbb{R}^n$ is an isometry if

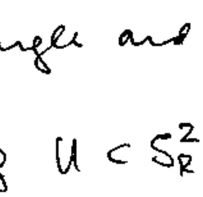
$\|f(P)-f(Q)\| = d_{S^2_R}(P,Q)$, $\forall P,Q \in U$.

Pf of thm. Any such isometry $f: \Delta \to \mathbb{R}^n$ would

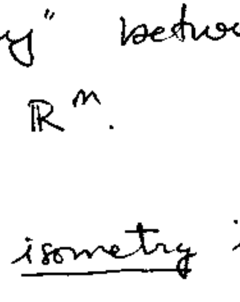
(a) preserve paths that minimize distance; i.e. it would send geodesics on S^2_R to straight lines in $\mathbb{R}^n$

(b) would preserve angles (& (c) would preserve areas)

But on S^2_R



$\xrightarrow{f}$ while

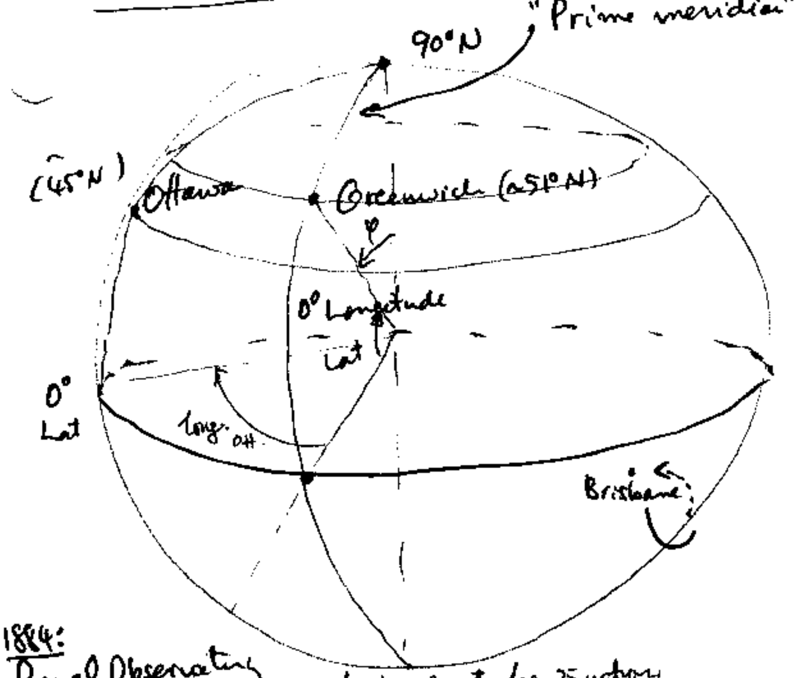


in $\mathbb{R}^n$

$\alpha+\beta+\gamma > \pi$ $\alpha+\beta+\gamma = \pi$

Hence, no such f can exist. □

Latitude, longitude, Bearing



1884: Royal Observatory in Greenwich;
x
(41 delegates from 25 nations met in Wash. DC in 1884: Green. won vote 22-1 with 2 abstentions (France, Brazil) San Domingo

cross hairs in eye piece of "Transit circle" telescope

Def^n ① Line of latitude is a $\{P \in S^2 \mid \varphi(P) = \varphi_0, \text{constant}\}$

② Line of longitude is a $\{P \in S^2 \mid \Theta(P) = \theta_0 \text{ or } \theta_0 + \pi\}$

However latitude is not φ; longitude lies between $-\pi$ & π; measured east & west of Greenwich.

eg **Ottawa**: Lat $45°24'$ N $\approx .252\pi$ N
$(1' = 1/60°)$ Long $75°43'$W $\approx .421\pi$ W

Brisbane Lat $27°28'$S $\approx .153\pi$ S
Long $130°51'$E $\approx .727\pi$ E

x-axis passes through 0° Lat, 0° Long.

Conversion to (θ, φ) coordinates:

φ (Ottawa) = $\pi/2$ - Lat (Ott) $\approx .248\pi$

θ (Ottawa) = 2π - long (Ott) $\approx 1.58\pi$

φ (Brisbane) = $\pi/2 + \underset{(\text{Lat S.})}{.153\pi} \approx .653\pi$

θ (Brisbane) = $.727\pi$

Exercise . lines of latitude and longitude are orthogonal

One could work out vectors for these cities

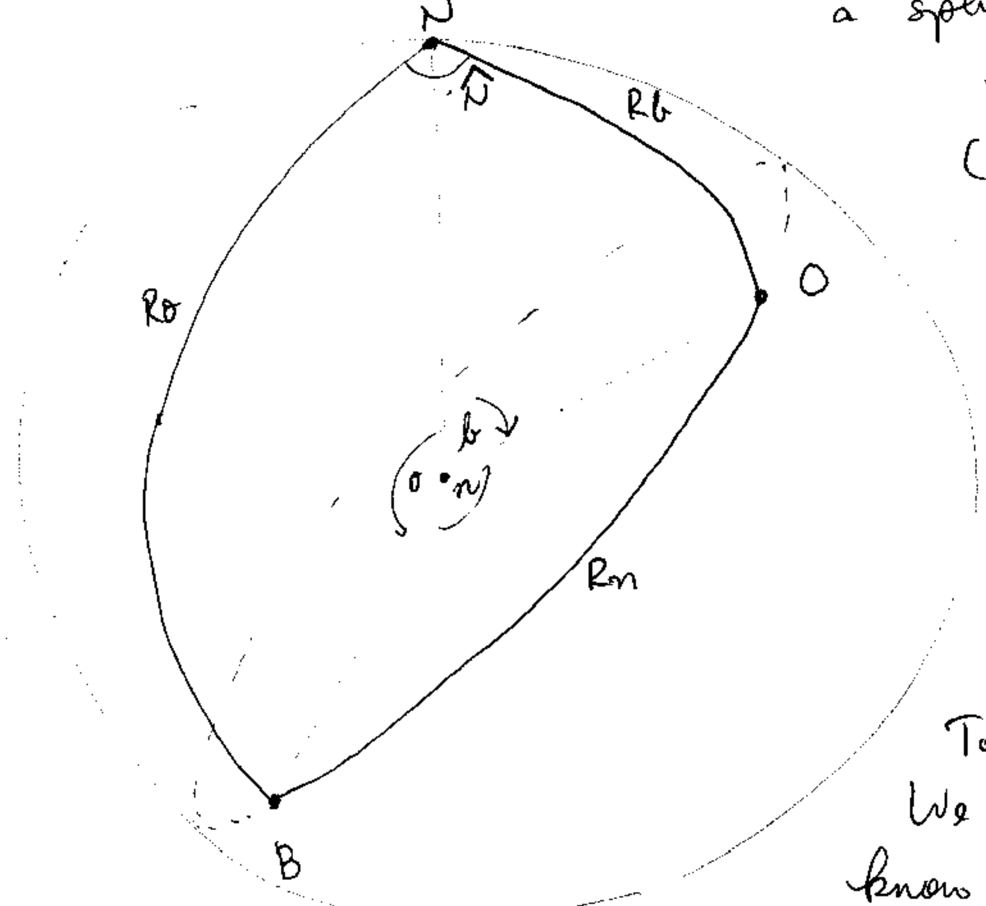
(vector Ottawa $\simeq R(.176, -.685, .707)$, R = Radius earth $\approx$ 6378 km).

• Spherical distance

How far is it from Ottawa to Brisbane? Construct a spherical Δ with vertices vertices N, Ottawa, Brisbane (using smallest arcs! as usual (for sides))

The cosine rule, for spherical Δ's doesn't need to be modified, as long as we remember that these angles o, b, n are subtended at the origin.

To compute Rn, We first need $\underline{\underline{n}}$. We know o, b and $\hat{N}$!



$$\cos n = \cos o \cos b + \sin o \sin b \cos . \hat{N}$$

where $o = \varphi(\text{Brisbane}) \approx .653\pi$

$b = \varphi(\text{Ottawa}) \approx .248\pi$

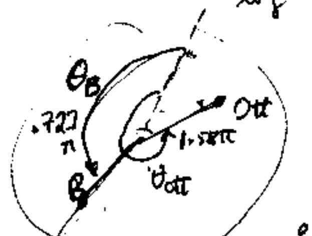
$\hat{N} \approx .853\pi \simeq (1.58 - .727)\pi \approx$

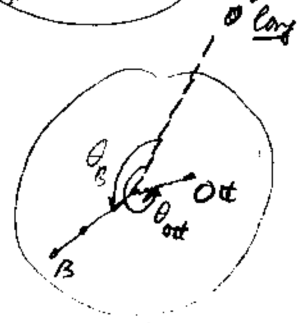
Using $\cos n = \cos o \cos b + \sin o \sin b \cos \hat{N}$, we find

$\therefore n \approx .846\pi$

$\therefore Rn \approx \overset{6380}{6378} \times .846\pi \approx 16951$ km.
≈ 17000 km

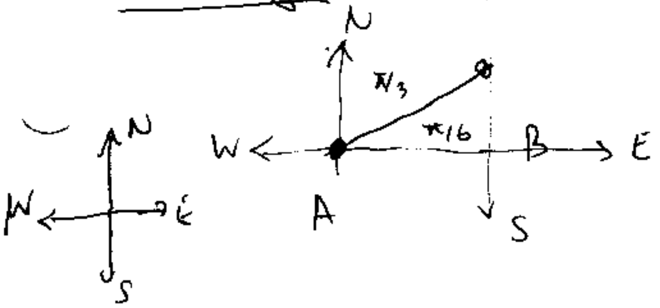
(No stops, 1000 km/h : 17.0 hrs in the air)





$\hat{N} = \theta_B - \theta_{ott}$

"Bearing" measures directions at a point "relative"



B is N π/3 E from A

A is S π/6 W from B.

etc.

- What's the bearing of Ottawa from Brisbane? (Head in this direction, in a "straight line").

$\hat{B}$ is the bearing i.e

Ottawa is N $\hat{B}$ E from Brisba

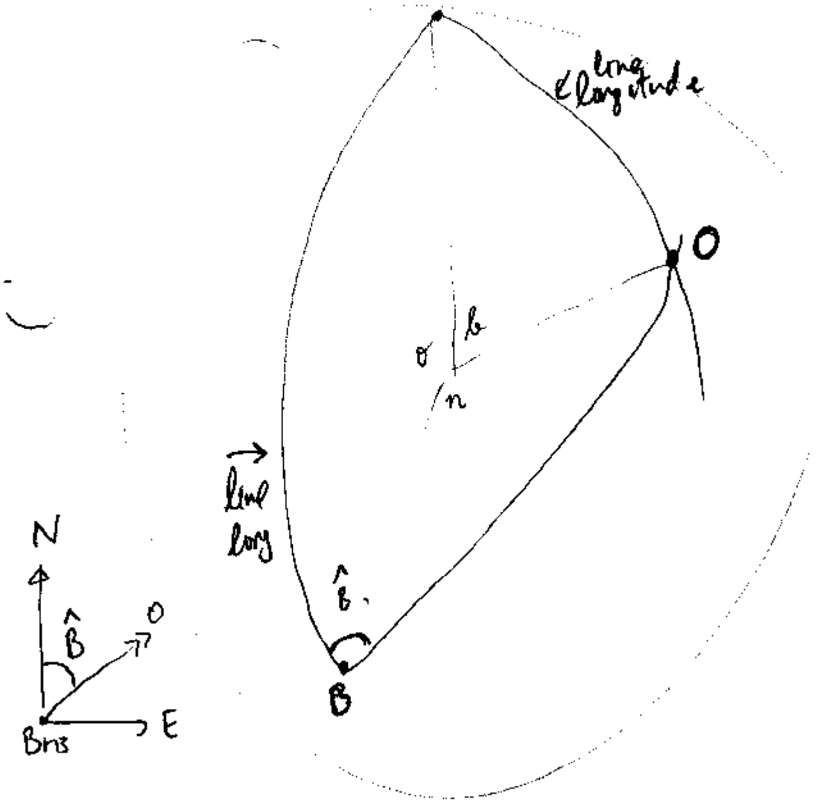
By the cosine rule (for sides)

$$\cos b = \cos n \cos \theta + \sin n \sin \theta \cos \hat{B}$$

$$\therefore \cos \hat{B} = \frac{\cos b - \cos n \cos \theta}{\sin n \sin \theta}$$

yields $\hat{B} \simeq .303\pi$

(54°32')

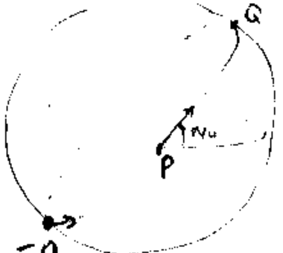


∴ Ottawa bears N 54°32' E from Brisba

exercise: find the bearing of Brisbane from Ottawa!
Careful!

− is $\widehat{BO}$ a line of constant bearing?...

probably not:-



N π/4 E at P
E! at −Q

2.9 Mapmaking

Saw that we can't have any isometry

$f: \overset{U,\ \text{area}\,U \neq 0}{S^2} \longrightarrow \mathbb{R}^{(n)!\ )}$, so "laying out"

the sphere will always involve some distortion.

We can preserve some things, not others

e.g. angle, but not area → similarity transf

area but not angle. → ex $\begin{bmatrix}1&1\\2&3\end{bmatrix}$ preserves area

We'll look briefly at 4 types.

"Gnomonic": H tang S^2 (half.)

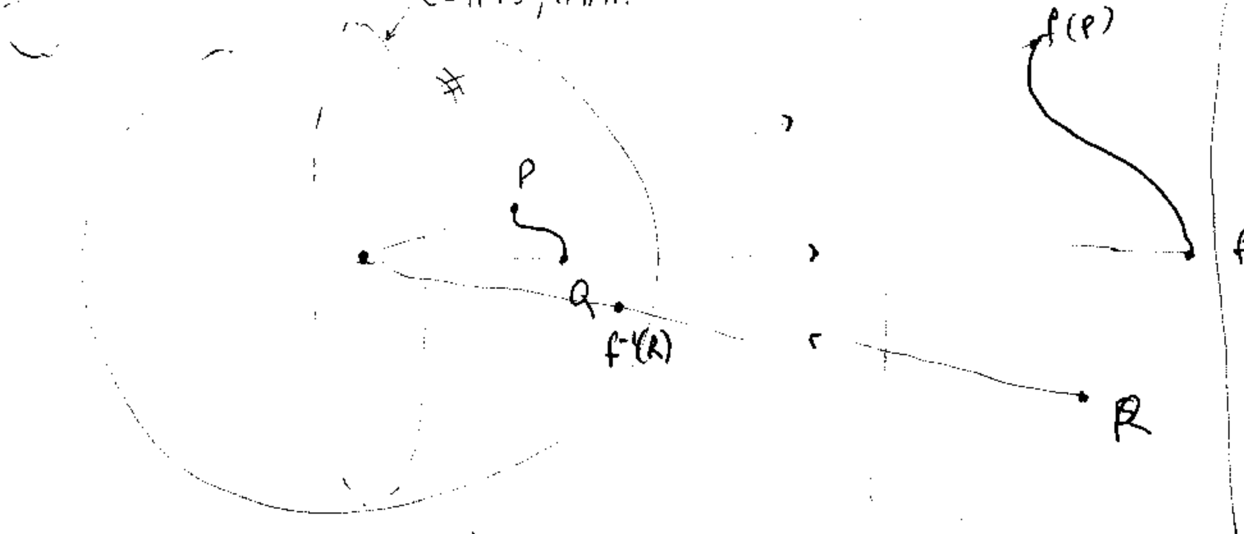
Central Projection $S^2_- \to H$

$H \not\ni 0$

Def^n

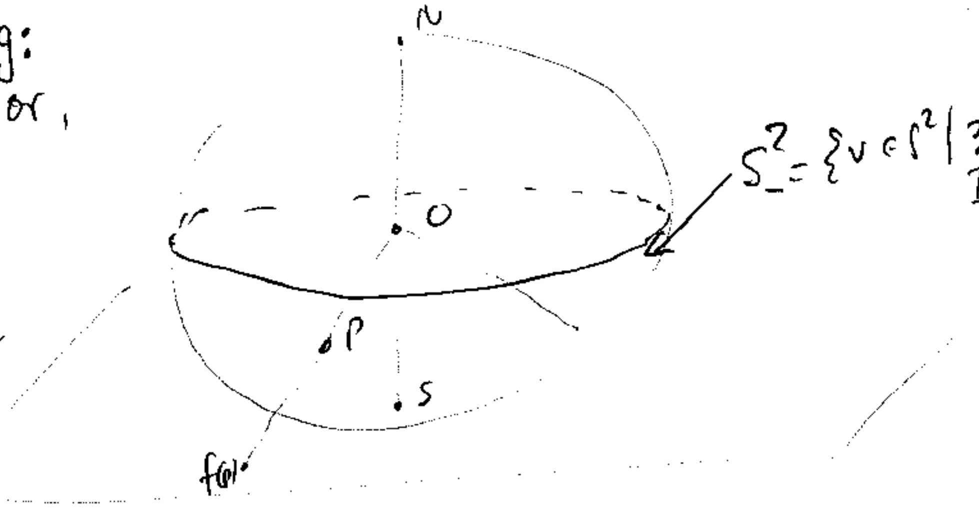
$f(P) :=$ intersection of $\overrightarrow{OP}$ with H.

$C = H' \cap S^2$, $H' \parallel H$.



Remark: an inverse for f is def'd by $f^{-1}(R) = \overline{OR} \cap S^2$.

e.g: or,



$S^2_- = \{v \in S^2 \mid z(v) \leq 0\}$

If the plane is tangent at S, say.

Note: can only get at most ½ the sphere this way

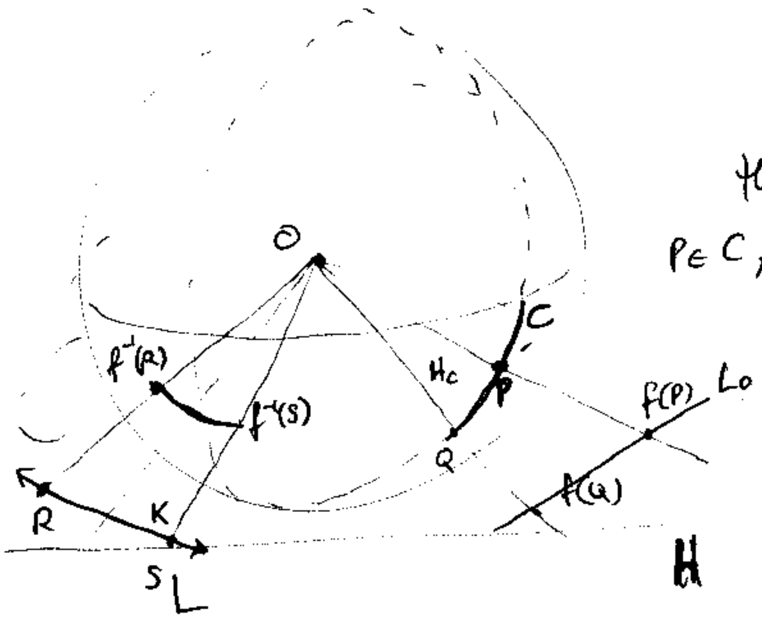
Good property: it sends great circles to lines! i.e.

Rmk: $f^{-1}(P) = \frac{P}{\|P\|}$ (independent of H!!) i.e. preserves $\overset{\text{shapes of}}{\text{geodesics}}$.

$f: S^2 \to H$

Prop^n ~~2.9.1~~ Central projection maps parts of great circles to parts of lines, and vice versa.

Pf. Suppose C is part of a great circle, and let H_C be the plane through O s.t. $C \subset H_c \cap S^2$. Then, since ~~if~~ $P \in C$ the line $\vec{OP}$ lies in H_C, (because O, P do,) Hence, if $P \in C$, $f(P) \in H_C$; and $f(P) \in H$, so $f(P) \in H_c \cap H$, which is a line in $\mathbb{R}^3$ (lying in H, of course).



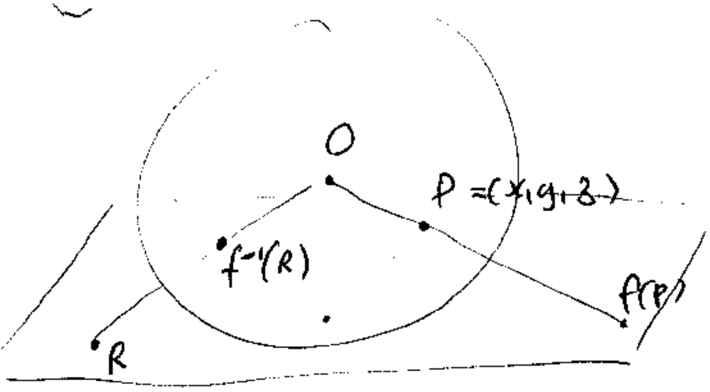
i.e. $f(C) \subset H_c \cap H = L_0$, a line.

Conversely, suppose L is a line in H, and let K be the plane containing O and L.

Then, all the pts $\{f^{-1}(R) \mid R \in L\}$ lie in the plane K, and in S^2 i.e. $\{f^{-1}(R) \mid R \in L\} \subset K \cap S^2$, which is a great circle.

∎

(Can be used as an aid in drawing great circles) e.g. find formulae for f, f^{-1} if $H = \{(x,y,-1) \mid x,y \in \mathbb{R}^2\}$.



The line $\vec{OP}$ is $\{s(x,y,z) \mid s \in [0,\infty)\}$

so $f(P) \doteq \vec{OP} \cap H$ satisfies

$f(P) = (sx, sy, sz)$ & $sz = -1$

$\therefore s = -\frac{1}{z}$ so $f(x,y,z) = -\frac{1}{z}(x,y,z)$

Moreover; $f^{-1}(R) = \frac{R}{\|R\|}$ so $f^{-1}(x,y,-1) = \frac{(x,y,-1)}{\|(x,y,-1)\|}$

$$= \frac{(x,y,-1)}{\sqrt{x^2+y^2+1}}$$

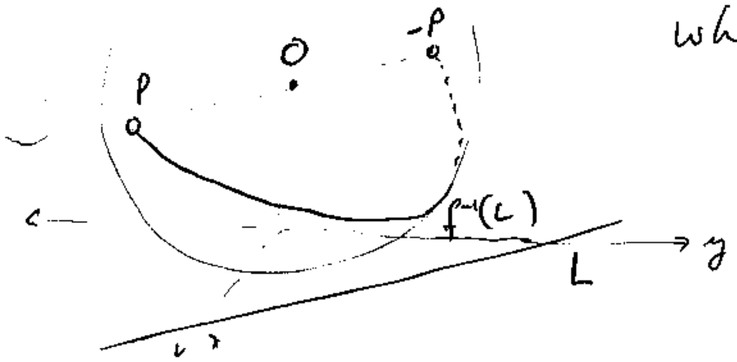
---

e.g. The image of $L = \{(x,y,z) \in \mathbb{R}^3 \mid x+y=1,\ z=-1\} \subset H$ via f^{-1} is:

$$f^{-1}(L) = \left\{ \frac{v}{\|v\|} \mid v \in L \right\}$$

Now $L = \{(1-s, s, -1) \mid s \in \mathbb{R}\}$ so $f^{-1}(L)$

$$\therefore f^{-1}(L) = \left\{ \frac{(1-s, s, -1)}{\|(1-s,s,-1)\|} \mid s \in \mathbb{R} \right\}$$ parametric repn.



what are P, $-P$?

$$P = \lim_{s \to -\infty} \frac{(1-s, s, -1)}{\|(1-s,s,-1)\|}$$

$$= \lim_{s \to} \frac{(1-s, s, -1)}{\sqrt{2s^2-2s+2}} = \frac{1}{\sqrt{2}}(1,-1,0)$$

Note: 1) $\lim_{s \to \infty} \frac{(1-s, s, -1)}{\|(1-s, s, -1)\|} = \frac{1}{\sqrt{2}}(-1, 1, 0) = -P$.

2) $P, -P \notin f^{-1}(L)$!

3) $\overleftrightarrow{P(P)}$ is parallel to L. 1) same plane 2) doesn't meet. Does this always happen? If $L = \{ u_0 + s v_0 \mid u_0 \in \mathbb{R}^3, 0 \neq v_0 \in \mathbb{R}^3, s \in \mathbb{R} \}$

then $f^{-1}(L) = \left\{ \frac{u_0 + s v_0}{\| u_0 + s v_0 \|} \mid s \in \mathbb{R} \right\}$

$$\text{so } \lim_{s \to +\infty} \frac{u_0 + s v_0}{\| u_0 + s v_0 \|} = \frac{s\left(\frac{u_0}{s} + v_0\right)}{s\left(\| \frac{u_0}{s} + v_0 \|\right.} = \frac{v_0}{\|v_0\|}$$

$|s| = s$

$$\left(\lim_{s \to -\infty} \frac{u_0 + s(v_0)}{\| u_0 + s v_0 \|} = \frac{s\left(\frac{u_0}{s} + v_0\right)}{-s\left(\| \frac{u_0}{s} + v_0 \|\right.} = -\frac{v_0}{\|v_0\|} \right)$$

$|s| = -s$

so line through $\frac{v_0}{\|v_0\|}$, 0, $\frac{-v_0}{\|v_0\|}$ is certainly

1) in the same plane as L

2) doesn't meet L. (same dir. vector; $L \not\ni 0$.)

- Central projection **preserves**
  shape of geodesics only: not their
  length: length $f^{-1}(L) < \infty$ $= \pi$
  while leng $L = \infty$

* * *

- doesn't preserve angles (ass #6!)

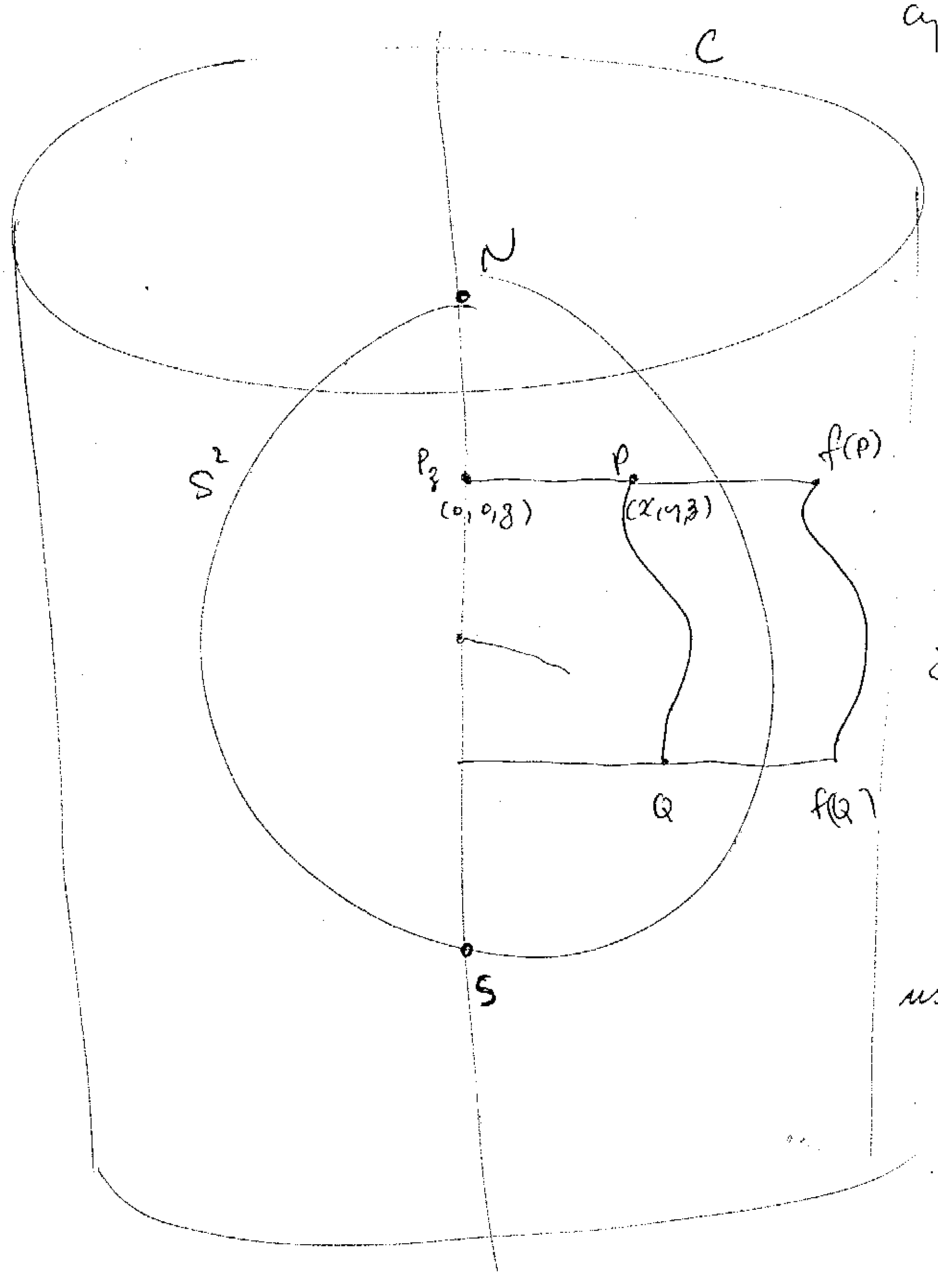
- doesn't preserve area
  area $f^{-1}(H) < \infty$ $(= 2\pi)$
  area $(H) = \infty$.

However, there is a projection that preserves area (though not the shapes of (most!) geodesics, nor angle):

Cylindrical Projections

: Suppose we take a cylinder C, axis the z-axis, radius $R \geq 1$. The cylindrical projection is def'd as follows: Let $P = (x, y, z)$ and $P_z = (0, 0, z)$. Then, $f(P) = \overrightarrow{P_zP} \cap C$.

Only 2 points cannot be mapped using this, those where $P_z = P$ i.e. N, S.



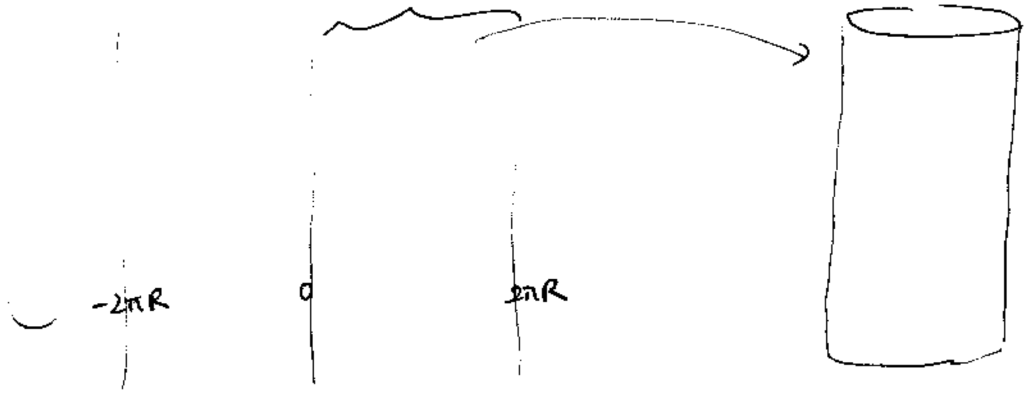
So $f: S^2 - \{N, S\} \longrightarrow C \cap \{(x,y,z) \in \mathbb{R}^3 \mid |z| \leq 1\}$

What are geodesics on a cylinder?

Cylindrical Geometry: one can cut, unwrap a cylinder to a flat strip, or, one can wrap (isometrically) the plane around a cylinder.

Define $\omega : \mathbb{R}^2 \longrightarrow C$ by

$$\underbrace{\omega(x, y)} = (R\cos(\tfrac{x}{R})\ R\sin(\tfrac{x}{R})\ y)$$



$x = \theta$
$y = z$

Theorem: ω is a local isometry; i.e. if $\alpha : [0,1] \to \mathbb{R}^2$ is a curve in $\mathbb{R}^2$, then $l(\alpha) = l(\omega \circ \alpha)$, where $\omega \circ \alpha : [0,1] \longrightarrow C$ is a curve in C.

Pf. $l(\alpha) = \int^1 \|\dot\alpha(t)\|\, dt$; Suppose $\alpha(t) = x(t), y(t)$.

$l(\gamma) = \int_0^1 \|\dot\gamma(t)\|\, dt$ Then $\|\dot\alpha(t)\| = \sqrt{\dot x^2(t) + \dot y^2(t)}$.

Then

$$\gamma(t) = \omega \circ \alpha(t) = (R\cos(\tfrac{x(t)}{R}), R\sin(\tfrac{x(t)}{R}), y(t))$$

$$\|\dot\gamma\| = \|(-\frac{R}{R}\dot x \sin(x(t)), \frac{R\dot x(t)}{R}\cos x(t), \dot y(t))\|$$

$$= \sqrt{\dot{x}^2(t) + \dot{y}(t)} = \dot{\alpha}(t).$$

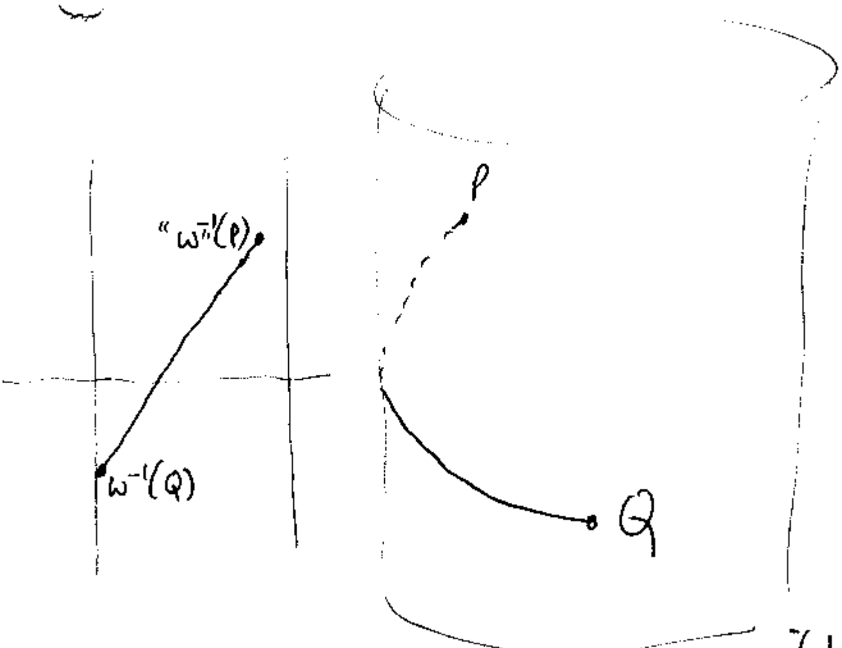
Thus, $\ell(w \circ \alpha) = \ell(\alpha)$.

$\square$

Rmk w is not an isometry globally

Defn (already seen) If $P, Q \in C$ then $d_C(P,Q)$ is the length of the shortest path (on C) from P to Q

Corollary w takes parts of straight lines to parts of geodesics on C.



Unwrap w(line segment) and you get a line segment.

if $S, T \in \mathbb{R}^2$ s.t. $x_S - x_T \leq \pi R$,

then $d_C(w(S), w(T)) = \|S - T\|$

Note: w has many "right inverses:"

$u_n : C \to \mathbb{R}^2$ $\quad \theta \in [R2n\pi, R2(n+1)\pi]$

$w \circ u_n = id_C$

$u_n(R\cos\theta, R\sin\theta, z) = (R\theta, z)$

$w \circ u_n = id_C$

ex. which isometries on $\mathbb{R}^2$ "descend" to C?

Back to

$$f : S^2 - \{N, S\} \longrightarrow C \xrightarrow{u_0} \mathbb{R}^2$$

ex. find a formula for $u_n \circ f$

Remarks : lines of longitude are sent to lines on C

equator ↦ "line segment" on C (circle in $\mathbb{R}^3$!)

indeed all other lines of latitude ↦ (not geodesics on S^2)

line segments on C (✓ are geodesics on C)

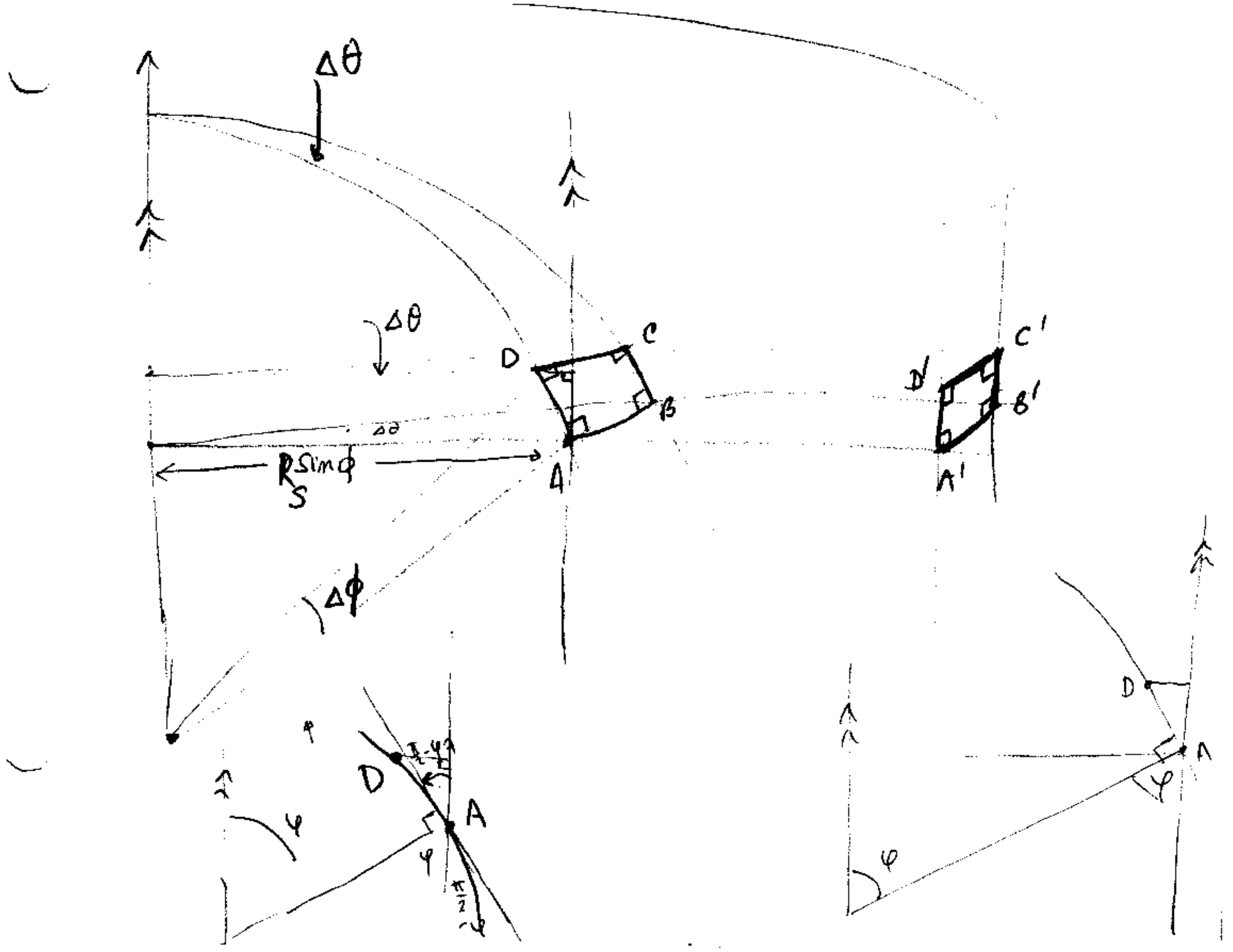
However, the cylindrical projection does preserve area!

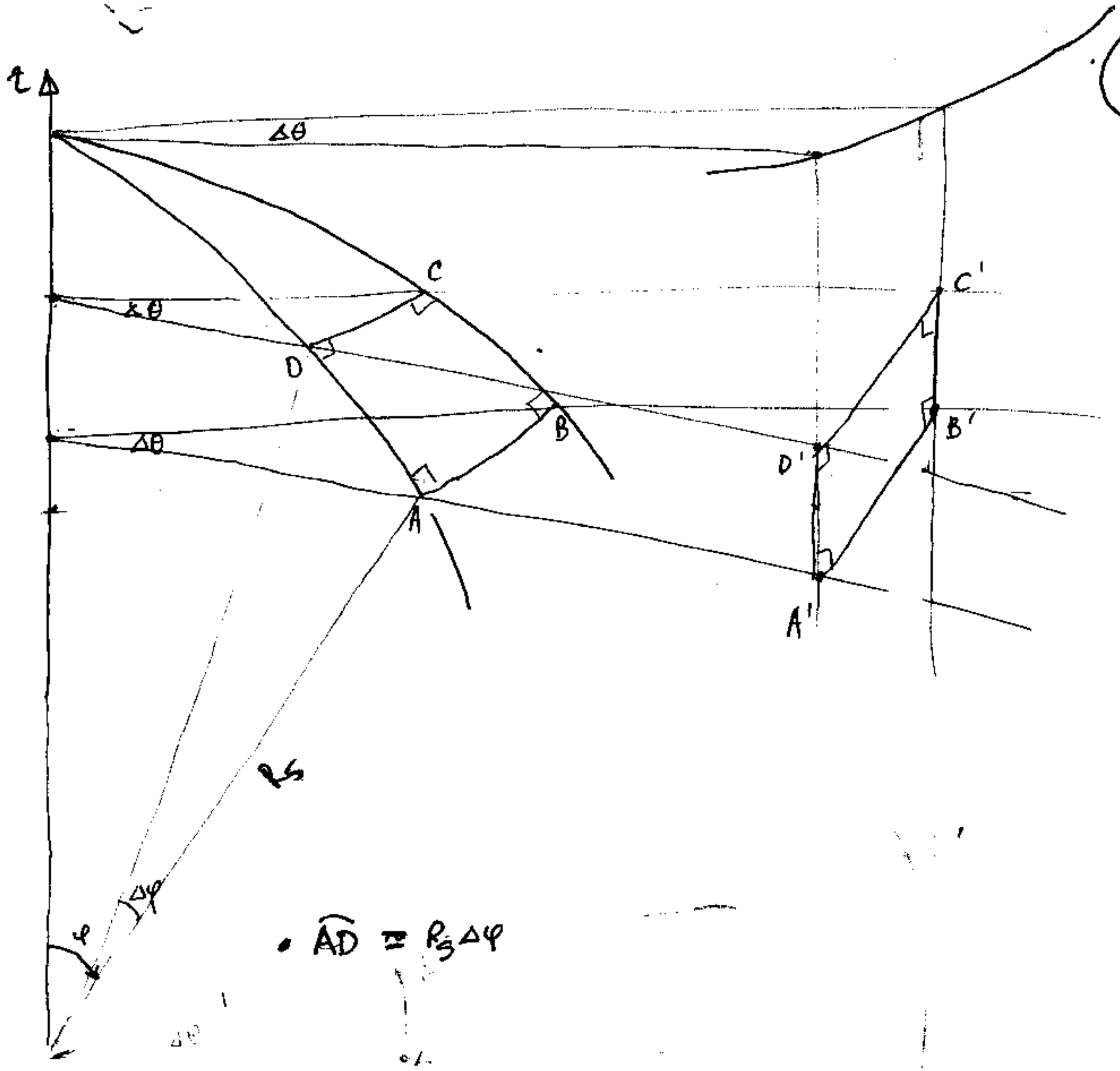
ex.

Propⁿ 2.9.2

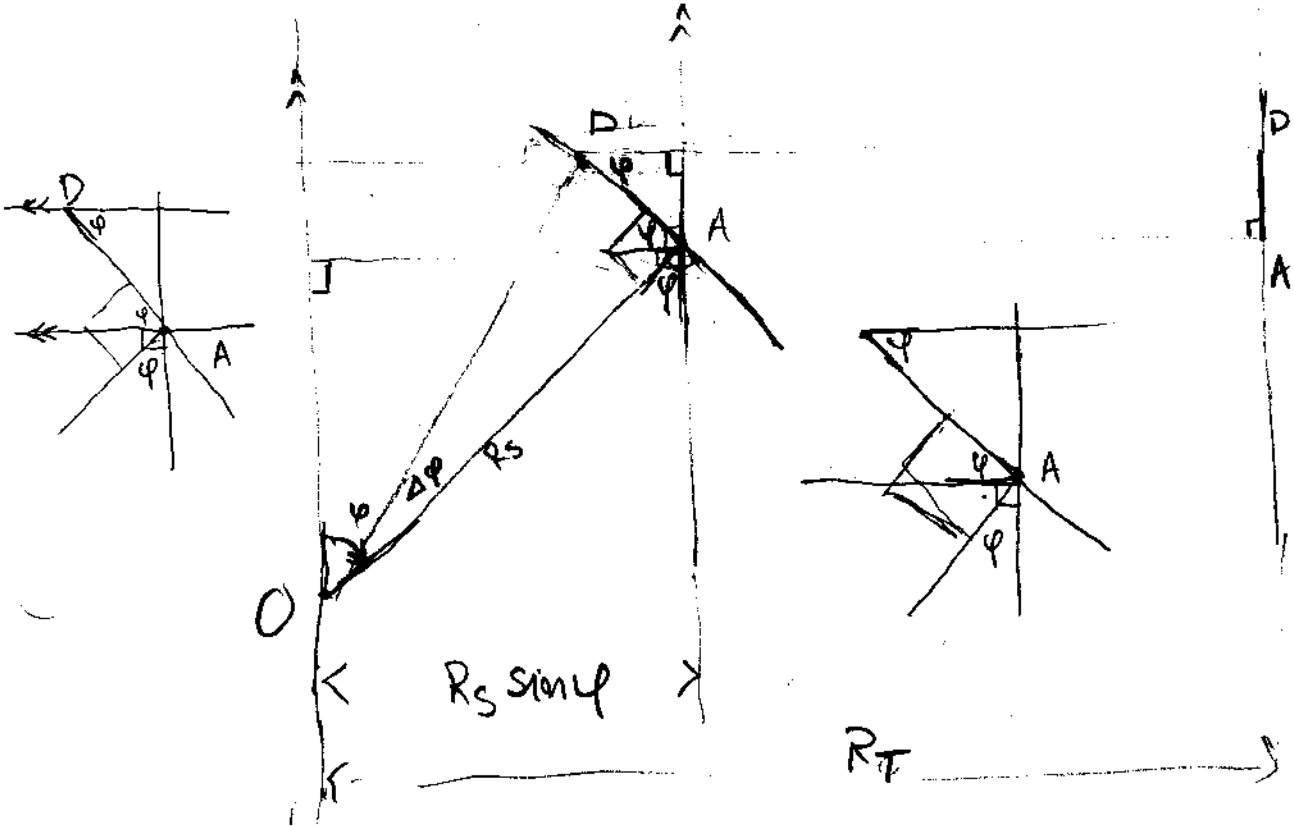
If the cylinder and the sphere are tangent, cylindrical projection preserves area! (If not, it multiplies them by a constant scaling factor $\frac{R_C}{R_S}$, where R_C = radius of cyl
R_S = " sphere.

Pf. We show that the area of a small rectangle on $S^2_{R_s}$ is $\frac{R_c}{R_s}$ ~~the same~~ (in the limit) as the area of its image - again a small rectangle - on C_{R_c}

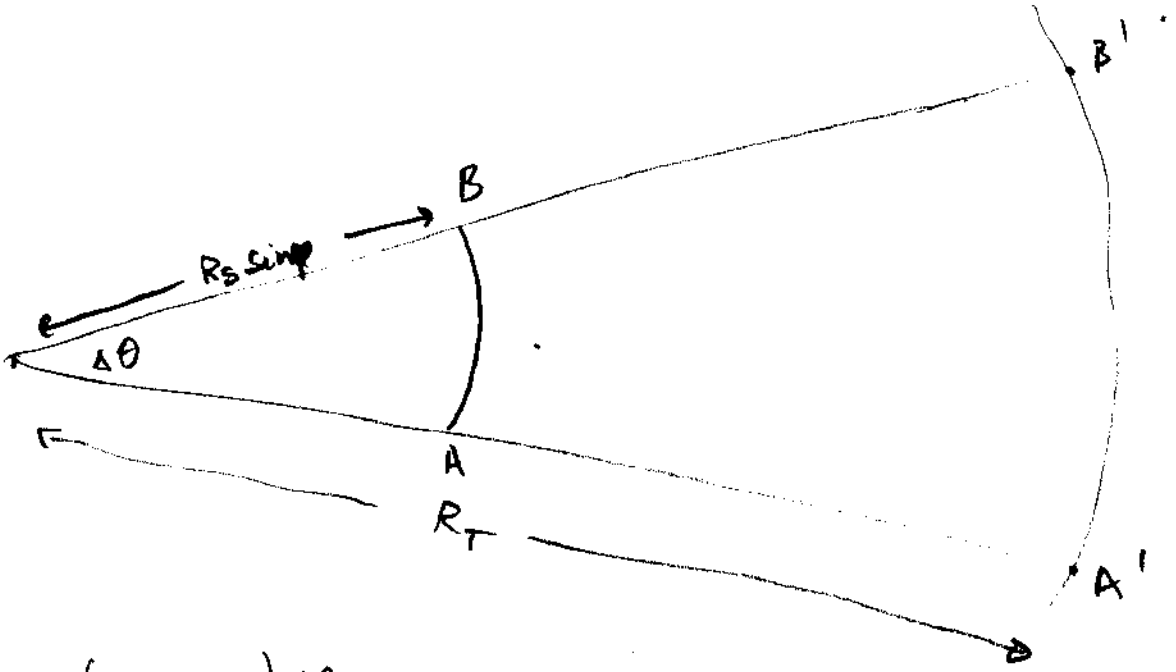




• $\widehat{AD} = R_S \Delta\varphi$



• $\widehat{A'D'} = \widehat{AD}\sin\varphi$



- $\widehat{AB} = (R_S \sin\varphi)\Delta\theta$
- $\widehat{A'B'} = R_T\,\Delta\theta$

$$\begin{aligned} \therefore\ \text{area } ABCD &\approx \widehat{AB}\cdot\widehat{AD} \\ &\approx (R_S \sin\varphi\,\Delta\theta)\, R_S\Delta\varphi \\ &\approx R_S^2 \sin\varphi\,\Delta\theta\,\Delta\varphi \end{aligned}$$

$$\begin{aligned} \text{area } A'B'C'D' &= \widehat{A'B'}\ \widehat{A'D'} \\ &= (R_T\Delta\theta)\ (R_S\Delta\varphi)\sin\varphi \\ &= R_T R_S \sin\varphi\,\Delta\theta\,\Delta\varphi \end{aligned}$$

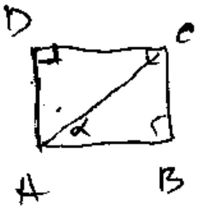
$\therefore$ Area $A'B'C'D' \approx \frac{R_S}{R_T}$. Area $ABCD$; equality in limit. □

Remark • $\widehat{A'B'} = \frac{R_T}{R_S}\cdot\frac{1}{\sin\varphi}\cdot\widehat{AB}$; $\widehat{A'D'} = \sin\varphi\ \widehat{AD}$

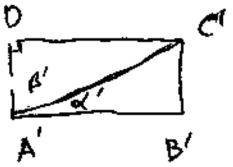
So

Effect on angles:

$\alpha+\beta=\pi/2$



$\alpha'+\beta'=\pi/2$



- $\widehat{A'B'} = \frac{R_T}{R_S}\cdot\frac{1}{\sin\varphi}\ \widehat{AB}$
- $\widehat{A'D'} = \sin\varphi\ \widehat{AD}$

$$\tan\alpha = \frac{\widehat{AD}}{\widehat{AB}} \qquad \tan\alpha' = \frac{\widehat{A'D'}}{\widehat{A'B'}} = \sin\varphi\ \widehat{AD}\ \frac{R_S}{R_T}\frac{\sin\varphi}{\widehat{AB}}$$

$$\therefore \quad \tan\alpha' = \frac{R_S}{R_T}\sin^2\varphi\ \tan\alpha$$

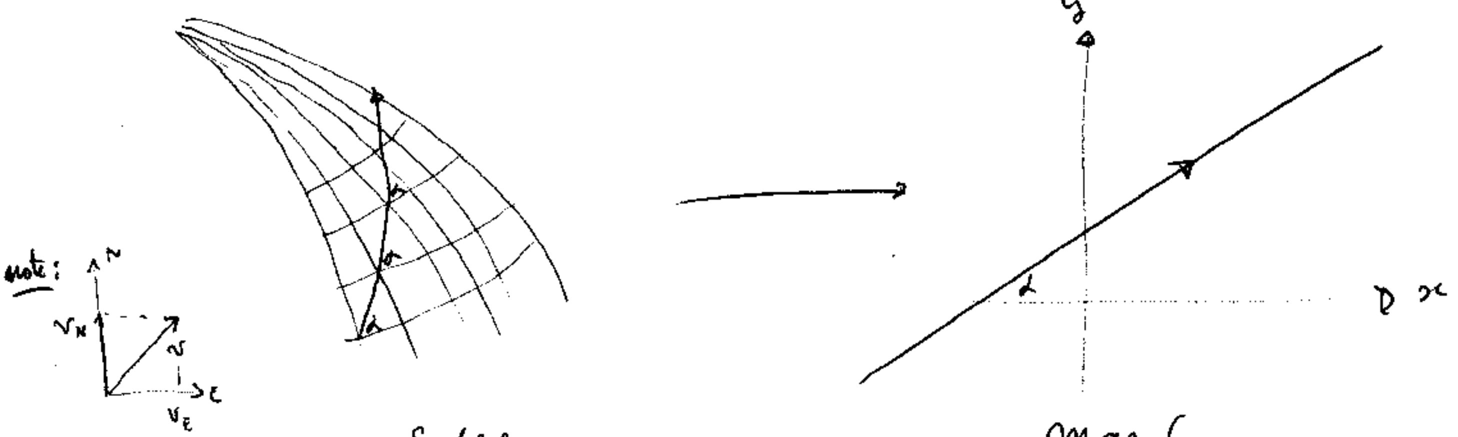
$\therefore$ Central proj changes angles, in general, (but at equator ($\varphi=\frac{\pi}{2}$) if $R_T = R_S$.

Mercator Projection

(10.1)

(1569 - Gerhard Kremer : his latin name was "Gerardus Mercator"

Idea: aid mariners by making lines of constant heading ("loxodrome", "rhumb line") map to straight lines on the map. (easy to navigate: keep the compass heading steady (but then, not travel on great circles...)



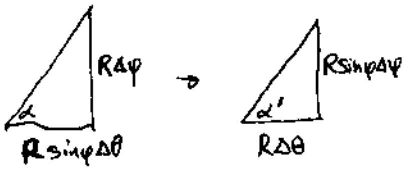
$v_N = |R_{earth}\dot{\varphi}|$

Sphere

Map ($\cdot R = 1$.

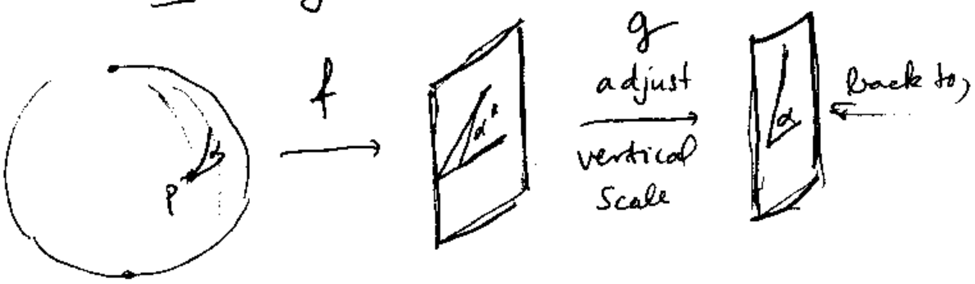
Want $\tan\alpha = \tan\alpha'$ in previous order. Take $R_C = R_S$ for simplicity

Already saw that at pt with lat. $(\frac{\pi}{2} - \varphi)$, distortion was



$\tan\alpha' = \frac{R_S}{R_C}\sin^2\varphi\tan\alpha$ (So $\alpha' \leq \alpha$)

So adjust vertical scale on map to compensate



Let (θ, z) be coords on C.

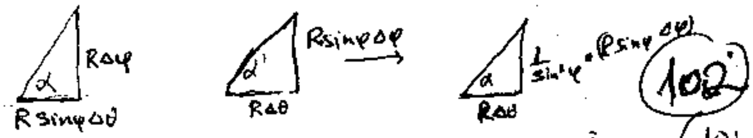
Let (φ, θ) be spherical polar coords of P;

Then $f(P) = (\theta, z)$

So we want $g: C \to C$ so angles are preserved by $g \circ f$.
(cylinder)

Set $g(\theta, z) = (\theta, h(z))$

Want



$$g(\theta, z+\Delta z) - g(\theta, z) \approx \Delta z \cdot \frac{1}{\sin^2\varphi}$$

i.e. $h(z+\Delta z) - h(z) \approx \Delta z \cdot \frac{1}{\sin^2\varphi}$ i.e. $\frac{dh}{dz} = \frac{1}{\sin^2\varphi}$

But $z = \cos\varphi$; (take $R=1$ for simplicity)

$$\therefore \text{ Solve } \left.\begin{array}{l} \frac{dh}{dz} = \frac{1}{1-z^2} \\ h(0) = 0 \end{array}\right\} \text{ for } h(z) = \ln\sqrt{\frac{1+z}{1-z}}$$

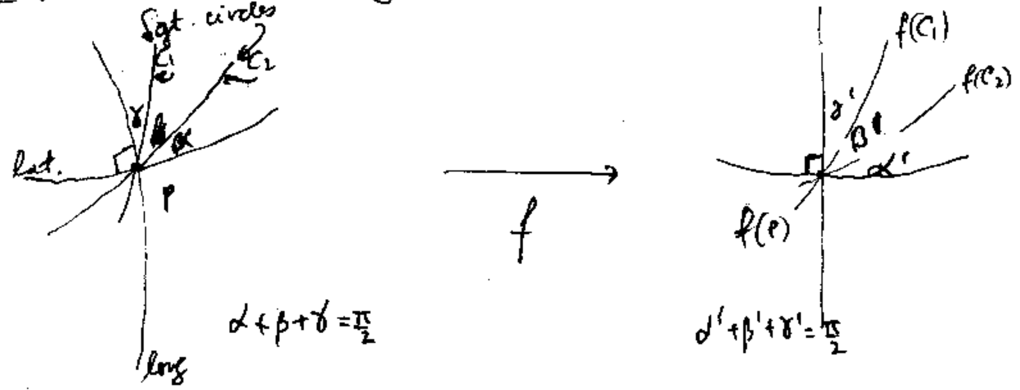
$$\text{or: } gf(\theta,\varphi) = \left(\theta, \ln\sqrt{\frac{1+\cos\varphi}{1-\cos\varphi}}\right) \qquad \text{Mercator}$$

Mercator projection: it is <u>conformal</u> ; $U \subseteq S^2$

<u>Def^n</u>: A map $f: U \to \mathbb{R}^2$ is <u>conformal</u> if it preserves angles everywhere

<u>Thm</u> The Mercator projn is conformal!

<u>Pf.</u> Consider angles at $P \in S^2$.



Note: $\alpha+\beta+\gamma = \pi/2$; $\alpha'+\beta'+\gamma' = \pi/2$

We know Mercator satisfies

① $\alpha = \alpha'$

② $\alpha+\beta = \alpha'+\beta'$ ("new" $\alpha = \alpha+\beta$)

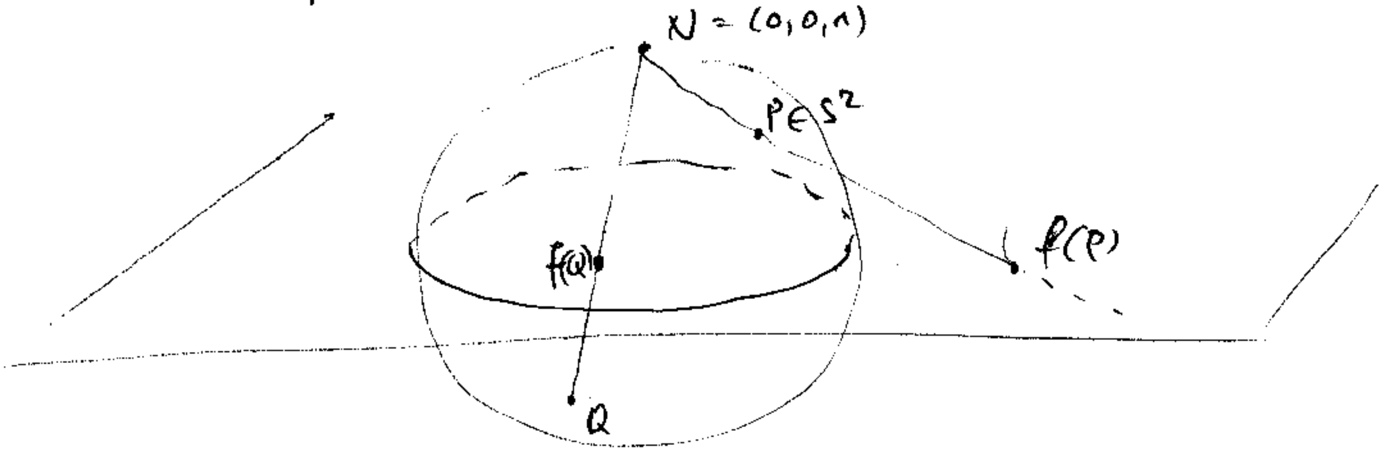
(③ $\alpha+\beta+\gamma = \alpha'+\beta'+\gamma'$)

$\therefore \beta = \beta'$

∎

Stereographic proj^n (Ptolemy 150)

is a map $f: S^2 - \{N\} \longrightarrow H =$ x-y plane $(z=0)$



and is defined by $f(P) = \overleftrightarrow{NP} \cap H$

Note: $f(S^2_+ - \{N\}) = \{v \in \mathbb{R}^2 \mid \|v\| > 1\}$

$f(\text{equator}) = \{v \in \mathbb{R}^2 \mid \|v\| = 1\}$

$f(S^2_-) = \{v \in \mathbb{R}^2 \mid \|v\| \leq 1\}$

- Can't preserve areas
- Doesn't preserve all geodesics, since $\underbrace{H \cap S^2}_{\text{geodesic on } S^2} \to \underbrace{\text{unit circle in } \mathbb{R}^2}_{\text{not geodesic in } \mathbb{R}^2}$

We can find a formula for f: If $P = (x, y, z)$,

we Solve $(0,0,1) + t\{(x,y,z) - (0,0,1)\} = (x', y', 0) = f(P)$

for t;

This gives $1 + t(z-1) = 0$ or $t = \frac{1}{1-z}$

$tx = x'$

$ty = y'$

$$\therefore f(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$$

<u>Remark</u> f is a bijection $S^2 - \{N\} \leftrightarrow H$

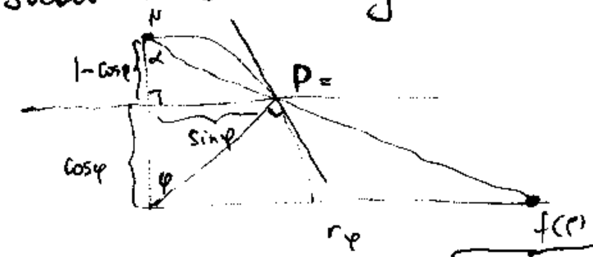
ex. find find a formula for $f^{-1} = g : H \to S^2 - \{N\}$

<u>ex.</u> • f takes lines of longitude to lines through the origin

• <u>ex</u> f takes lines of latitude (except $\varphi = \pi$) to circles through 0

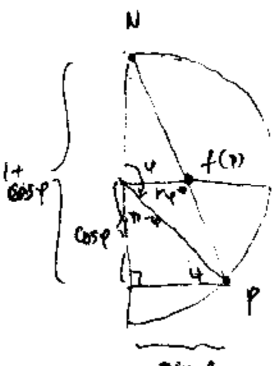
— Show that line of lat. corres to $\varphi \xrightarrow{f}$ circle, centre 0 radius $r(\varphi)$:

find $r(\varphi)$



$$\frac{r_\varphi}{1} = \tan\alpha ; = \frac{\sin\varphi}{1-\cos\varphi} \quad \therefore \boxed{r_\varphi = \frac{\sin\varphi}{1-\cos\varphi}}$$

$\varphi = \pi/2$; $r_\varphi = 1$ $\quad 0 \leq \varphi \leq \frac{\pi}{2}$

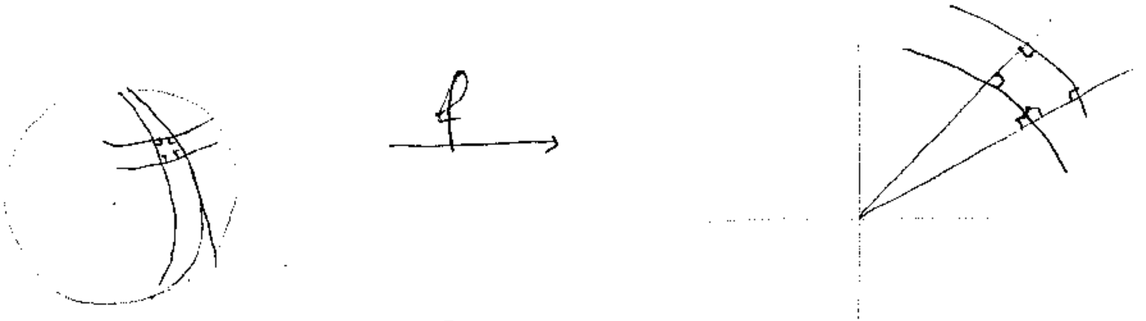


$$\frac{1}{r_\varphi} = \frac{1+\cos\varphi}{\sin\varphi}$$

$$\boxed{r_\varphi = \frac{\cos\varphi}{1-\sin\varphi} ; \pi \geq \varphi \geq \frac{\pi}{2}}$$

$\varphi \geq \frac{\pi}{2}$

$\psi + \frac{\pi}{2} + \pi - \varphi = \pi \quad \psi = \varphi - \pi/2$

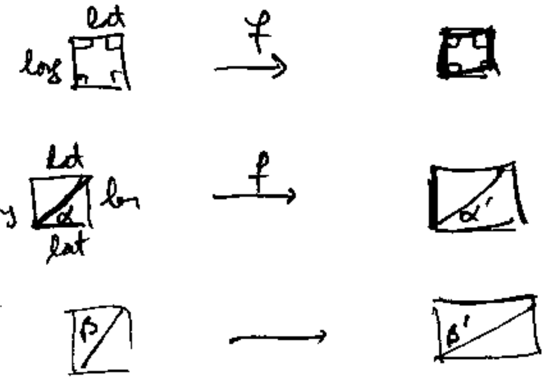


(Same as for
central projection).

L20
L21

"con" = same
"forma" = shape (Latin)

Theorem The stereographic proj^n is conformal!



Pf. As for Mercator,

So if we show

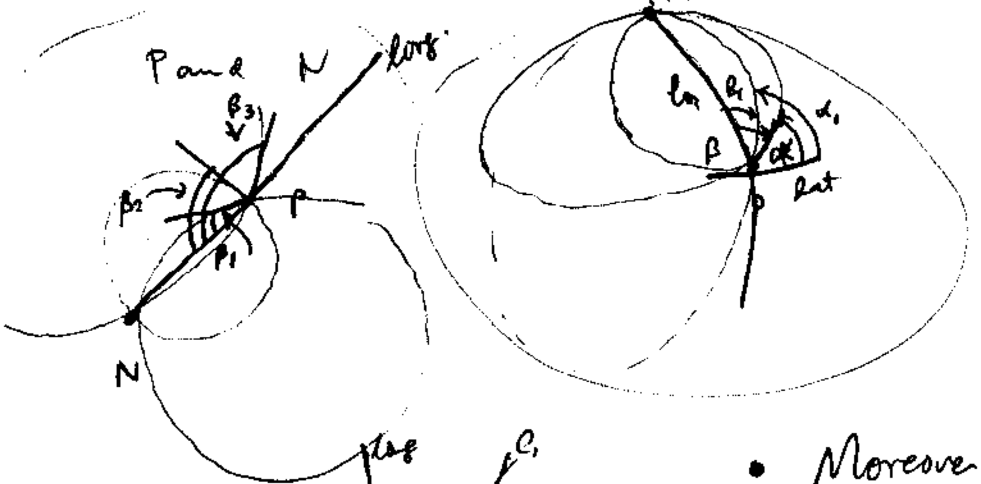
and α = α',

we've done. OR

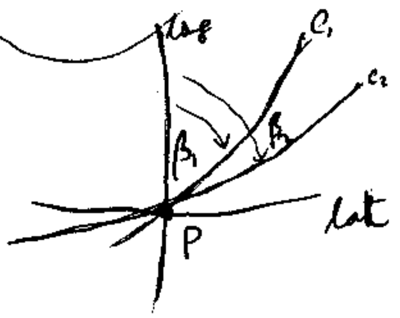
and β = β'

Note that at P ∈ S² there are ∞ly many circles on S² containing
(not nec. gt circle)

P and N



and any angle 0 ≤ α ≤ π/2
can be found between
this circle and line of
lat or long

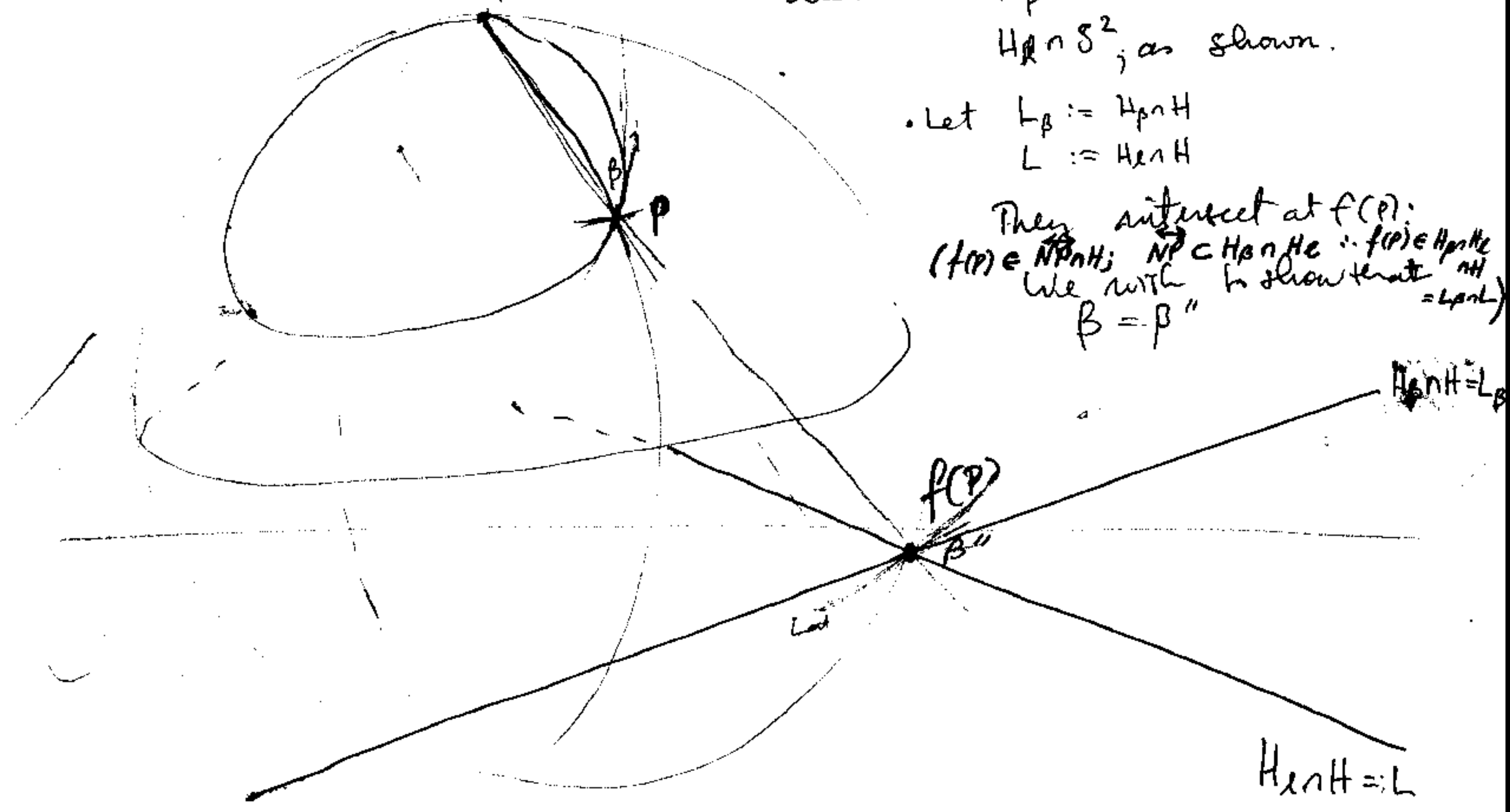


• Moreover, these circles are
the various intersection of S² with planes
passing through N and P. (See ass #6,
DGD of Rolin 35,36
or week)

So let, $P \in S_2$ and H_ℓ be the plane through N, P and O ie. $H_\ell \cap S^2$ is the line of longitude containing P. Now let H_β be the plane through N, P s.t. an angle β is made between $H_\beta \cap S^2 =: C_\beta$ and $H_\ell \cap S^2$, as shown.

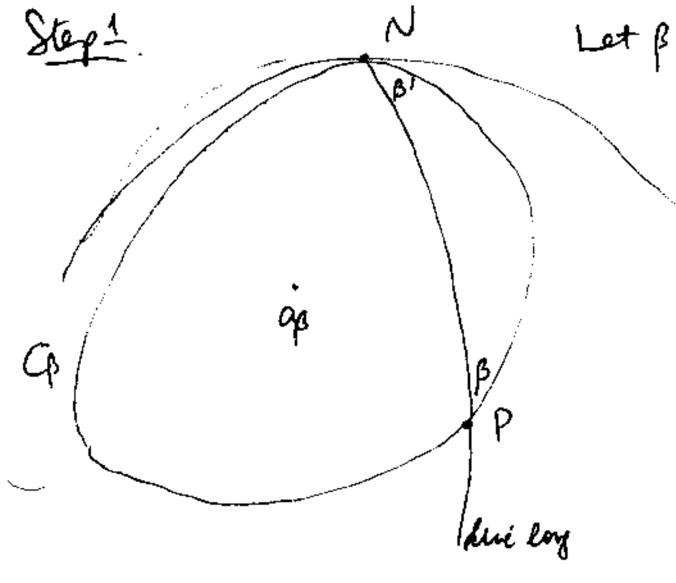
- Let $L_\beta := H_\beta \cap H$
  $L := H_\ell \cap H$

They intersect at $f(P)$:
($f(P) \in \overleftrightarrow{NP} \cap H$; $\overleftrightarrow{NP} \subset H_\beta \cap H_\ell$ $\therefore f(P) \in H_\beta \cap H_\ell \cap H = L_\beta \cap L$)
We wish to show that
$\beta = \beta''$



Note $f(C_\beta - \{N\}) = L_\beta$ (ex.)

Step 1.



Let β' be as shown. We first show $\beta = \beta'$.

Let M be the plane equidistant from N and P;

If a_β is the centre of the circle C_β, then
($\notin S^2$) $\|a_\beta - N\| = \|a_\beta - P\|$ because $P, N \in C_\beta$
(Euclidean) $H_\beta \cap \{v \mid \|v - a_\beta\| = r\}$
r radius C_β

$\therefore a_\beta \in M$. Moreover, we know $\overleftrightarrow{NP} \perp M$,
$\therefore H_\beta \perp M$.

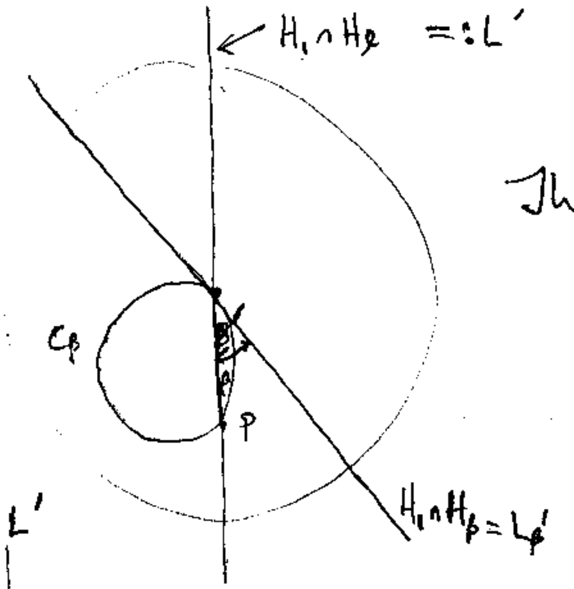
Hence, (ex. 37) $R_M(H_\beta) = H_\beta$; Thus $R_M(C_\beta) = C_\beta$!

Since R_M takes β to β', and is an isometry $\beta = \beta'$

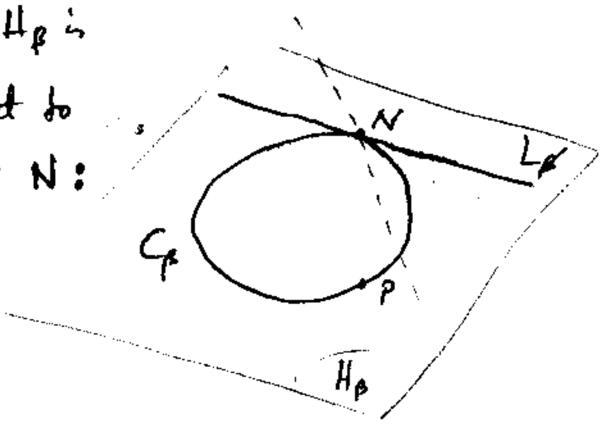
Step 2 View from top. Let H_1 = plane with eqn $z = 1$.

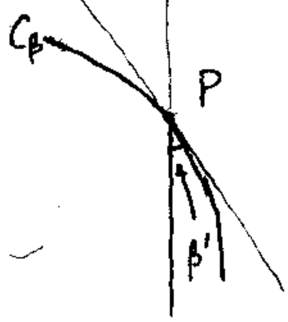
(H_1 is tangent to S^2 at N) Let $L' = H_1 \cap H_e$

$L_\beta' = H_1 \cap H_\beta$.

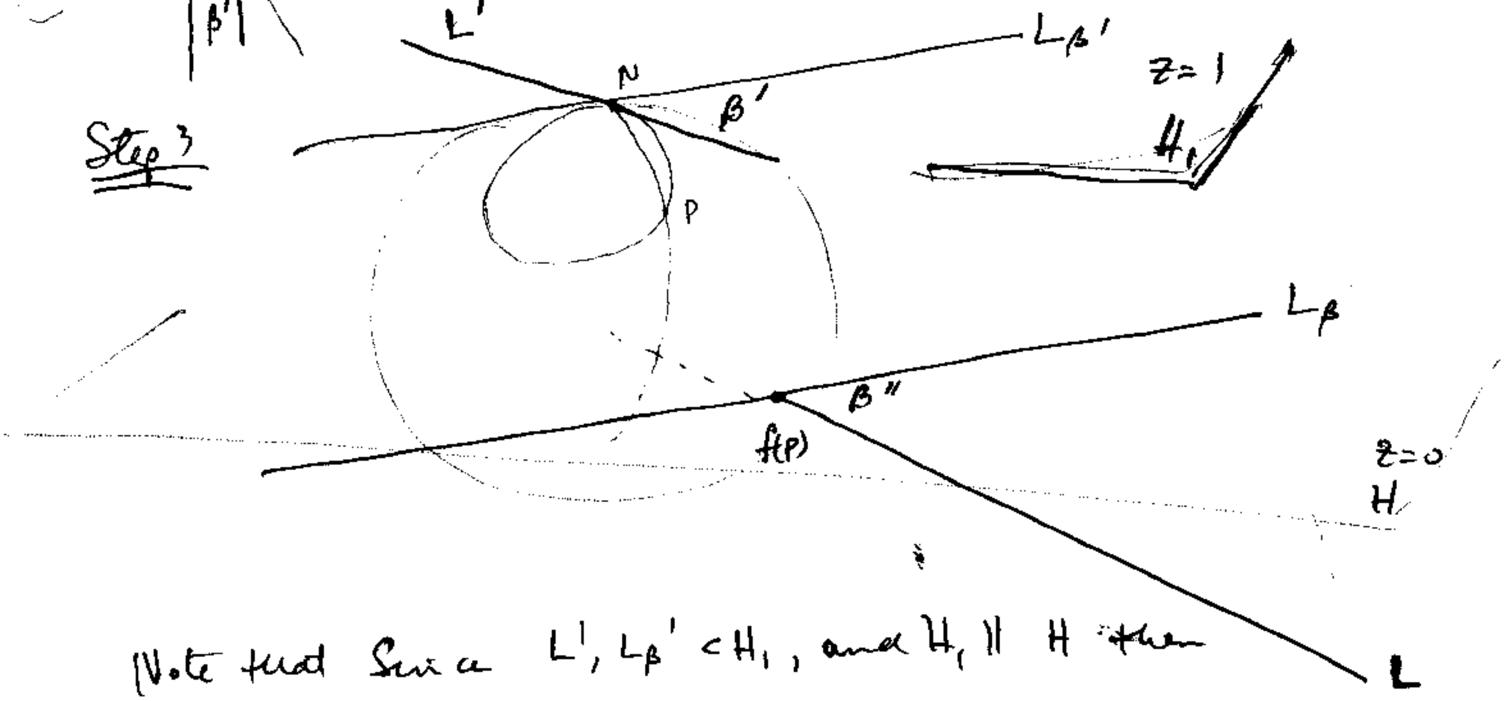


Then; β' is the angle as shown between L' and L_β', since $C_\beta \subset H_\beta$ is tangent to L_β' at N:





Step 3



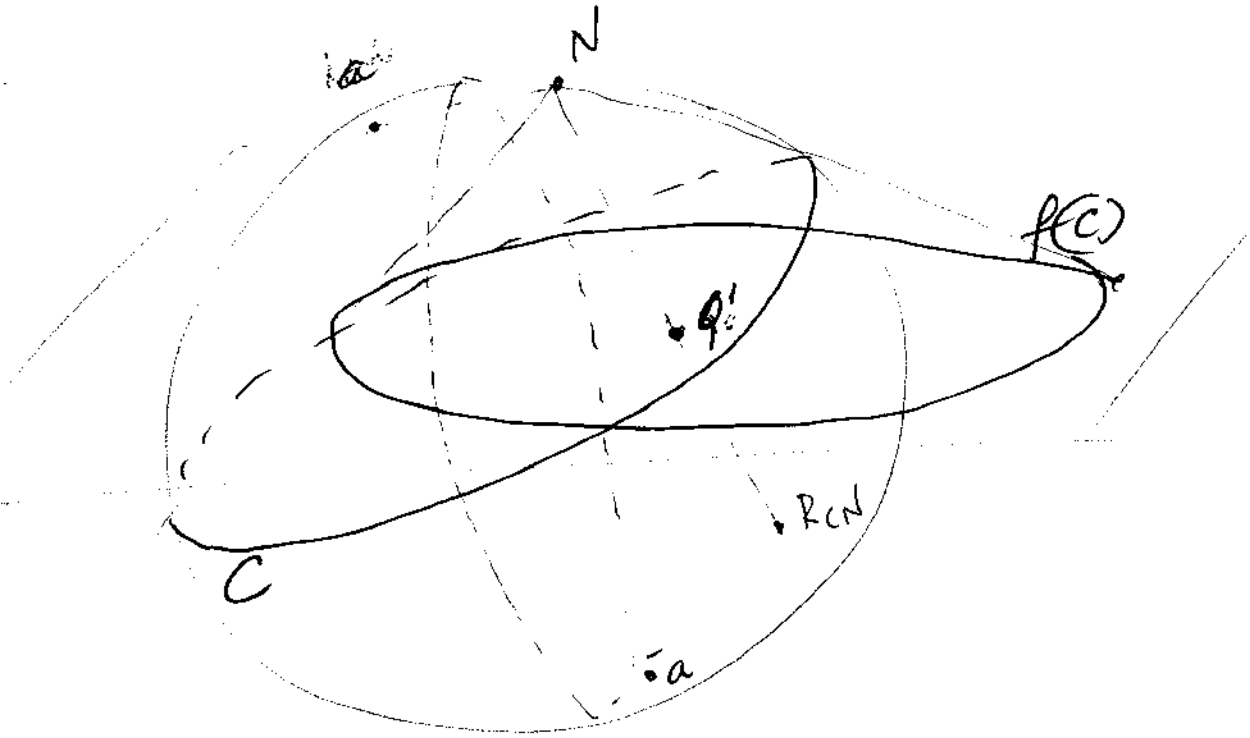
Note that since $L', L_\beta' \subset H_1$, and $H_1 \parallel H$ then

ex: $L' = H_1 \cap H_e \parallel H \cap H_e = L$

p to and $L_\beta' = H_1 \cap H_\beta \parallel H \cap H_\beta = L_\beta$

Now, the translation $g(v) = v + f(P) - N$ takes N to f(P), L_β' to L_β and L' to L $\therefore$ it takes β' to β''. Since g is an isom, $\beta'' = \beta' = \beta$ ☒

On an #6 you'll show f maps ~~great~~ circles on S^2 to circles or lines on H



In fact, if $a = (a_1, a_2, a_3) \in S^2$ and $C = \{ v \in S^2 / a \cdot v = 0 \}$

Then, 1) if $a_3 = 0$, $f(C)$ is the line

$$a_1 u + a_2 v = 0 \quad \text{on } H = \{(u,v,0) \mid u,v \in \mathbb{R}\}$$

2) if $a_3 \neq 0$ $f(C)$ is the circle

centre $Q' = -\left(\frac{a_1}{a_3}, \frac{a_2}{a_3}, 0\right)$, radius $\frac{1}{|a_3|}$.

Suppose $a_3 \neq 0$.

Consider the effect on H of $\tilde{f} = f R_C f^{-1}$. Now, since f is not defined at N, we must exclude $Q_0 = f R_C(N)$ from the domain of $\tilde{f}$ (since $\tilde{f}(Q_0) = f R_C f^{-1}(f R_C N) = f(N)$, which is not defined.

What is this point? Well, $R_C N = N - 2(a \cdot N)a$

$$= (0,0,1) - 2a_3(a_1, a_2, a_3) = (-2a_1a_3 - 2a_2a_3,\ 1 - 2a_3^2)$$

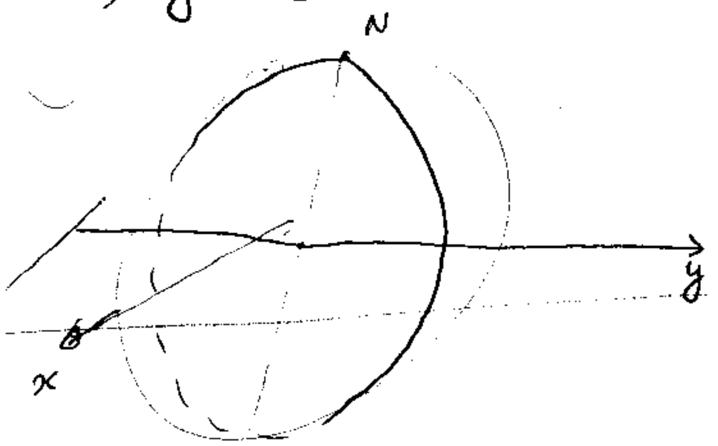
Thus $f R_C N = \left(\frac{-2a_1a_3}{2a_3^2}, \frac{-2a_2a_3}{2a_3^2}, 0\right) = -\left(\frac{a_1}{a_3}, \frac{a_2}{a_3}, 0\right)$!

$f(x,y,z) = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$

That is, the excluded pt is the centre of the circle $f(C)$ above.

If $a_3 = 0$, $N \in C$, $f R_C N = f(0,0,1) =$ not def'd i.e. Since $f^{-1} : H \to S^2 - \{N\}$, $R_C : S^2 - \{N\} \to S^2 - \{N\}$, so $f R f^{-1}$ is def'd everywhere!

e.g. $C = \{v \in S^2 \mid (1,0,0)\cdot v = 0\}$ longit. (90° E, W).



If $P = (0,y,z) \in C$

$f(P) = \left(0, \frac{y}{1-z}, 0\right)$

Hence $f(C)$ is the y-axis.

Then $R_C(x,y,z) = (x,y,z) - 2x(1,0,0) = (-x,y,z)$

So that $fR_Cf^{-1}(u,v,0) = ?$

$f^{-1}(u,v,0) = \frac{1}{(u^2+v^2+1)}(2u, 2v, u^2+v^2-1)$, so $\sigma = 1+u^2+v^2$

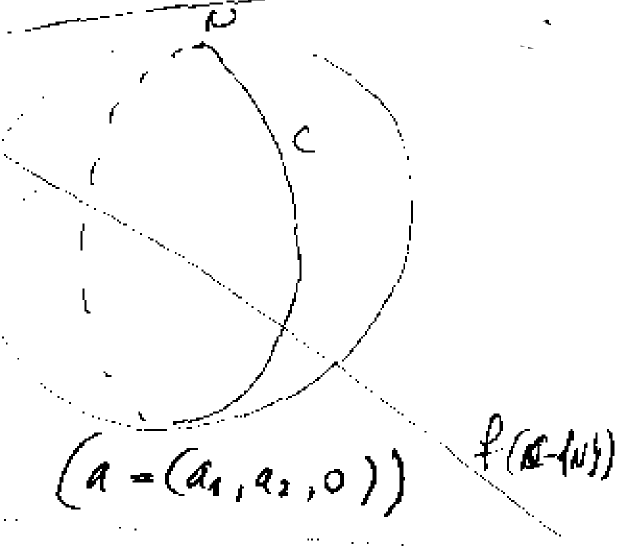
$$fR_cf^{-1}(u,v,0) = fR_c\left(\frac{2u}{1+u^2+v^2}, \frac{2v}{1+u^2+v^2}, \frac{u^2+v^2-1}{1+u^2+v^2}\right)$$

$$= f\left(-\frac{2u}{\sigma}, \frac{2v}{\sigma}, \frac{\sigma-2}{\sigma}\right) \qquad 1-z = \frac{\sigma-\sigma+2}{\sigma} = \frac{2}{\sigma}$$

$$= \left(\frac{-2u}{\sigma\cdot 2/\sigma}, v, 0\right)$$

$$= (-u, v, 0);$$

In H (ignoring the last coord)
This is reflection in the line $u=0$, i.e. the "y-axis".



Theorem If $a_3 = 0$, C is a st circle through N, $\tilde{f}$ is defined everywhere and is reflection in $f(C)$

$$P = f^{-1}(Q) = (x, y, z)$$

Pf. Again, let $Q = \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$.

Then if $R_C P = P' = (x', y', z')$, we know

$$(x', y', z') = (x, y, z) + s(a_1, a_2, 0)$$

with $s = -2a \cdot P = -2(a_1 x + a_2 y)$.

Hence, $\tilde{f}(Q) - Q = f R_C P - Q = f(P') - Q$

$$= \left(\frac{x'}{1-z} - \frac{x}{1-z}, \frac{y'-y}{1+z}, 0\right)$$

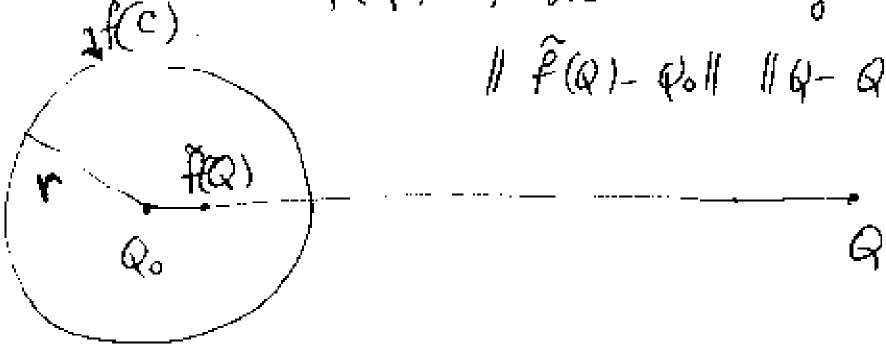
$$= \frac{1}{1-z}(sa_1, sa_2, 0) = \frac{s}{1-z}(a_1, a_2, 0)$$

$$= -\frac{2(a_1 x + a_2 y)}{1-z}(a_1, a_2, 0) = -2((a_1, a_2, 0) \cdot Q)\, a$$

i.e. $\tilde{f}(Q) = Q + 2((a_1, a_2, 0) \cdot Q)a$ — 'reflection' in $f(C) = L = \{(u, v, 0) \mid a_1 u + a_2 v = 0\}$

Recall; $a_3 \neq 0$; $f(C)$ is circle centre $\underset{Q_0 \,''}{-\left(\frac{a_1}{a_3}, \frac{a_2}{a_3}, 0\right)}$, radius $\frac{1}{|a_3|}$.

Theorem If $a_3 \neq 0$,
$\tilde{f}(Q)$ is on the ray $\overrightarrow{Q_0 Q}$ s.t.
$\| \tilde{f}(Q) - Q_0 \| \, \| Q - Q_0 \| = r^2$



Pf. Let $Q = (u, v, 0)$, $P = f^{-1}(Q) = (x, y, z)$

$= \left(\frac{x}{1-z}, \frac{y}{1-z}, 0\right)$ $\quad = \frac{1}{(1+u^2+v^2)}(2u, 2v, u^2+v^2-1)$

We know $Q_0 = \left(-\frac{a_1}{a_3}, -\frac{a_2}{a_3}\right)$;

Let $R_e P = P' = P - 2(a \cdot P)\, a = P + sa \quad (s = -2(a \cdot P))$.

$= (x', y, z') = (x + sa_1, y + sa_2, z + sa_3)$

Now, $\tilde{f}(Q) = f(P') = \left(\frac{x + sa_1}{1 - z - sa_3}, \frac{y + sa_2}{1 - z - sa_3}, 0\right)$

$\therefore \tilde{f}(Q) - Q_0 = \left(\frac{x + sa_1}{1 - z - sa_3} + \frac{a_1}{a_3}, \frac{y + sa_2}{1 - z - sa_3} + \frac{a_2}{a_3}, 0\right)$

$$= \left(\frac{a_3 x + s a_1 a_3 + a_1 - a_1 z - s a_1 a_3}{1 - z - sa_3}, *, 0\right)$$

$$= \left(\frac{a_3 x + a_1 - a_1 z}{1 - z - sa_3}, \frac{a_3 y + a_2 - a_2 z}{1 - z - sa_3}, 0\right)$$

On the other hand,

$$Q-Q_0 = \left(\frac{x}{1-z}+\frac{a_1}{a_3}, \frac{y}{1-z}+\frac{a_2}{a_3}, 0\right)$$

$$= \left(\frac{a_3x+a_1-a_1z}{1-z}, \frac{a_3y+a_2-a_2z}{1-z}, 0\right)$$

$$\therefore \tilde{f}(Q)-Q_0 = \frac{(1-z)}{(1-z-sa_3)}\cdot(Q-Q_0)$$ i.e.

$\tilde{f}(Q)$ is on the ray $\overrightarrow{QQ_0}$

Moreover, $\|\tilde{f}(Q)-Q_0\|\,\|Q-Q_0\| = \frac{|1-z|}{|1-z-sa_3|}\cdot\|Q-Q_0\|^2$

Switch to (u,v) coords :

But $\|Q-Q_0\|^2 = \left\|\left(u+\frac{a_1}{a_3}, v+\frac{a_2}{a_3}\right)\right\|^2 = u^2+\frac{2a_1}{a_3}u+\frac{a_1^2}{a_3^2}+v^2+\frac{2a_2}{a_3}v+\frac{a_2^2}{a_3^2}$

Now, $1-z = \frac{1+u^2+v^2-u^2-v^2+1}{1+u^2+v^2} = \frac{2}{1+u^2+v^2}$

$1-z-sa_3 = \frac{2}{1+u^2+v^2} + 2a_3(a\cdot P)$ $\quad (s=-2(a\cdot P))$

But $a\cdot P = \frac{(a_1, a_2, a_3)\cdot(2u, 2v, u^2+v^2-1)}{1+u^2+v^2}$

$$= \frac{2ua_1+2va_2+a_3u^2+a_3v^2-a_3}{1+u^2+v^2}$$

$$\therefore\ 1 - z - Sa_3 = \frac{\overset{2(a_1^2+a_2^2+a_3^2)}{\overset{\shortparallel}{2}} + 4ua_1a_3 + 4va_2a_3 + 2a_3^2u^2 + 2a_3^2v^2 - 2a_3^2}{1+u^2+v^2}$$

$$= \frac{2a_3^2\left(\frac{a_1^2}{a_3^2} + \frac{a_2^2}{a_3^2} + 2\frac{ua_1}{a_3} + 2\frac{va_2}{a_3} + u^2 + v^2\right)}{1+u^2+v^2}$$

$$= \frac{2a_3^2\,\|Q-Q_0\|^2}{1+u^2+v^2}$$

$$\therefore\ \frac{|1-z|}{|1-z-Sa_3|}\cdot\|Q-Q_0\|^2 = \frac{2}{(1+u^2+v^2)}\ \frac{(1+u^2+v^2)}{2a_3^2\,\|Q-Q_0\|^2}\cdot\|Q-Q_0\|^2$$

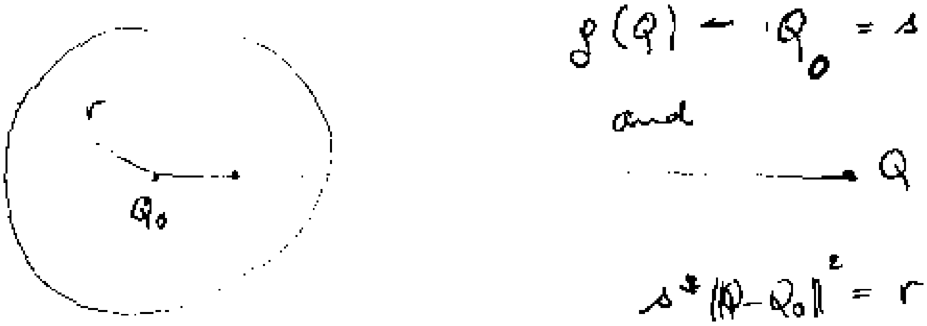
$$= \frac{1}{a_3^2} = r^2 \quad \text{as req'd!}$$

Remark: This map $H \longrightarrow H$ induced by the isometry R_c of S^2 is not an isometry of H; it is however (ex.) conformal!

Def^n. If C is a circle in R^2, inversion in the circle C is defined as before. i.e.

$$\rho(Q) - Q_0 = s(Q - Q_0)$$

and



$$s^2 \|Q - Q_0\|^2 = r^2$$

i.e. $s = \frac{r^2}{\|Q - Q_0\|^2}$

i.e. $\rho(Q) = Q_0 + \frac{r^2}{\|Q - Q_0\|^2}(Q - Q_0)$

Rmk: This has a simple formula in complex coordinates.

$$\rho(z) = z_0 + \frac{r^2}{|z - z_0|^2}(z - z_0) = z_0 + \frac{r^2}{(z - z_0)(\bar{z} - \bar{z}_0)}(z - z_0)$$

$$= z_0 + \frac{r^2}{\bar{z} - \bar{z}_0}$$

e.g. $z_0 = 0$, $r = 1$; $\rho(z) = \frac{1}{\bar{z}}$

Remark

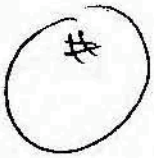
§ The Hyperbolic Plane

Plane 

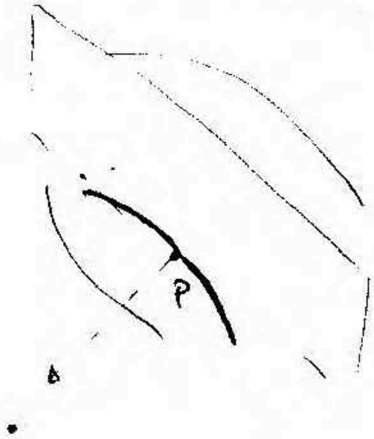
~~Torus~~

Flat (as is the torus)  globally; locally if written as $\mathbb{R}^2/\Gamma$.)

cylinder, etc 

sphere 

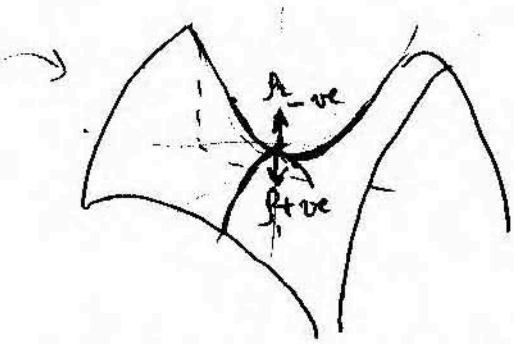
curved:

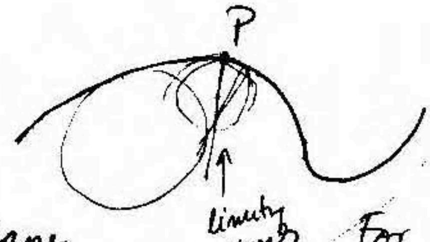


normal plane at P

centre of sphere

$z = x^2 - y^2$





The radius of curvature of a plane curve is the radius of a circle which most closely approximates the curve at P (Newton)

$$\text{Curvature} := \frac{1}{\text{radius of curvature}} =: \rho$$

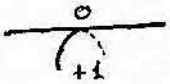
Signed to distinguish

so here, $\rho_1\rho_2$ is a measure.

For a surface with a well-defined inside & outside (unlike K, eg), we ~~say~~ can cut ~~[illegible]~~ at P with many normal planes. For each plane, we get a ρ. When ρ_1 & ρ_2 are maximum & minimum (this occurs at planes which are in fact $\perp$ ~~normal~~ to each other, too!) we call $K = \frac{1}{\rho_1\rho_2}$ the Gaussian curvature.

plane : $K = 0$

cylinder $K = 0$



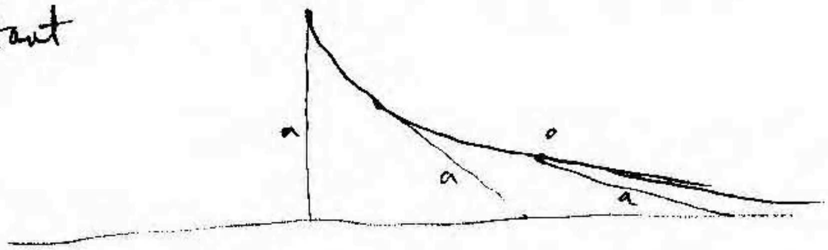
$0 \cdot 1 = 0$

Sphere $K = +1$.

Search for a surface of constant negative curvature first yielded an analogue of the cylinder; it's called the pseudosphere. (Minding 1839)

It is a surface of revolution, of a curve known as the tractrix

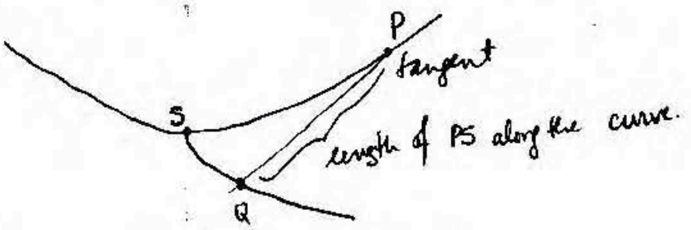
The tractrix was known to Newton (1676), who defined it as the curve with the property ~~such~~ the length of its tangent from point of contact to the x-axis is constant

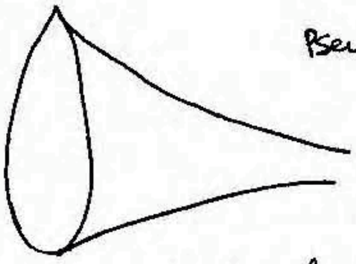


or, path of a stone pulled by a string of length a as you walk along the x-axis. Huygens

Another def^n is as the involute of the catenary (shape a rope or chain makes when freely suspended from its ends.) ~~freely~~

$v = \cosh u = \frac{e^u + e^{-u}}{2}$





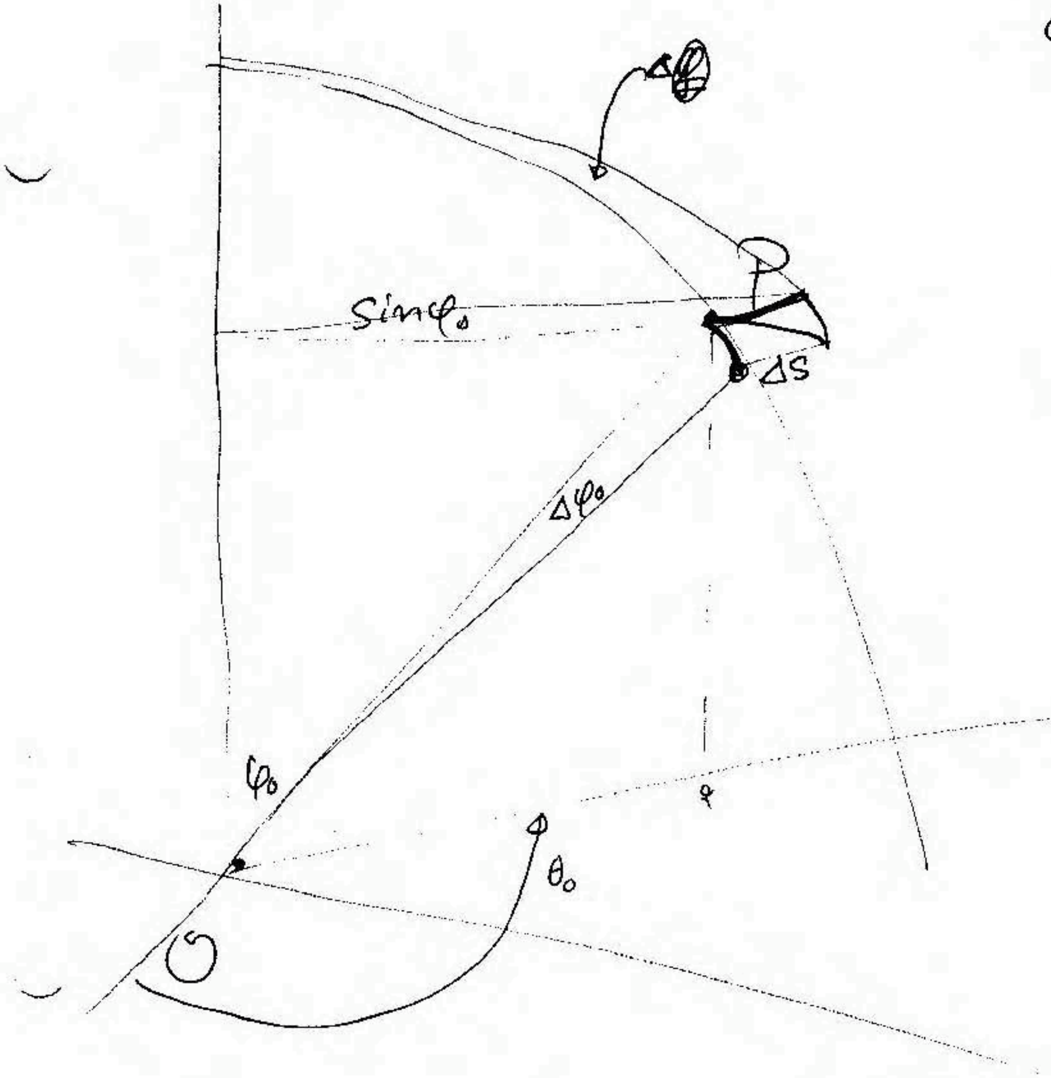
Pseudosphere.

(has finite area, volume)

It turns out that the centre of the curvature at Q is P! (of course)

— When looking for coords on Pseudosphere that make the distance function simple, end up with $\mathbb{H}^n$ – half plane on S^2

alternative way to describe distance on S^2:

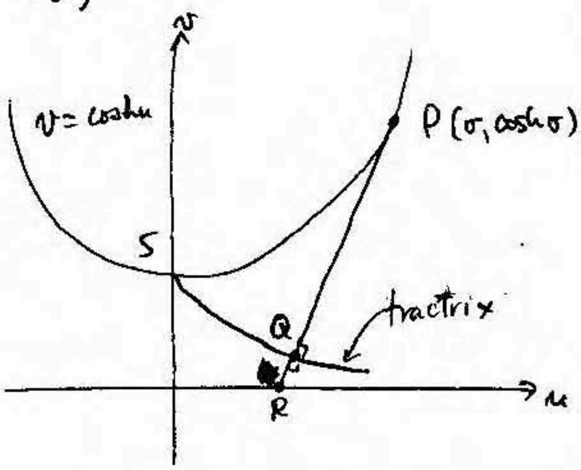


$$\Delta S^2 \approx \Delta\varphi_0^2 + \sin^2\varphi_0 (\Delta\theta)^2$$

$$ds^2 = d\varphi_0^2 + \sin^2\varphi_0 \, d\theta^2$$

Theorem 4.1 The pseudosphere has constant curvature = −1, and is locally isometric to $\mathbb{H}^2$, which is defined to be the upper half plane with ~~has~~ distance function (locally) defined by $ds^2 = \frac{dx^2+dy^2}{y^2}$.

pf.



$$PQ = \overset{\text{length}}{\text{arc}}\,PS$$

$$= \int_0^\sigma \sqrt{\underbrace{\underbrace{1+\left(\frac{dv}{du}\right)^2}_{1+\sinh^2}}_{\cosh}}\;du = \sinh\sigma$$

~~Hence~~ $R = (\sigma - \coth\sigma, 0)$

(since $\frac{y-\cosh\sigma}{x-\sigma} = \sinh\sigma$, and $y = 0$, so $(x-\sigma) = -\coth\sigma$
$\therefore x = \sigma - \coth\sigma$)

Hence, $PR = \sqrt{\coth^2 + \cosh^2\sigma} = \cancel{\sqrt{\coth^2 \ldots}} = \cosh\sigma\sqrt{1+\operatorname{cosech}^2\sigma}$
$= \cosh\sigma\,\coth$
$= \frac{\cosh^2\sigma}{\sinh\sigma}$

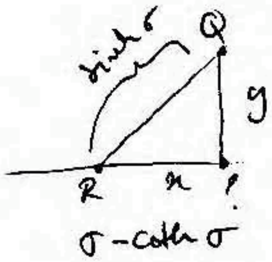
~~so QR =~~

Thus, $QR = PR - PQ = \frac{\cosh^2\sigma - \sinh^2\sigma}{\sinh\sigma} = \frac{1}{\sinh\sigma} = \frac{1}{PQ}$.

However, PQ is the radius of curvature $\underline{\underline{\text{at } Q}}$ of the normal section cut by the (u, v)-plane, and QR is one cut by the plane normal to (u, v) through PQ.

These are in fact the max & min, & have opposite signs, so Gaussian curvature $= PQ \cdot \frac{-1}{PQ} = -1$.

To get coord, first find coords of Q:



$\frac{y}{x} = \sinh\sigma =: s$

$x^2 + y^2 = \frac{1}{s^2}$ $\quad \therefore \; x^2(1+s^2) = \frac{1}{s^2}$

$\therefore \; x^2 = \frac{1}{c^2 s^2}$

$\therefore \; y = \frac{1}{cs}$

hence, $P = \sigma - \coth\sigma + \frac{1}{cs} = \sigma - \left(\frac{c^2 - 1}{cs}\right) = \sigma - \frac{s^2}{cs}$

$= \sigma - \tanh\sigma$

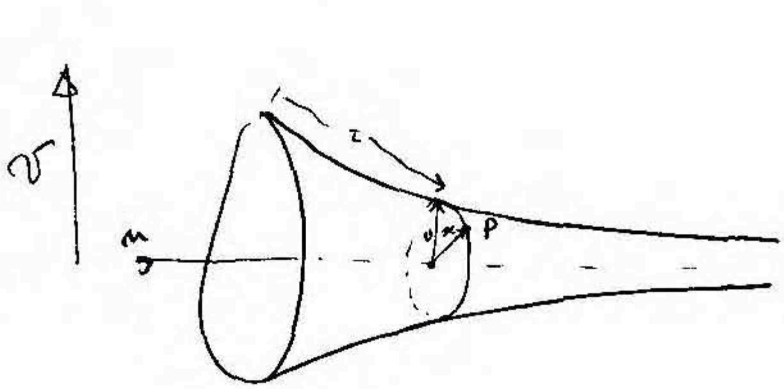
Hence, $y = s.x = \frac{1}{c} = \operatorname{sech}\sigma$. Hence $u = \sigma - \tanh\sigma$, $v = \operatorname{sech}\sigma$ parametrizes tractrix.

The arc length along v-colm the tractrix is

$$\tau = \int_0^\sigma \sqrt{du^2 + dv^2} = \int\sqrt{(d\sigma - \operatorname{sech}^2\sigma\, d\sigma)^2 + \left(- \frac{\sinh\sigma}{\cosh^2\sigma}\, d\sigma\right)^2}$$

$$= \int_0^\sigma \tanh\sigma\, d\sigma = \log\cosh\sigma \quad ; \quad \text{hence}$$

$\cosh\sigma = e^\tau$, so $v = e^{-\tau}$



Take τ and angle x as coords.

So length subtended by angle dx at P is $vdx = e^{-\tau}dx$, hence infinitesimal distance between (x,τ) & $(x+dx, \tau+d\tau)$ is

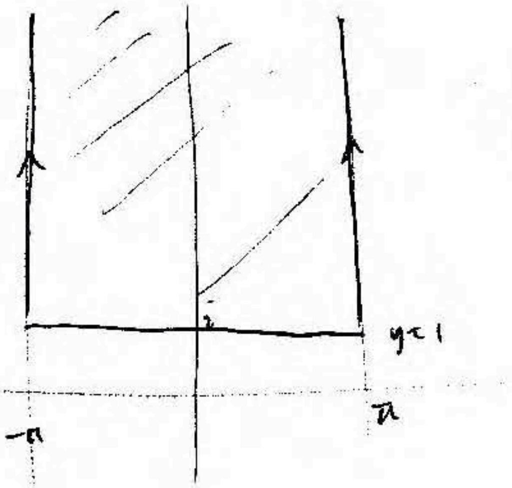
$$ds^2 = e^{-2\tau}dx^2 + d\tau^2$$
(real distance)

So let variable $y = e^{\tau}$ $\left(= \frac{1}{v}\right)$

Then $dy = e^{\tau}d\tau$ so $ds^2 = e^{-2\tau}.(dx^2+dy^2)$

$$= \frac{dx^2+dy^2}{y^2}.$$

Since $y = e^{\tau} > 0$, only the upper half plane is relevant.



N.B.

Remark: Since $ds = \frac{\sqrt{dx^2+dy^2}}{y_0}$ at (x_0, y_0), is the $\frac{\text{Euclidean distance}}{y_0 = \text{const}}$, the angles at (x_0,y_0) will be the same, measured using either metric.

reflect in y-axis

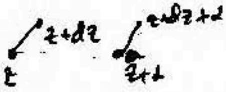
$$ds = \frac{|dz|}{\text{Im } z}$$ ✓ ✓

$z \mapsto -\bar{z}$ ✓ ✓

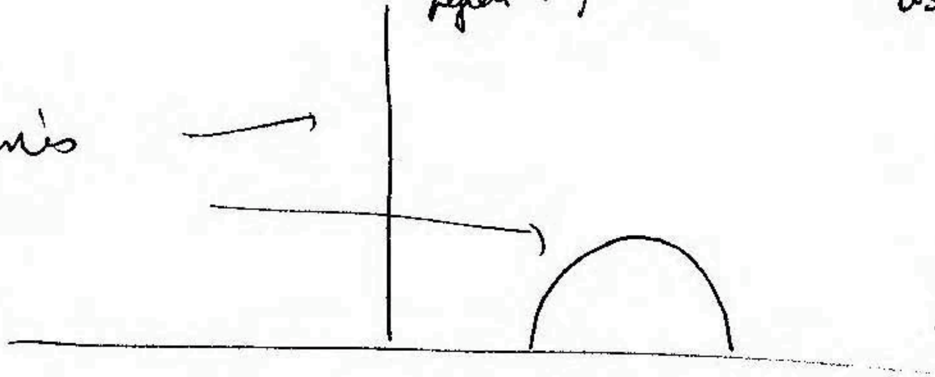
gives y-axis

$\leftarrow z \mapsto z + \alpha, \ \alpha \in \mathbb{R}$

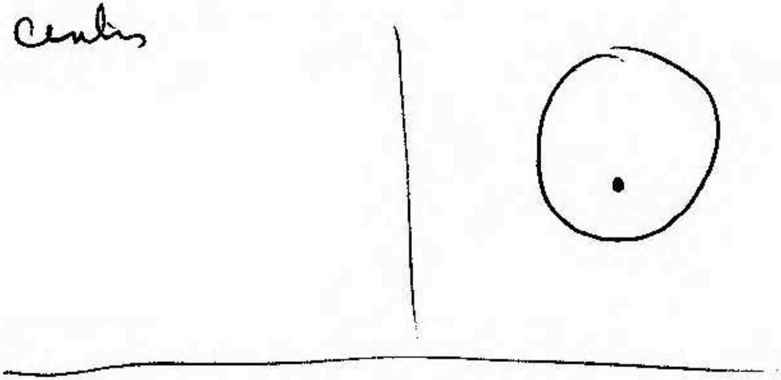
all lines



lines



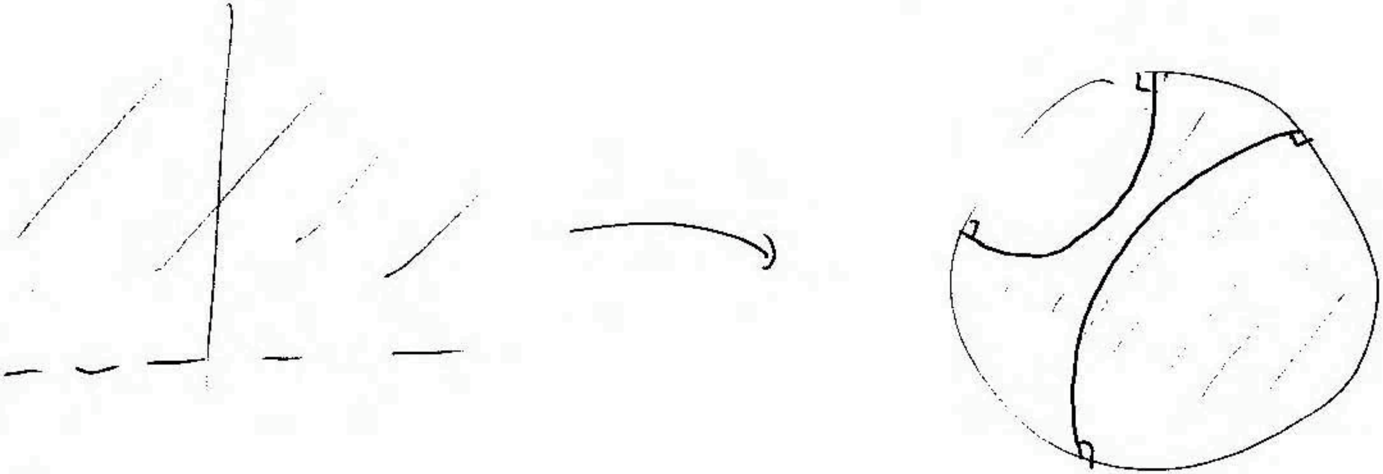
circles are Euclidean, but have differen centers

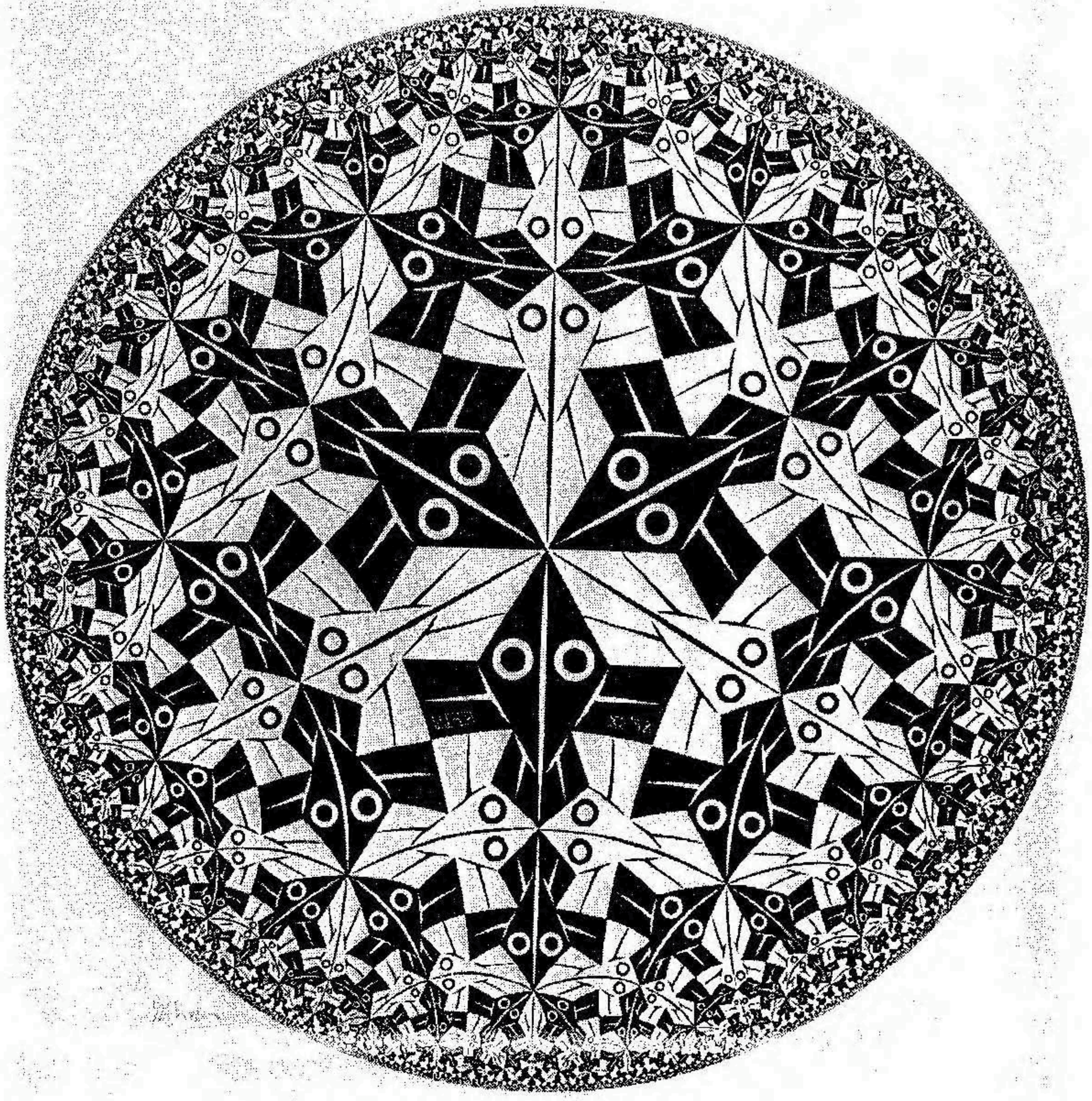


$$\mathbb{H}^2 \longrightarrow \mathbb{D}^2$$

Poincaré disc model.

$$z \longmapsto \frac{iz+1}{z+i}$$





GURE 4.10. Picture by M.C. Escher. Used with permission of the Collection aags Gemeentemuseum—The Hague.