Math 302, assignment 7 solutions

1. If X is a $N(0,1)$ r.v., find the p.d.f. of X^{-2}. (Hint: use the CDF).

   solution. We have that $\mathbf{P}(X^{-2} \le t) = \mathbf{P}(|X| \ge 1/\sqrt{t}) = 2\mathbf{P}(X \ge 1/\sqrt{t})$. Therefore we get
   $$\mathbf{P}(X^{-2} \le t) = 2\int_{1/\sqrt{t}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx.$$
   Taking the derivative gives the density
   $$f(t) = \frac{1}{\sqrt{2\pi} t^{3/2}} e^{-1/2t}.$$

2. If X is a Cauchy r.v. (density $\frac{\pi/2}{1+x^2}$), find the p.d.f. of $\sqrt{|X|}$. (Hint: same as for 1.)

   solution.
   $$\mathbf{P}(\sqrt{|X|} \le t) = \mathbf{P}(|X| < t^2) = \int_{-t^2}^{t^2} \frac{1}{\pi} \frac{1}{1+x^2} dx = \frac{1}{\pi}(\arctan(t^2) - \arctan(-t^2)).$$
   Taking aderivative we get the density (for $t > 0$):
   $$f(t) = \frac{2t}{\pi}\left(\frac{1}{1+t^4} + \frac{1}{1+t^4}\right) = \frac{4t}{\pi(1+t^4)}.$$

3. a. Prove that if $\mathbb{E}X = 0$ then $\mathbb{E}(X-t)^2$ is minimized at $t = 0$.
   b. Prove that if $\mathbb{E}X = \mu$ then $\mathbb{E}(X-t)^2$ is minimized at $t = \mu$.

   solution. a. This is a special case of b.
   b. By linearity of expectation, $\mathbb{E}(X-t)^2 = t^2 - 2(\mathbb{E}X)t + \mathbb{E}X^2 = t^2 - 2\mu t + \mathbb{E}X^2$. The derivative is $2t - 2\mu$, so the extremal value is at $t = \mu$. (As $t \to \pm\infty$, the quantity tends to $+\infty$.)

4. A gambler wins each bet with probability 0.48. They gain or lose a dollar each time. Estimate the probability that after 1000 bets the gambler breaks even or better. (total profit ≥ 0).

   solution. We want $\mathbf{P}(\text{Bin}(1000, 0.48) \ge 500)$. Using the normal approximation, this is roughly
   $$\mathbf{P}(N(480, 1000 \cdot 0.48 \cdot 0.52) \ge 500) = \mathbf{P}(N(0,1) \ge \frac{500-480}{\sqrt{249.6}}) = \Phi(1.266...).$$
   With the continuity correction this becomes
   $$\Phi(\frac{499.5-480}{\sqrt{249.6}}).$$

Extra practice problems (do not hand in): Chapter 4, problems 2,3,4,5,7,8,9,12,17,31,36