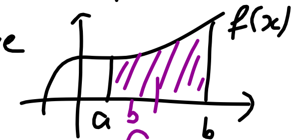
50 points, 16 from after the 2nd midterm!!

Review: Applications of integrals

Area between curves

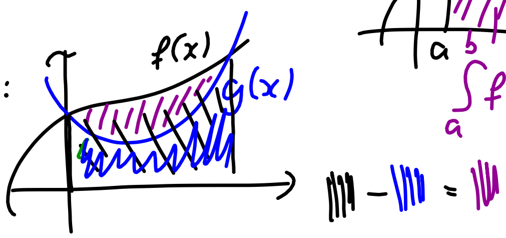
$\int_a^b f(x)\,dx$ is the area between $f(x)$ and $y=0$ (x-axis) from a to b.

→ area under a curve



$\int_a^b f(x)\,dx$

area between curves:



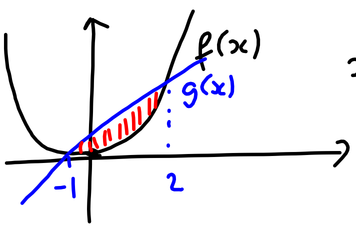
$$\int_a^b f(x)-g(x)\,dx, \quad f(x) \geq g(x)$$

area between $f(x)$ and $g(x)$.

Ex $f(x)=x^2$, $g(x)=x+2$

check which one is bigger:

do they intersect? $f(x)=g(x)$ for any x?



$x^2=x+2$ solve for x

$x^2-x-2=0 \rightsquigarrow x=\begin{cases}2\\-1\end{cases}$ quadratic equation

check at $0\in(-1,2)$ if $f(x)\geq g(x)$ or $g(x)\geq f(x)$: $f(0)=0$

$g(0)=0+2=2$

$\Rightarrow g$ is bigger in $(-1,2)$

inters. points

$$\int_{-1}^{2} g(x)-f(x)\,dx$$

$$=\int_{-1}^{2}(x+2)-x^2\,dx=\int_{-1}^{2}x+2-x^2\,dx$$

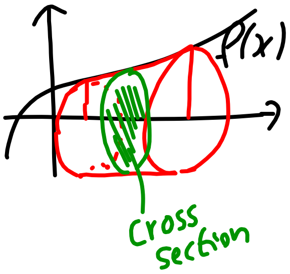
$$=\left[\frac{x^2}{2}+2x-\frac{x^3}{3}\right]_{-1}^{2}=\left(\frac{2^2}{2}+2\cdot 2-\frac{2^3}{3}\right)-\left(\frac{(-1)^2}{2}+2\cdot(-1)-\frac{(-1)^3}{3}\right)$$

$$=\underbrace{2+4}_{=\frac{36}{6}}-\frac{8}{3}-\left(\frac{1}{2}-2+\frac{1}{3}\right)=\frac{36}{6}-\frac{16}{6}-\frac{3}{6}+\frac{12}{6}-\frac{2}{6}$$

need 6th-fraction

$$=\frac{27}{6}=\frac{9}{2}=4.5$$

Volumes rotational solids
- rotation of a curve
- rotation of a section enclosed by 2 curves.

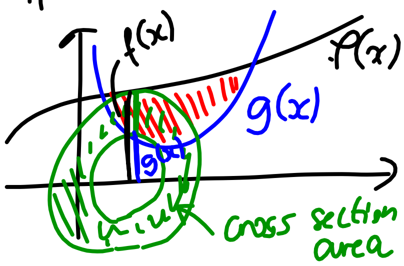


$A(x)$ = cross section area function

$$V=\int_a^b A(x)\,dx$$

If we rotate a curve $f(x)$, then $\underline{A(x)=(f(x))^2\cdot\pi}$ area of a circle w. rad $f(x)$

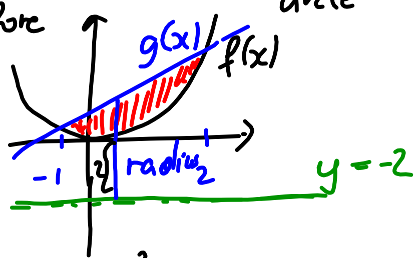
If we rotate a section between 2 curves $f(x), g(x)$:



outer radius: $f(x)$
inner radius: $g(x)$

$$A(x)=\pi\cdot(f(x))^2-\underbrace{\pi\cdot(g(x))^2}_{\text{hole in circle}}$$

Ex $f(x)=x^2$, $g(x)=x+2$ as before
rotate around $y=-2$:
adjust for rotation around $y=-2$:



Cross section area fct:

$$\underline{A(x)}=\pi(g(x)+2)^2-\pi\cdot(f(x)+2)^2$$
$$=\underline{\pi((g(x)+2)^2-(f(x)+2))^2}$$

$$V=\int_{-1}^{2}\pi((g(x)+2)^2-(f(x)+2)^2)\,dx$$

$$=\pi\cdot\int_{-1}^{2}(x+2+2)^2-(x^2+2)^2\,dx$$

$$=\pi\int_{-1}^{2}x^4+3x^2-8x-12\,dx$$

$$=\pi\left[\frac{x^5}{5}+\frac{3x^3}{3}-\frac{8x^2}{2}-12x\right]_{-1}^{2}=\underline{\underline{\frac{16}{5}\pi}}$$

WORK Springs

natural length

Δx need force to stretch

Hooke's Law:

$$F = k \cdot \Delta x$$

constant depending on spring

Δx = difference betw. natural and stretched length

Ex 

20 cm unstretched

given: need 40N to extend to 30cm.

30 cm

this allows us to calculate k: here 10 cm = 0.1 m

$$F = k \cdot \Delta x$$

difference in length

$$40\,N = k \cdot 0.1\,m \Rightarrow \underline{k = 400\,N/m}$$

Q How much work to stretch it to 40cm from 30cm.

$$W = \int_{0.3m}^{0.4m} F(x)\,dx = \int_{0.3}^{0.4} k \cdot \Delta x\,dx = \int_{0.3}^{0.4} 400(x-0.2)\,dx$$

force

$\Delta x = (x - x_0)$

natural length

$$= \int_{0.3}^{0.4} 400x - 80\,dx = \left[\overset{200}{\cancel{400}} \cdot \frac{x^2}{\cancel{2}} - 80x\right]_{0.3}^{0.4}$$

$$= \left(200 \cdot \left(\frac{4}{10}\right)^2 - 80 \cdot \frac{4}{10}\right) - \left(200 \cdot \left(\frac{3}{10}\right)^2 - 80 \cdot \frac{3}{10}\right)$$

$$= \cancel{32} - \cancel{32} - (18 - 24) = \underline{\underline{6\,J}} \quad (\text{Joules})$$

Differential equations $y' = F(x,y)$ some equation

Ex $\frac{dy}{dx} = \frac{3x^2}{5y^2}$ separate variables

$\int 5y^2\,dy = \int 3x^2\,dx$ then integrate

$5\cdot\frac{y^3}{3} = \frac{\cancel{3}x^3}{\cancel{3}} + C \;\Big/\cdot\frac{3}{5}$ then solve for y

$\underline{\underline{y}} = \sqrt[3]{\frac{3x^3+3C}{5}} = \sqrt[3]{\frac{3x^3}{5} + \frac{3C}{5}} = \underline{\underline{\sqrt[3]{\frac{3x^3}{5} + K}}}$

$K = \frac{3C}{5}$

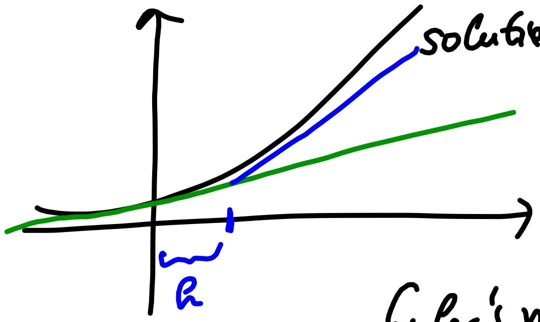
assume $y(0)=1$ (initial value problem)

→ allows us to calculate K.

$y(0) = \sqrt[3]{\frac{3\cdot 0^3}{5} + K} = \sqrt[3]{K} = 1 \Rightarrow K = 1$

$$\boxed{y(x) = \sqrt[3]{\frac{3x^3}{5} + 1}}$$

if we can't solve a diff. equation, approximate function values of $y(x)$. → Euler's method



solution of $y' = F(x,y)$, $y(0) = 1$ ↖ need initial value

use tangent line as approximation

Euler's method: adjust direction at check-points

Ex $y' = x + y$, step size 0.2 $(=h)$, $y(0) = 1$

$\Rightarrow h = 0.2$, $0 = x_0$, $1 = y_0$

formula: $x_n = x_{n-1} + h$

$$y_n = y_{n-1} + h \cdot \underbrace{F(x_{n-1}, y_{n-1})}$$

$= y'$ at x_{n-1}, y_{n-1}
slope of y!!

estimate $y(0.4)$:

$$x_1 = x_0 + h = 0 + 0.2 = 0.2$$

$$y(0.2) \approx y_1 = y_0 + h \cdot F(0,1) = 1 + 0.2 \cdot (0 + 1) = 1.2$$

$$x_2 = x_1 + 0.2 = 0.4$$

$$y(0.4) \approx y_2 = y_1 + h \cdot F(x_1, y_1) = 1.2 + 0.2 \cdot (0.2 + 1.2)$$

$$= 1.48$$

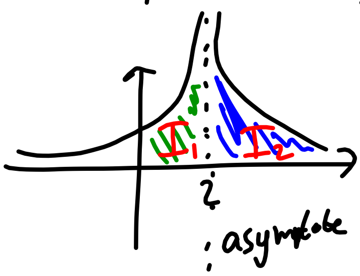
smaller step size = better approximation!!

Improper Integrals

TYPE 1: eg $\displaystyle\int_0^\infty \frac{1}{e^x}\,dx$ open interval

$$\parallel\ \lim_{t\to\infty}\int_0^t \frac{1}{e^x}\,dx$$

TYPE 2: eg $\displaystyle\int_0^5 \frac{1}{(x-2)^2}\,dx$ discontinuous integrand!!



split into $\displaystyle\underbrace{\int_0^2 \frac{1}{(x-2)^2}\,dx}_{I_1} + \underbrace{\int_2^5 \frac{1}{(x-2)^2}\,dx}_{I_2}$

I_1: 2 is not in domain of $\frac{1}{(x-2)^2}$ $\Rightarrow$ use a limit

$$\lim_{t\to 2^-}\int_0^t \frac{1}{(x-2)^2}\,dx = \lim_{t\to 2^-}\left[-\frac{1}{x-2}\right]_0^t =$$

$$\lim_{t\to 2^-}\left(-\frac{1}{t-2} - \left(-\frac{1}{0-2}\right)\right) = \lim_{t\to 2^-}\left(-\frac{1}{t-2}\right) + \left(\frac{1}{-2}\right)$$

$$= \lim_{t\to 2^-}\frac{-1}{t-2} - \frac{1}{2} = \underline{\underline{+\infty}}$$

$\underline{t<2}$ so $t-2<0$

So I_1 diverges and I_2 is positive, so $I_1+I_2=\displaystyle\int_0^5\frac{1}{(x-2)^2}\,dx$ diverges.