MAT3320 Assignment 1

Total: 10 marks. Due date: Tuesday, Sept 29, before 6:00pm. In MATH Department (585 King Edward), there is a Drop-Box. You need to put your assignment into the box **before 6:00pm** on the due date. Late assignments will not be accepted.

1. (5 marks) Consider the following differential equation $y'' + xy' - 3y = 0$, near $x_0 = 0$.

   (a) (2 marks) Find the coefficient recursion relation for the general series solution.

   (b) (2 marks) Find two linearly independent solutions by solving the recursive relation.

   (c) (1 mark) Find the particular solution y such that $y(0) = 1$, $y'(0) = 3$.

   Solution: (a) Note that $x_0 = 0$ is an ordinary point. Hence the general solution is given by $y(x) = \sum_{m=0}^{\infty} a_m x^m$. Then

$$\begin{aligned} y'(x) &= \sum_{m=0}^{\infty} m a_m x^{m-1}, \\ y''(x) &= \sum_{m=0}^{\infty} m(m-1) a_m x^{m-2}. \end{aligned}$$

   Substituting these into DE we have

$$\sum_{m=0}^{\infty} m(m-1)a_m x^{m-2} + x\sum_{m=0}^{\infty} m a_m x^{m-1} - 3\sum_{m=0}^{\infty} a_m x^m = 0,$$

   i.e.,

$$\sum_{m=0}^{\infty}(m+2)(m+1)a_{m+2}x^m + \sum_{m=0}^{\infty} m a_m x^m - 3\sum_{m=0}^{\infty} a_m x^m = 0.$$

   Collecting like powers of x we get

$$\sum_{m=0}^{\infty} \left[(m+2)(m+1)a_{m+2} + a_m(m-3)\right] x^m = 0.$$

   By the property of VAC,

$$(m+2)(m+1)a_{m+2} + (m-3)a_m = 0, \quad m \geq 0; \Rightarrow$$

$$a_{m+2} = -\frac{(m-3)a_m}{(m+2)(m+1)}, \quad m \geq 0.$$

(b) This implies that

$$a_2 = \frac{3}{2}a_0, \quad a_3 = \frac{1}{3}a_1, \quad a_4 = \frac{1}{4(3)}a_2 = \frac{3(1)}{4(3)2}a_0,$$
$$a_5 = 0, a_7 = 0, ..., a_n = 0 \quad \text{for all odd } n \geq 5;$$
$$a_6 = -\frac{a_4}{6(5)} = -\frac{3(1)}{6!}a_0, \quad a_8 = \frac{3(1)3}{8!}a_0;$$
$$a_{2n} = 3(-1)^n \frac{(2n-5)!!}{(2n)!}a_0, \quad n \geq 3.$$

Thus we get

$$y = a_1(x + \frac{1}{3}x^3) + a_0 \left[1 + \frac{3}{2}x^2 + \frac{3}{4!}x^4 + 3\sum_{n=3}^{\infty}(-1)^n \frac{(2n-5)!!}{(2n)!}x^{2n} \right],$$

where

$$y_1 = x + \frac{1}{3}x^3, \quad y_2 = 1 + \frac{3}{2}x^2 + \frac{3}{4!}x^4 + 3\sum_{n=3}^{\infty}(-1)^n \frac{(2n-5)!!}{(2n)!}x^{2n}.$$

(c) From $y(0) = 1$, $y'(0) = 3$ we imply that $a_0 = y(0) = 1$, $a_1 = y'(0) = 3$. Thus

$$y = 1 + 3x + \frac{3}{2}x^2 + x^3 + \frac{3}{4!}x^4 + 3\sum_{n=3}^{\infty}(-1)^n \frac{(2n-5)!!}{(2n)!}x^{2n}.$$

2. (2 marks) Legendre's equation $(1-x^2)y'' - 2xy' = 0$. Show that $y(x) = \ln\frac{1+x}{1-x}$ is a solution of the equation.

Solution: From $y(x) = \ln\frac{1+x}{1-x}$ we imply that

$$y'(x) = \frac{1}{1+x} + \frac{1}{1-x}; y''(x) = -\frac{1}{(1+x)^2} + \frac{1}{(1-x)^2}.$$

Substitute them into the differential equation,

$$\begin{aligned}(1-x^2)y'' - 2xy' &= (1-x^2)\left(-\frac{1}{(1+x)^2} + \frac{1}{(1-x)^2}\right) - 2x\left(\frac{1}{1+x} + \frac{1}{1-x}\right) \\ &= -\frac{1-x^2}{(1+x)^2} + \frac{1-x^2}{(1-x)^2} - \frac{2x}{1+x} - \frac{2x}{1-x} \\ &= -\frac{1-x}{1+x} + \frac{1+x}{1-x} - \frac{2x}{1+x} - \frac{2x}{1-x} \\ &= -\frac{1+x}{1+x} + \frac{1-x}{1-x} = -1 + 1 = 0.\end{aligned}$$

3. (1 mark) What is the polynomial solution of the following Legendre's equation: $(1 - x^2)y'' - 2xy' + 6162y = 0$?

   Solution: From $n(n+1) = 6162$ we get a positive solution $n = 78$. Hence the polynomial solution is $P_{78}(x)$.

4. (2 marks) Let $f(x) = x^3$, $2 < x < 4$. Find the Fourier-Legendre expansion.

   Solution: $P_n(x)$ are only defined on $-1 < x < 1$. So we need to make a linear transformation from (2,4) to (-1,1). Let $s = ax + b$. Then $-1 = a(2) + b, 1 = a(4) + b$. Then $a = 1, b = -3$, $s = x - 3$. Let

$$g(s) = f(x) = f(s+3) = (s+3)^3 = s^3 + 9s^2 + 27s + 27.$$

   Note that

$$s^3 = \frac{2}{5}P_3(s) + \frac{3}{5}s; \quad s^2 = \frac{2}{3}P_2(s) + \frac{1}{3}; s = P_1(s); 1 = P_0(s).$$

   We imply that

$$g(s) = \frac{2}{5}P_3(s) + \frac{3}{5}s + 9[\frac{2}{3}P_2(s) + \frac{1}{3}] + 27s + 27 = \frac{2}{5}P_3(s) + 6P_2(s) + \frac{138}{5}s + 30$$

$$= \frac{2}{5}P_3(s) + 6P_2(s) + \frac{138}{5}P_1(s) + 30P_0(s), \Rightarrow$$

$$f(x) = \frac{2}{5}P_3(x-3) + 6P_2(x-3) + \frac{138}{5}P_1(x-3) + 30P_0(x-3).$$