Review II

Series

$\sum_{n=1}^{\infty} a_n$ — converge / diverge

- comparison test, limit comparison
- ratio test
- root test
- integral test

$\sum_{n=1}^{\infty} (-1)^n a_n$ alternating series

converges / diverges

converges ⇓ alternating series test

$$\left[\begin{array}{l} \cdot\ a_{n+1} \le a_n \quad \text{for all } n > N \\ \cdot\ \lim_{n\to\infty} a_n = 0 \end{array}\right]$$

absolute convergence: if $\sum_{n=1}^{\infty} |a_n|$ converges

→ treat it as $\sum_{n=1}^{\infty} a_n$, any of the test above.

<u>Ex</u> $\sum_{n=1}^{\infty} \frac{n+2}{n^2+3n+5}$ ← exponent 1, ↖ exponent 2

So $\frac{n+2}{n^2+3n+5} \sim \frac{n}{n^2} = \frac{1}{n}$ (behaviour for 'big' n)

→ gives an intuition what it does

$\sum_{n=1}^{\infty} \frac{n+2}{n^2+3n+5} \sim \sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series → diverges

now prove both series diverge:

limit comparison thm: $\sum a_n$, $\sum b_n$ if $\lim_{n\to\infty} \frac{|a_n|}{|b_n|} = C > 0$

then both series converge or diverge.

$$\lim_{n\to\infty} \frac{\left|\frac{n+2}{n^2+3n+5}\right|}{\left|\frac{1}{n}\right|} = \lim_{n\to\infty} \frac{|(n+2)\cdot n|}{|(n^2+3n+5)\cdot 1|}$$

$$= \lim_{n\to\infty} \frac{n^2+2n}{n^2+3n+5} \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}} = \lim_{n\to\infty} \frac{1+\overset{\to 0}{\frac{2}{n}}}{1+\underset{\to 0}{\frac{3}{n}}+\underset{\to 0}{\frac{5}{n^2}}} = 1 > 0$$

So both $\sum_{n=1}^{\infty} \frac{n+2}{n^2+3n+5}$ and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverge because $\sum \frac{1}{n}$ does.

($\sum_{n=1}^{\infty} \frac{n+2}{n^2+3n-7}$ here n^2+3n-7 is negative for $n=1$, so start at $n=2$ to check limit comparison)

Power Series $\sum_{n=0}^{\infty} a_n \cdot (x-a)^n$

a_n: coefficient $\in \mathbb{R}$; a: constant; x: variable (any value in $\mathbb{R}$)

Question: for which values of x does the series converge?
- radius of convergence (interval of convergence)

Ex $\sum_{n=1}^{\infty} \underbrace{\frac{n^2}{3^n}}_{a_n} (x-3)^n$ — want: interval & radius of convergence

$b_n = \frac{n^2}{3^n}(x-3)^n$, constant $a = 3$

check for convergence with ratio test:

$$\lim_{n\to\infty} \frac{|b_{n+1}|}{|b_n|} = \lim_{n\to\infty} \left| \frac{\frac{(n+1)^2}{3^{n+1}} (x-3)^{n+1}}{\frac{n^2}{3^n}(x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left| \frac{(n+1)^2 \cdot \cancel{3^n} \cdot (x-3)^{\cancel{n+1}}}{3^{\cancel{n+1}} \cdot n^2 \cdot \cancel{(x-3)^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \cdot \underbrace{\frac{(x-3)}{3}}_{\text{independent of } n} \right|$$

$$= \frac{|x-3|}{3} \cdot \lim_{n\to\infty} \left| \frac{(n+1)^2}{n^2} \right| = \frac{|x-3|}{3} \lim_{n\to\infty} \left| \frac{n^2+2n+1}{n^2} \right| \cdot \frac{\frac{1}{n^2}}{\frac{1}{n^2}}$$

$$= \frac{|x-3|}{3} \cdot \underbrace{\lim_{n\to\infty} \left| \frac{1+\frac{2}{n}+\frac{1}{n^2}}{1} \right|}_{=1} = \frac{|x-3|}{3}$$

ratio test: $\lim \frac{|b_{n+1}|}{|b_n|} = L$ — if $L<1$: $\sum b_n$ converg.
$L>1$: diverges
$L=1$: no conclusion $\to$ try another test

For $\sum_{n=1}^{\infty} \frac{n^2}{3^n}(x-3)^n$ to converges,

$\lim \frac{|b_{n+1}|}{|b_n|} = \frac{|x-3|}{3}$ has to be less than 1!

$$\frac{|x-3|}{3} < 1 \quad / \cdot 3$$

$$|x-3| < 3 \leftarrow \text{radius of convergence!}$$

for the interval, solve $|x-3|<3$ for x:

2 cases: (I) $x-3<3$
$\underline{x<6}$

(II) $-(x-3)<3$
$-x+3<3$
$-x<0 \quad /\cdot(-1)$
$\underline{x>0}$

$\Rightarrow$ interval: check edge points first!!

put $x=0$ and $x=6$ into series and check convergence!

$x=0$: $\sum_{n=1}^{\infty} \frac{n^2}{3^n}(0-3)^n = \sum_{n=1}^{\infty} \frac{n^2 \cdot (-1)^n \cdot \cancel{3^n}}{\cancel{3^n}} = \sum_{n=1}^{\infty} (-1)^n \cdot n^2$ diverges (AST fails)

$x=6$: $\sum_{n=1}^{\infty} \frac{n^2}{3^n}(6-3)^n = \sum_{n=1}^{\infty} \frac{n^2}{\cancel{3^n}} \cdot \cancel{3^n} = \sum_{n=1}^{\infty} n^2 = \infty$ diverges too

if the series at $x=0$ diverges, then $x=0$ is not included in the interval of convergence. (same $x=6$)

$\Rightarrow$ interval of convergence: $\underline{0<x<6}$

Integration with power series

compute $\int \sin(2x^2)\,dx$, by expressing it as a power series!

first : remember $\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot x^{2n+1}}{(2n+1)!}$

then we substitute:

$$\int \sin(2x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n \cdot (2x^2)^{2n+1}}{(2n+1)!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n+1} \cdot x^{2(2n+1)}}{(2n+1)!}$$

$$= \int \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n+1} \cdot x^{4n+2}}{(2n+1)!}$$

also: $\int \sin(2x^2)\,dx = \sum_{n=0}^{\infty} \left(\int \frac{(-1)^n \cdot 2^{2n+1} \cdot x^{4n+2}}{(2n+1)!}\,dx \right)$

$$= \sum_{n=0}^{\infty} \left(\underbrace{\frac{(-1)^n \cdot 2^{2n+1}}{(2n+1)!}}_{\text{no } x \text{ in these term}} \cdot \int x^{4n+2}\,dx \right)$$

$$= \sum_{n=0}^{\infty} \left(\frac{(-1)^n \cdot 2^{2n+1}}{(2n+1)!} \cdot \frac{x^{4n+2+1}}{4n+3} \right)$$ simplify a bit

$$= \sum_{n=0}^{\infty} \frac{(-1)^n \cdot 2^{2n+1}}{(2n+1)! \cdot (4n+3)} x^{4n+3} + C$$

now a power series representing $\int \sin(2x^2)\,dx$!

functions in 2 variables

$z = f(x,y)$

- sketch graph
- partial der
- tangent planes (linear approximation)
- gradient vector

Ex $f(x,y) = 2x^2 + 3y^2 - 2$

contour lines (level curves)

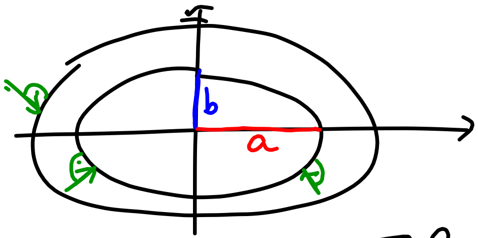
set

$\underset{\uparrow \text{value}}{k} = 2x^2 + 3y^2 - 2 \to$ draw curve

$k = 2: \quad 2 = 2x^2 + 3y^2 - 2$ or $\underbrace{4 = 2x^2 + 3y^2}_{\text{ellipse}}$

$1 = \frac{x^2}{2} + \frac{3y^2}{4}$

level curves are ellipses:



general: $1 = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

gradient vector of f: $\nabla f = \langle f_x, f_y \rangle$

$f_x = 4x, \quad f_y = 6y \Rightarrow \nabla f = \langle 4x, 6y \rangle$

direction of steepest incline!

$\nabla f = \langle 4x, 6y \rangle$ is always orthogonal to the level curves!!