Series (alternating series)

recall: Series $\sum_{i=1}^{\infty} a_i$, $a_i \geq 0$ so far

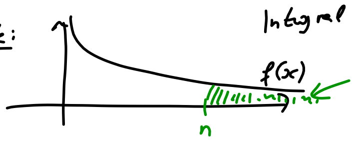
- convergent, divergent
  partial sums: $S_n = \sum_{i=1}^{n} a_i$
  if $\lim_{n\to\infty} S_n = L$, $L \in \mathbb{R}$
  then $\sum a_i$ converges (if not, it diverges)
- Comparison thm
- limit comparison: $\sum a_i$, $\sum b_i$
  if $\lim \frac{a_i}{b_i} = L \neq 0$ then either both converge or diverge.

Estimating sums

$$\sum a_i = \underbrace{a_1 + \dots + a_n}_{\text{estimation for series}} + \underbrace{a_n + \dots}_{\text{estimate remainder (want this to be small)}} \dots$$

if $\sum a_i$ converges, $\lim a_i = 0$.
So $a_n + a_{n+1} + \dots$ gets very small if n is big.

Estimate: Integral test



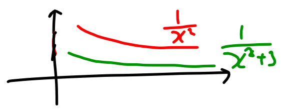
gives an upper bound for $a_n + a_{n+1} + \dots = \sum_{k=n}^{\infty} a_k$
if $f(i) \geq a_i$

Ex $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ converges: exercise: comparison thm with $\sum \frac{1}{n^2}$

now: remainder $\sum_{n=k}^{\infty} a_n \leq \int_k^{\infty} f(x)\,dx$

where $f(x) = \frac{1}{x^2+3}$ could choose this, but: we need to integrate it!!

so easier: $f(x) = \frac{1}{x^2+1}$ or even $f(x) = \frac{1}{x^2}$!



balance between making easy to int. and still converging!!

$$\sum_{k=n}^{\infty} a_k = \sum_{k=n}^{\infty} \frac{1}{k^2+3} \leq \int_k^{\infty} f(x)\,dx = \int_k^{\infty} x^{-2}\,dx \quad (*)$$

(1) $k > 0$ so $\frac{1}{x^2}$ continuous on $[k, \infty)$ ✓

(2) deal with upper bound ∞:

$$\lim_{t\to\infty} \int_k^t x^{-2}\,dx = \lim_{t\to\infty} \left[-x^{-1}\right]_k^t = \lim_{t\to\infty}\left(-t^{-1} - (-k^{-1})\right)$$

indep. of t

$$= \underbrace{\lim_{t\to\infty}(-t^{-1})}_{\to 0} + \frac{1}{k} = 0 + \frac{1}{k} = \frac{1}{k}$$

$\left(\lim_{t\to\infty} \frac{-1}{t}\right)$

This means: from $(*)$ we see that
$$\sum_{k=n}^{\infty} a_k \leq \frac{1}{k}.$$

Choose k big enough to get desired precision.

Alternating series (§11.4 ??)

$\sum_{i=1}^{\infty} a_i$ where $a_i \geq 0$ for i even, $a_i \leq 0$ for i odd, or v.v.

a_i jump between positive and negative.

Ex (1) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} \ldots\ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \underbrace{\frac{1}{n}}_{a_n}$

alternating signs

alternating sign! $(-1)^{n+2} = (-1)^n$

(2) $1 - 1 + 1 - 1 + 1 - 1 \ldots\ldots = \sum_{n=1}^{\infty} (-1)^{n+1}$

Still wanna know: converges or diverges?

partial sums tricky to express (in part. hard to compute limit)

had so far: integral test, comparison test, limit comparis.

→ none of them work here (no straight forward)

But we have something else: (much easier)

Alternating series test: (p ...?)

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot a_n = a_1 - a_2 + a_3 - a_4 + a_5 - \ldots\ldots$$

now: $a_n \geq 0$

if (C1) $a_{n+1} \leq a_n$ for all $n \geq N$, $N \in \mathbb{N}$.

tail end matters!

(C2) $\lim_{n\to\infty} a_n = 0$

then series converges.

First, check basic examples:

(Ex 1) $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} \ldots = \sum_{n=1}^{\infty} (-1)^{n-1} \cdot \underbrace{\frac{1}{n}}_{= a_n}$

check (C1): $a_{n+1} \leq a_n$

$\frac{1}{n+1} \leq \frac{1}{n} \quad / \cdot n(n+1)$

$n \leq n+1$ ✓ true for all n.

(C2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓ ok too.

So $\sum (-1)^{n-1} \cdot \frac{1}{n}$ converges.

had $\sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series → diverges.

But now $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$ converges!!

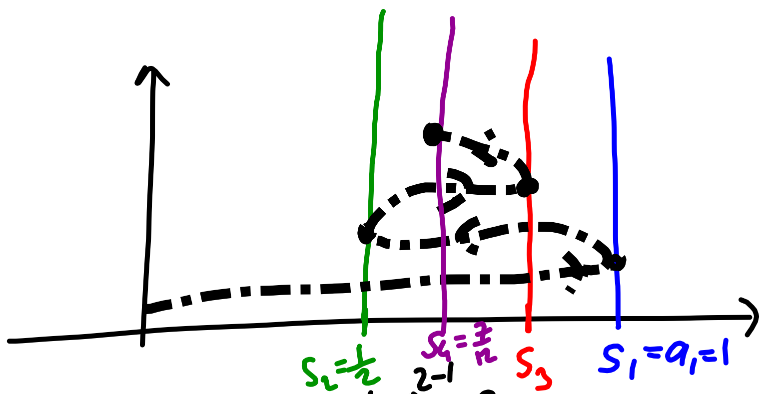
→ more later!

Why does the alternating series test work?

(C1) $a_{n+1} \leq a_n$ monotonically decreasing

(C2) $\lim a_n = 0$.

$$\sum_{n=1}^{\infty} (-1)^{n-1} \cdot a_n$$



partial sums:

$$S_k = \sum_{n=1}^{k} (-1)^{n-1} \cdot a_n$$

$$S_1 = \underbrace{(-1)^{1-1}}_{=1} \cdot a_1 = a_1$$

look at $\sum_{n=1}^{\infty} (-1)^{n-1} \cdot \frac{1}{n}$

then: $S_1 = 1$

next step: $S_2 = S_1 + (-1)^{2-1} \cdot a_2$

$= 1 + (-1) \cdot \frac{1}{2} = 1 - \frac{1}{2} = \frac{1}{2}$

$S_3 = S_2 \underset{\text{alt. signs}}{+} a_3 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$

$S_4 = S_3 - \frac{1}{4} = \frac{5}{6} - \frac{1}{4} = \frac{10}{12} - \frac{3}{12} = \frac{7}{12}$

limit assures gap width goes to 0!

(C1) assures: "we only bounce back and forth between the first two lines"

detailed: textbook.

Understanding it makes it easier to remember!!

(proof will not be examined)

Ex (1) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n \cdot 3n}{4n-1} = \sum_{n=1}^{\infty} (-1)^n \cdot \underbrace{\frac{3n}{4n-1}}_{a_n}$

(C1) : $a_{n+1} \le a_n$

(C2) : $\lim_{n\to\infty} a_n = 0$

check (C1): first, determine $a_n = \frac{3n}{4n-1}$

$$\underline{a_{n+1}} = \frac{3(n+1)}{4(n+1)-1} = \frac{3n+3}{4n+4-1} = \underline{\frac{3n+3}{4n+3}}$$

bracket!!!!

now compare: check!! a_n

$$a_{n+1} = \frac{3n+3}{4n+3} \le \frac{3n}{4n-1} \quad \Big/ \cdot (4n+3)(4n-1)$$

$$(3n+3)(4n-1) \le 3n \cdot (4n+3)$$ simplify

$$\cancel{12n^2} + \cancel{9n} - 3 \le \cancel{12n^2} + \cancel{9n} \quad \Big/ - 12n^2 \quad \Big/ - 9n$$

$$-3 \le 0 \checkmark$$ OK for (C1)

check (C2): $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n}{4n-1} = \lim_{n\to\infty} \frac{\frac{3\cancel{n}}{\cancel{n}}}{\frac{4n-1}{n}}$

divide by highest power of n

$$= \lim_{n\to\infty} \frac{3}{\frac{4\cancel{n}}{\cancel{n}} - \frac{1}{n}} = \lim_{n\to\infty} \frac{3}{4 - \underbrace{\frac{1}{n}}_{\to 0}} = \underline{\underline{\frac{3}{4}}} \ne 0$$

C2 not satisfied!!!

$$\Rightarrow \sum_{n=1}^{\infty} (-1)^n \cdot \frac{3n}{4n-1} \text{ diverges}$$

Ex2 $\sum_{n=1}^{\infty} (-1)^{n+3} \cdot \frac{\sqrt{n}}{n+4} = \sum_{n=1}^{\infty} (-1)^{n+1} \cdot \underbrace{\frac{\sqrt{n}}{n+4}}_{a_n}$

$(-1)^{n+3} = (-1)^{n+1} \cdot \underbrace{(-1)^2}_{=1} = (-1)^{n+1}$

(C1): $a_{n+1} \leq a_n$

$a_{n+1} = \frac{\sqrt{n+1}}{(n+1)+4} = \frac{\sqrt{n+1}}{n+5} \leq \underbrace{\frac{\sqrt{n}}{n+4}}_{a_n}$!

Checking straightforward not easy.

Instead: use a function $f(x) = \frac{\sqrt{x}}{x+4}$ so $f(n) = \frac{\sqrt{n}}{n+4} = a_n$

Check in/decreasing via 1st derivative of $f(x)$:

$f(x)' = \left(\frac{\sqrt{x}}{x+4}\right)' \underset{\substack{\uparrow \\ \text{quotient rule}}}{=} \frac{\frac{1}{2\sqrt{x}} \cdot (x+4) - \sqrt{x} \cdot 1}{(x+4)^2} =$

$\dots = \frac{\frac{x+4}{2\sqrt{x}} - \sqrt{x}}{(x+4)^2}$ only need sign of $f'(x)$ for $x \in [1, \infty)$ (or: enough $[N, \infty)$, $\in \mathbb{N}$)

den. always pos!!

So sign of $f(x)'$ is given by sign of

$\frac{x+4}{2\sqrt{x}} - \sqrt{x} = \frac{\cancel{x}^{=\sqrt{x}\cdot\sqrt{x}}}{2\cancel{\sqrt{x}}} + \frac{\cancel{4}\,2}{\cancel{2}\sqrt{x}} - \sqrt{x}$

$= \frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}} - \sqrt{x} = \underbrace{\frac{2}{\sqrt{x}}}_{\to 0 \text{ as } x\to\infty} - \frac{\sqrt{x}}{2} \to -\infty$

$= -\frac{\sqrt{x}}{2}$

as $x \to \infty$: goes to ∞

For suff. large x, this is negative!!

can show: $4 \leq x$ need this. C1 true for $n \geq N = 4$

C2

$\lim_{n\to\infty} a_n = \lim \frac{\sqrt{n}}{n+4} \underset{\substack{\uparrow \\ \text{divide by } n}}{=} \lim \frac{\frac{\sqrt{n}}{n}}{\frac{n+4}{n}}$ $\left(\frac{\sqrt{n}}{n} = \frac{\sqrt{n}}{\sqrt{n}\cdot\sqrt{n}} = \frac{1}{\sqrt{n}}\right)$

$= \lim_{n\to\infty} \frac{\frac{1}{\sqrt{n}} \to 0}{1 + \frac{4}{n} \to 0} = 0$ ✓ C2 is OK

$\to$ Series converges.