review: tangent planes $f(x,y)$, at point (x_0, y_0, z_0):

$\uparrow = f(x_0, y_0)$

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

tangent plane equation $\swarrow z_0 = f(x_0, y_0)$

linear approximation of f at (x_0, y_0)

$$z = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + \underline{z_0}$$

only difference.

two cases of chain rule
(may be asked explicitly)

Implicit Differentiation:

CASE I: $F(x, y(x)) = 0$

$$y' = \frac{\partial y}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$$

Ex $x^3 + y^3 = 6xy$

What is F? $F(x,y) = x^3 + y^3 - 6xy = 0$

$$\frac{\partial y}{\partial x} = -\frac{3x^2 - 6y}{3y^2 - 6x}\begin{matrix} = f_x \\ = f_y \end{matrix} = -\frac{\cancel{3}(x^2 - 2y)}{\cancel{3}(y^2 - 2x)}$$

$\uparrow$!

$$= -\underline{\underline{\frac{x^2 - 2y}{y^2 - 2x}}}$$

CASE II $F(x, y, z) = 0$ where $z = z(x, y)$

z is a function in $\underline{\underline{2}}$ variables, so get 2 partial der.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} \qquad \frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}}$$

Ex $x^3 + y^3 + z^3 + 6xyz = 1$

Find $F(x,y,z) = 0$:

$$F(x,y,z) = x^3 + y^3 + z^3 + 6xyz - 1 = 0$$

$$\frac{\partial F}{\partial x} = 3x^2 + 0 + 0 + 6 \cdot 1 \cdot yz = \underline{\underline{3x^2 + 6yz}}$$

$$\frac{\partial F}{\partial y} = 0 + 3y^2 + 0 + 6xz = \underline{\underline{3y^2 + 6xz}}$$

$$\frac{\partial F}{\partial z} = 0 + 0 + 3z^2 + 6xy = \underline{\underline{3z^2 + 6xy}}$$

as before cancel 3.

$$\frac{\partial z}{\partial x} = -\frac{\frac{\partial F}{\partial x}}{\frac{\partial F}{\partial z}} = -\frac{3x^2 + 6yz}{3z^2 + 6xy} \overset{\swarrow}{=} -\underline{\frac{x^2 + 2yz}{z^2 + 2xy}}$$

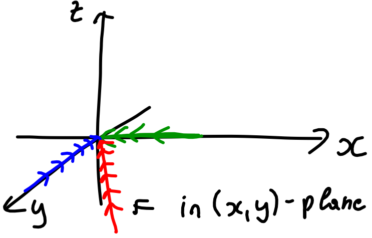
$$\frac{\partial z}{\partial y} = -\frac{\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial z}} = -\frac{3y^2 + 6xz}{3z^2 + 6xy} = -\underline{\frac{y^2 + 2xz}{z^2 + 2xy}}$$

Directional Derivatives (§ 14.6)

recall: $f_x(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h,\, y_0) - f(x_0,y_0)}{h}$

(along) x-axis

$f_y(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0\,,\, y_0+h) - f(x_0,y_0)}{h}$

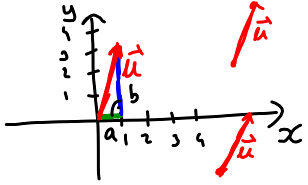


today: approch along a line $y = k\cdot x$ (e.g. $y=2x$

Solution: add 'some' h to both x_0 and y_0 in the definition.

Vectors 'arrows' pointing in a direction $\vec{u} = \langle 1,3\rangle$

2-dim



can attach it any where

LENGTH: $\vec{u} = \langle a,b\rangle$ then its length is

Pythagoras: $|\vec{u}| = \sqrt{a^2+b^2}$

unit vector has length 1.

directional derivative: $\vec{u}=\langle a,b\rangle$ at (x_0,y_0) $-f(x_0,y$

$$D_{\vec{u}} f(x_0,y_0) = \lim_{h\to 0} \frac{f(x_0+h\cdot a,\, y_0+h\cdot b)}{h}$$

need $\vec{u}$ has length 1! $\sqrt{a^2+b^2}=1$

(or in general for $\langle a,b\rangle$) (or $a^2+b^2=1$)

$\vec{u}=\langle a,b\rangle$ with $a^2+b^2=1$, $\vec{u}$ a unit vector!

$$\boxed{D_{\vec{u}} f(x,y) = f_x(x,y)\cdot a + f_y(x,y)\cdot b}$$

What do we do if $\vec{u}$ is not a unit vector? (too long or too short)

If we multiply a vector by a constant C, it becomes longer ($C>1$) or shorter ($C<1$).

$\vec{u}=\langle a,b\rangle$ has length $\sqrt{a^2+b^2}$

to make $\vec{u}$ into a unit vector, divide each component by its length:

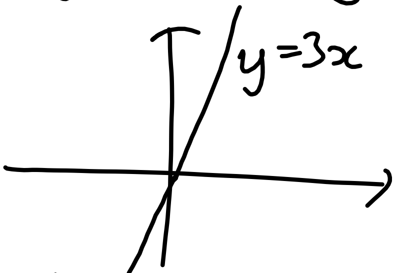
$$\vec{u}_e = \left\langle \frac{a}{\sqrt{a^2+b^2}}, \frac{b}{\sqrt{a^2+b^2}} \right\rangle$$ has length 1.

↑ unit vector in direction $\vec{u}$

why: $\sqrt{\left(\frac{a}{\sqrt{a^2+b^2}}\right)^2 + \left(\frac{b}{\sqrt{a^2+b^2}}\right)^2}$

$= \sqrt{\frac{a^2}{a^2+b^2} + \frac{b^2}{a^2+b^2}} = \sqrt{\frac{\cancel{a^2+b^2}}{\cancel{a^2+b^2}}} = 1$

Ex Der. of $f(x,y) = x^3 - 3xy + 4y^2$ along line $y = 3x$.



first, need a vector: pick any point on the line, then turn into a unit vector.

a point: $(\underset{\substack{\uparrow \\ \text{can choose}}}{1}, 3\cdot 1)$ or any point $(x, \underset{\substack{\shortparallel \\ y}}{3x})$

So we get $\vec{u} = \langle 1, 3 \rangle$

now turn $\vec{u}$ into a unit vector:

$$\underline{\underline{\vec{u}_e}} = \left\langle \frac{1}{\sqrt{1^2+3^2}}, \frac{3}{\sqrt{1^2+3^2}} \right\rangle = \underline{\underline{\left\langle \frac{1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle}}$$

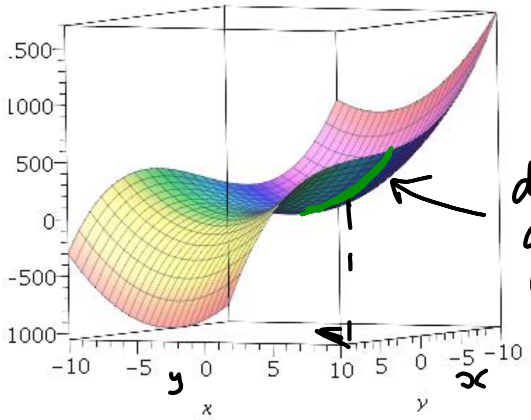
has length 1.

Now we can use the formula: $\vec{u}_e = \langle a, b \rangle$

$$D_{\vec{u}_e} f(x,y) = f_x(x,y)\cdot a + f_y(x,y)\cdot b$$

$$= (3x^2 - 3y)\cdot \frac{1}{\sqrt{10}} + (-3x + 8y)\cdot \frac{3}{\sqrt{10}}$$

(can simplify)

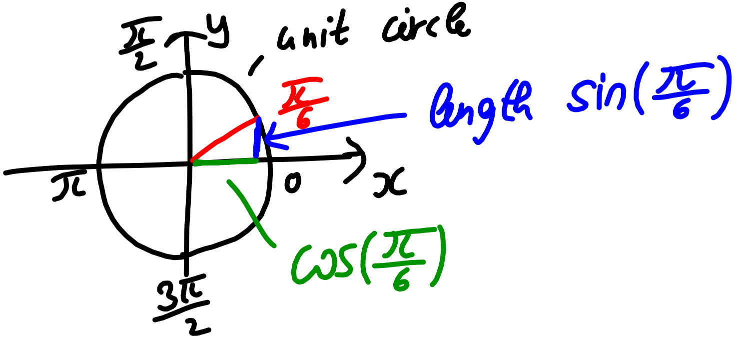


directional derivative is the derivative of the curve that we get if we set $y=3x$.

<u>Ex</u> approach f at $(2,4)$ from an angle $\theta = \frac{\pi}{6}$.

$$f(x,y) = x^3 - 3xy + 4y^2$$

first, find a unit vector with angle $\frac{\pi}{6}$.



So $\vec{u} = \langle \cos(\frac{\pi}{6}), \sin(\frac{\pi}{6}) \rangle$ has angle $\frac{\pi}{6}$.

check length: $\sqrt{\cos^2(\frac{\pi}{6}) + \sin^2(\frac{\pi}{6})} = \sqrt{1} = 1$

(recall: $\sin^2(x) + \cos^2(x) = 1$ for <u>ALL</u> x)

$$\text{So } \underbrace{D_{\vec{u}} f(x,y)}_{x^3-3xy+4y^2} = (3x^2-3y)\cdot\cos(\tfrac{\pi}{6}) + (-3x+8y)\cdot\sin(\tfrac{\pi}{6})$$

↑ as above

$$= \underline{(3x^2-3y)\cdot\frac{\sqrt{3}}{2} + (-3x+8y)\cdot\frac{1}{2}}$$ can simplify

now fill in $(2,4)$:

$$\underline{D_{\vec{u}} f(2,4)} = \underbrace{(3\cdot 2^2 - 3\cdot 4)}_{=0}\cdot\frac{\sqrt{3}}{2} + (-3\cdot 2 + 8\cdot 4)\cdot\frac{1}{2}$$

$$= 0 + (-6+32)\cdot\frac{1}{2} = \underline{13}$$

Gradients & maximising dir. der.

gradient (vector) of $f(x,y)$ $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y) \rangle$

notation: $$= \frac{\partial f}{\partial x} \cdot \underset{\overset{\uparrow}{=\langle 1,0 \rangle}}{i} + \frac{\partial f}{\partial y} \cdot \underset{\overset{\|}{\langle 0,1 \rangle}}{j}$$

$\boxed{\text{Turns out } D_{\vec{u}} f(x,y) \text{ is maximised if } \vec{u} = \nabla f(x,y)}$

WORKS FOR ALL (x,y) as $\nabla f(x,y)$ is different for different points.

CAREFUL: maximises the derivative not the function!!

Ex max. rate of change at $(2,0)$ of $f = x \cdot e^y$

maximise $D_{\vec{u}} f$

first: $\nabla f(x,y) = \langle f_x, f_y \rangle = \langle e^y, x \cdot e^y \rangle$

to get $\vec{u}$, fill in $(2,0)$:

$$\underline{\nabla f(2,0)} = \langle e^0, 2 \cdot e^0 \rangle = \underline{\langle 1,2 \rangle}$$

$\vec{u} = \langle 1,2 \rangle$ is not a unit vector!!

$$\vec{u}_e = \left\langle \frac{1}{|\vec{u}|}, \frac{2}{|\vec{u}|} \right\rangle = \left\langle \frac{1}{\sqrt{1^2+2^2}}, \frac{2}{\sqrt{1^2+2^2}} \right\rangle$$
$$= \left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$$

$$D_{\vec{u}_e} f(x,y) = e^y \cdot \frac{1}{\sqrt{5}} + x \cdot e^y \cdot \frac{2}{\sqrt{5}}$$ general directional derivative along $\left\langle \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}} \right\rangle$

now fill in $(2,0)$:

$$\underline{D_{\vec{u}_e} f(2,0)} = \underbrace{e^0}_{=1} \cdot \frac{1}{\sqrt{5}} + 2 \cdot \underbrace{e^0}_{=1} \cdot \frac{2}{\sqrt{5}} = \frac{1}{\sqrt{5}} + \frac{4}{\sqrt{5}} = \frac{5}{\sqrt{5}}$$
$$= \frac{\sqrt{5}^2}{5} = \frac{\sqrt{5} \cdot \cancel{\sqrt{5}}}{\cancel{\sqrt{5}}} = \underline{\sqrt{5}}$$