

THE UNIVERSITY OF BRITISH COLUMBIA

# Mid-Term Test I

Course Code:	MATH221:104 Winter 2016
Paper Title:	Matrix Algebra

Time allowed: FORTY-FIVE MINUTES

Number of pages: 7

Read these instructions carefully

- Answer all 3 questions.
- Not all questions carry equal marks.
- Show all working.
- Write your answers in the spaces provided.
- Extra paper is available upon request.
- Use black or blue ink only, no pencil.
- Do not use ink erasers or white-out.
- Calculators, books or notes are not permitted.

Surname:	Model Answers for [A]
Given name:	_____
Student ID number:	_____

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Q1	Q2	Q3	Total Marks
<input type="text"/>	<input type="text"/>	<input type="text"/>	<input type="text"/>

## Question 1 (10 marks)

Consider the problem

$$\begin{pmatrix} -1 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

(a) (1 mark) Write down the row representation of this system as two simultaneous equations.

$$-x_1 + 2x_2 = 1 \quad \text{(Equation 1)}$$

$$x_1 + x_2 = 2 \quad \text{(Equation 2)}$$

(b) (1 mark) Write down the column representation of the linear system as a vector equation.

$$x_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + x_2 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{(Equation 3)}$$

When you multiply a scalar with a vector, you should write the scalar in front of the vector. NOT:  $\begin{pmatrix} -1 \\ 1 \end{pmatrix} x_1$

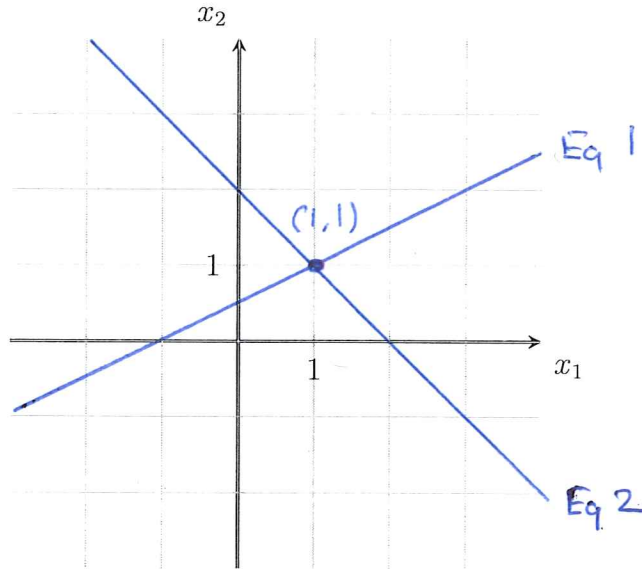
(c) (2 marks) By visual inspection, or otherwise, find the solution of this linear system. Remember to justify your answer (show all working).

$$1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + 1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \Rightarrow x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Most answers were based on Gaussian elimination, which is also fine (just longer).

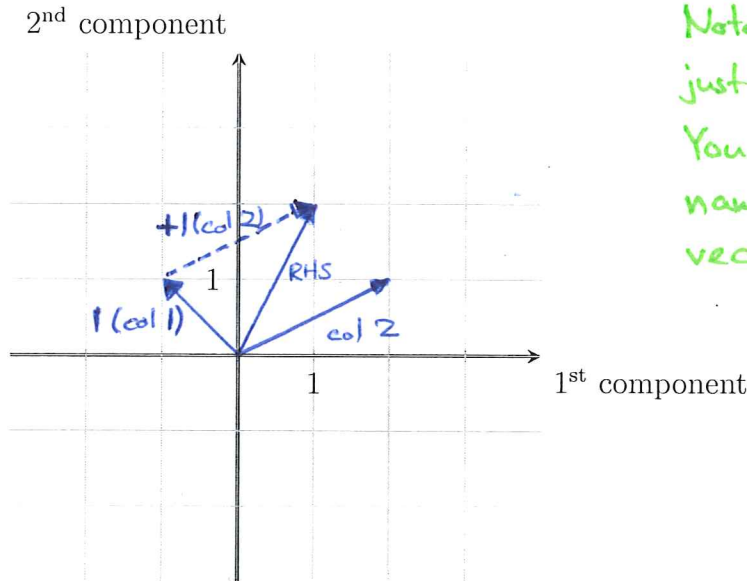
Question 1 continued...

(d) (3 marks) Make a clean sketch of the set of all points that satisfy Equation 1 and the set of all points that satisfy Equation 2 in the row representation. Clearly label all important elements of your graph.



These lines should be straight and show the right intercepts.

(e) (3 marks) Sketch the three vectors which occur in the column representation. Illustrate in your graph how the correct linear combination solves the problem in Equation 3. Don't forget proper labelling.



Note that  $x_1$  and  $x_2$  are just real numbers, not vectors. You have to give different names to the column vectors.

## Question 2 (5 marks)

Find all solutions of the problem

$$\begin{pmatrix} 2 & 1 & -1 & 1 \\ -2 & 0 & 1 & 1 \\ 4 & 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 2 \\ -2 \\ 10 \end{pmatrix}$$

and give your answer in vector form.

Augmented matrix:

$$\left( \begin{array}{cccc|c} 2 & 1 & -1 & 1 & 2 \\ -2 & 0 & 1 & 1 & -2 \\ 4 & 3 & -2 & 1 & 10 \end{array} \right) \begin{array}{l} \downarrow + \\ \downarrow + \\ \downarrow + \end{array} \begin{array}{l} x(-2) \\ \\ \end{array}$$

$$\left( \begin{array}{cccc|c} 2 & 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & -1 & 6 \end{array} \right) \begin{array}{l} \\ \downarrow + \\ \downarrow + \end{array} \begin{array}{l} \\ x(-1) \\ \end{array}$$

$$\left( \begin{array}{cccc|c} 2 & 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & -3 & 6 \end{array} \right) \begin{array}{l} \times 1/2 \\ \\ \times (-1/3) \end{array}$$

$$\left( \begin{array}{cccc|c} 1 & 1/2 & -1/2 & 1/2 & 1 \\ 0 & 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right) \begin{array}{l} \\ \downarrow + \\ \downarrow + \end{array} \begin{array}{l} \\ \\ x(-2) \\ x(-1/2) \end{array}$$

$$\left( \begin{array}{cccc|c} 1 & 1/2 & -1/2 & 0 & 2 \\ 0 & 1 & 0 & 0 & 4 \\ 0 & 0 & 0 & 1 & -2 \end{array} \right) \begin{array}{l} \downarrow + \\ \\ \end{array} \begin{array}{l} \\ \\ x(-1/2) \end{array}$$

$$\left( \begin{array}{cccc|c} \boxed{1} & 0 & -1/2 & 0 & 0 \\ 0 & \boxed{1} & 0 & 0 & 4 \\ 0 & 0 & 0 & \boxed{1} & -2 \end{array} \right) \begin{array}{l} x_1 = 1/2 x_3 \\ x_2 = 4 \\ x_4 = -2 \end{array}$$

p = pivot col  
f = free col

↓ ↓ ↓ ↓  
P P f P

$$x = \begin{pmatrix} 1/2 x_3 \\ 4 \\ x_3 \\ -2 \end{pmatrix} = \begin{pmatrix} 0 \\ 4 \\ 0 \\ -2 \end{pmatrix} + x_3 \begin{pmatrix} 1/2 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

 $(x_3 \in \mathbb{R})$ 

both are fine

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$  as this problem has 4 unknowns. In some answers,  $x$  only had 3 components.

You should always remember to identify all free variables and these may assume any real value, not just integer numbers.

Question 2 continued...

If you have spare time at the end and you want to check your answer, you don't have to do the entire elimination all over again. You can simply calculate  $Ax$  and check whether that gives you the RHS:

$$\begin{aligned} \begin{pmatrix} 2 & 1 & -1 & 1 \\ -2 & 0 & 1 & 1 \\ 4 & 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2}x_3 \\ 4 \\ x_3 \\ -2 \end{pmatrix} &= \frac{1}{2}x_3 \begin{pmatrix} 2 \\ -2 \\ 4 \end{pmatrix} + 4 \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} + x_3 \begin{pmatrix} -1 \\ 1 \\ -2 \end{pmatrix} - 2 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\ &= \begin{pmatrix} x_3 \\ -x_3 \\ 2x_3 \end{pmatrix} + \begin{pmatrix} 4 \\ 0 \\ 12 \end{pmatrix} + \begin{pmatrix} -x_3 \\ x_3 \\ -2x_3 \end{pmatrix} + \begin{pmatrix} -2 \\ -2 \\ -2 \end{pmatrix} \\ &= \begin{pmatrix} 2 \\ -2 \\ 10 \end{pmatrix} \quad \checkmark \end{aligned}$$

## Question 3 (10 marks)

(a) (2 marks) Find all values of  $s, t \in \mathbb{R}$  that render the system

$$\begin{aligned} 2x_1 + x_2 &= 4 \\ sx_2 &= t \end{aligned}$$

inconsistent.

$$s = 0 \quad \text{and} \quad t \neq 0$$

The correct notation for the set of all real numbers except zero is

$$\mathbb{R} \setminus \{0\}$$

the set of real numbers  $\uparrow$  without (minus sign for sets)  $\leftarrow$  the set that only contains 0

(b) (2 marks) Write down a matrix  $A \in \mathbb{R}^{3 \times 2}$  and a vector  $b \in \mathbb{R}^3$  such that  $x = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$  is a solution of the linear system  $Ax = b$ .

$$A = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix} \quad b = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \text{1st col of } A$$

Some answers gave a  $2 \times 3$  matrix, but still a vector  $b \in \mathbb{R}^3$ . It doesn't make sense to have 2 equations on the LHS and 3 equations on the RHS...

(c) (1 mark) A general linear system  $Ax = b$  has either no solution, a unique solution or infinitely many solutions. What about linear systems of the form  $Ax = 0$  and why?

$$x = 0 \quad \text{always solves } Ax = 0.$$

$\Rightarrow Ax = 0$  either has a unique solution (namely the "trivial solution"  $x = 0$ ) or infinitely many solutions (one of which is  $x = 0$ ).

Some students wrote that  $Ax = 0$  means either  $A = 0$  or  $x = 0$ . This is NOT true. Take e.g.  $A$  from Question 2 which is not 0, and any  $x = x_3 \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}$  ( $x_3 \in \mathbb{R}$ ) will solve  $Ax = 0$ . This zero-product rule only works for real numbers, but not with vectors and matrices.

Question 3 continued...

(d) (3 marks) Consider the five vectors

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad u_2 = \begin{pmatrix} 2 \\ 3 \\ 0 \\ 0 \end{pmatrix}, \quad u_3 = \begin{pmatrix} 4 \\ 5 \\ 0 \\ 0 \end{pmatrix}, \quad u_4 = \begin{pmatrix} 6 \\ 7 \\ 8 \\ 0 \end{pmatrix}, \quad u_5 = \begin{pmatrix} 9 \\ 10 \\ 11 \\ 12 \end{pmatrix}.$$

Are these vectors linearly independent? Do they span the space  $\mathbb{R}^4$ ?

Let  $U = (u_1, u_2, u_3, u_4, u_5) \in \mathbb{R}^{4 \times 5}$  be the matrix composed of these vectors.  $U$  has

$m = 4$  rows  
 $n = 5$  columns  
 $r = 4$  pivots.

← You should explain what  $m, n$  and  $r$  mean. Someone who didn't attend our lectures wouldn't know what you mean otherwise.

Since all 4 rows contain a pivot,  $\text{span}(u_1, u_2, u_3, u_4, u_5) = \mathbb{R}^4$ .

Since not all 5 columns contain a pivot, the vectors must be linearly dependent.

You also have to explain why these vectors are l.i. and span the full space. Just yes/no is not sufficient.

(e) (2 marks) Let  $a_1, a_2 \in \mathbb{R}^2$  be linearly dependent. Show that  $\text{span}(a_1, a_2) \neq \mathbb{R}^2$ .

Let  $A = (a_1, a_2) \in \mathbb{R}^{2 \times 2}$  be the matrix composed of these two vectors.  $a_1, a_2$  are l.d.  $\Rightarrow$

Not both columns contain a pivot ( $r=0$  or  $r=1$ ),  $\Rightarrow$

Not both rows contain a pivot  $\Rightarrow$

$\text{span}(a_1, a_2) \neq \mathbb{R}^2$

Some students considered a special case. If you did that correctly, you could still get 1 mark.

