Series

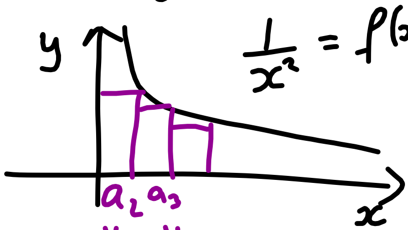
recall: series, infinite sums $\sum_{i=1}^{\infty} a_i$

- convergence: sum is finite
- divergence: $\sum_{i=1}^{\infty} a_i$ is infinite

partial sums: $S_n = \sum_{i=1}^{n} a_i$

study limit of S_n to check for conv. or divergence.

Integral test: Series $\sum_{n=1}^{\infty} \frac{1}{n^2}$



$\frac{1}{x^2} = f(x)$

$a_2 = \frac{1}{2^2}$, $a_3 = \frac{1}{3^2}$ ....

Idea: if area under $f(x)$ is finite, then the sum of the area of the bars is finite too.

Ex $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ want to use integral test

$f(x) = \frac{1}{x^2+1}$ integrate from 1 to ∞.

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t \to \infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t \to \infty} \arctan(x)\Big]_1^t$$

$$= \underbrace{\lim_{t \to \infty} \arctan(t)}_{\frac{\pi}{2}} \underbrace{- \arctan(1)}_{-\frac{\pi}{4}} = \frac{\pi}{4}$$

(last time)

Ex $\sum_{n=1}^{\infty} n^2 \cdot e^{-n^3}$, int. test $f(x) = x^2 \cdot e^{-x^3}$

same but x instead of n!

approach via integral:

$$\int_1^{\infty} x^2 \cdot e^{-x^3}\,dx = \lim_{t\to\infty} \int_1^t x^2 \cdot e^{-x^3}\,dx =$$

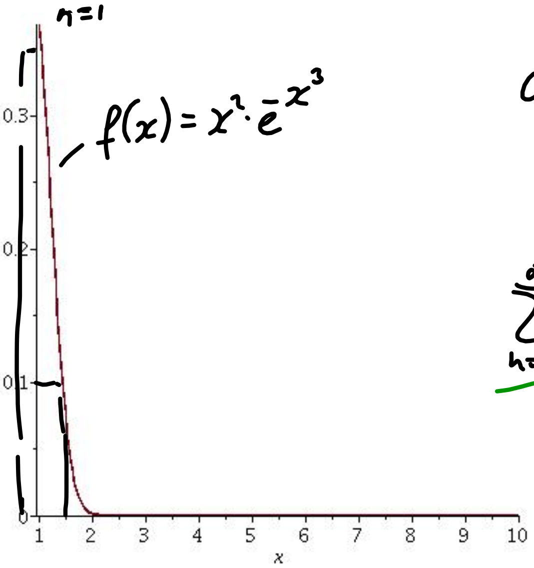
subst. $-x^3$

$$= \left| \begin{array}{l} u = -x^3 \quad g(x)=u \\ du = -3x^2\,dx \\ \text{or: } dx = \frac{du}{-3x^2} \end{array} \right| = \lim_{t\to\infty} \int_{g(1)}^{g(t)} \cancel{x^2} \cdot e^u \cdot \frac{du}{-3\cancel{x^2}}$$

$$= \lim_{t\to\infty} -\frac{1}{3} \int_{-1}^{-t^3} e^u\,du = \lim_{t\to\infty} \left(-\frac{1}{3} e^u\right)\Big|_{-1}^{-t^3}$$

$$= -\frac{1}{3} \lim_{t\to\infty} \left[\underbrace{e^{-t^3}}_{\to 0} - \underbrace{e^{-1}}_{\text{indep. of } t}\right] = -\frac{1}{3}\left(0 - e^{-1}\right) = \frac{1}{3e}$$

Goal: $\sum_{n=1}^{\infty} n^2 \cdot e^{-n^3}$ need to relate this to $\int_1^{\infty} x^2 \cdot e^{-x^3}\,dx$



$a_1 = 1^2 \cdot e^{-1^3} = \frac{1}{e}$

$a_2 = 2^2 \cdot e^{-2^3} = \frac{4}{e^8}$

$$\sum_{n=1}^{\infty} n^2 \cdot e^{-n^3} = 1^2 \cdot e^{-1} + \sum_{n=2}^{\infty} n^2 \cdot e^{-n^3}$$

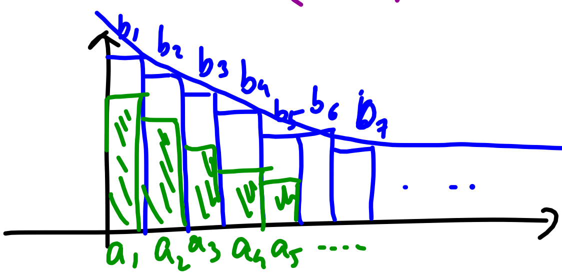
$$\le \frac{1}{e} + \int_1^{\infty} x^2 \cdot e^{-x^3}\,dx$$

$$= \frac{1}{e} + \frac{1}{3e} = \frac{4}{3e}.$$

new today: Compare a series with other series that we know (§11.4)

Idea: Have $\sum_{i=1}^{\infty} a_i$ and $\sum_{i=1}^{\infty} b_i$. If all $a_i \le b_i$ (& pos) and $\sum_{i=1}^{\infty} b_i$ converges, then $\sum_{i=1}^{\infty} a_i$ converges too.

(Comparison test)



$f(x)$ for $\sum_{i=1}^{\infty} b_i$ ← converges

If the sum of the areas of the blue bars is finite, then the added up areas of the green bars is finite too.

need a list of converging series:

(1) $\sum_{n=1}^{\infty} \frac{1}{n^p}$, $p > 1$ converges (last time)

(2) $\sum_{n=1}^{\infty} q^n$, $|q| < 1$ converges.

$\boxed{Ex}$ $\sum_{n=1}^{\infty} \frac{1}{n^2+3}$ show it converges by comparison!

each $a_n = \frac{1}{n^2+3}$ can be estimated as:

$$\frac{1}{n^2+3} \le \frac{1}{n^2}$$

needs to be of shape $\frac{1}{n^p} \cdot C$ ← constant

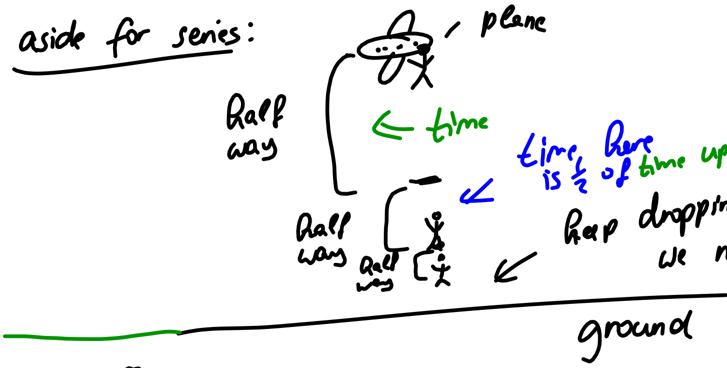
"if I want to make the fraction bigger, I need to make the denom. smaller".

(can be anything between (1, 2])

now: $\sum_{n=1}^{\infty} \frac{1}{n^2+3} \le \sum_{n=1}^{\infty} \frac{1}{n^2}$ ← converges (last time, integral test)

$\Rightarrow \sum_{n=1}^{\infty} \frac{1}{n^2+3}$ converges too.

aside for series:



$\left(\sum_{n=1}^{\infty} \frac{1}{2^n}\right)$

keep dropping half way, we never hit ground.

Ex $\sum_{n=1}^{\infty} \frac{1}{n!}$ use comparison test $\sum \frac{1}{n^p}$, $\sum q^n$

$n! = 1\cdot 2\cdot 3 \cdot \ldots \cdot \underbrace{(n-1)\cdot n}_{\text{use these factors}}$

$\frac{1}{n!} = \frac{1}{\underbrace{1\cdot 2\cdot 3\cdot \ldots}_{\text{drop these to make fraction smaller}} \cdot (n-1)\cdot n} \leq \frac{1}{(n-1)\cdot n}$

$\leq \frac{1}{\frac{n}{2}\cdot n}$ – made denom. smaller.

$\uparrow$ if $n>1$, then $n-1 \geq \frac{n}{2}$ eg: $2-1 \geq \frac{2}{2}$, $1 \geq 1$

$\leq \frac{1}{\frac{n^2}{2}} = \frac{2}{n^2}$ So all together: $\frac{1}{n!} \leq \frac{2}{n^2}$ if $n>1$.

Now: $\sum_{n=1}^{\infty} \frac{1}{n!} \leq \frac{1}{1!} + \sum_{n=2}^{\infty} \frac{2}{n^2} = 1 + \sum_{n=2}^{\infty} \frac{2}{n^2}$

$= 1 + 2\cdot \underbrace{\sum_{n=2}^{\infty} \frac{1}{n^2}}_{\text{converges.}}$ (We start at $n=2$)

No fixed rules for estimations!

Can use comparison to show divergence:

Idea: $\sum a_i$, $\sum b_i$, if $b_i \geq a_i$ and $\sum a_i$ diverges, then $\sum b_i$ diverges too. (comparison test part II)

List of diverging series: ← prime candidate

(1) $\sum_{n=1}^{\infty} \frac{1}{n}$ harmonic series, it diverges

(2) $\sum_{n=1}^{\infty} a_i$ where $\lim_{i\to\infty} a_i \neq 0$, (more theoretical)

Ex (1) $\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$ converges or diverges?

↑ Looks like $\sum_{k=1}^{\infty} \frac{1}{k}$ plus some stuff.

↑ diverges

show it diverges:

here need to make the fraction smaller for a lower bound.

$$\frac{\ln(k)}{k} \geq \frac{1}{k}$$

↑ for $k \geq 3$

$\ln(1) = 0$

$\ln(2) = < 1$

$\ln(3) > 1$

So: $\sum_{k=1}^{\infty} \frac{\ln(k)}{k} = \frac{0}{1} + \frac{\ln(2)}{2} + \sum_{k=3}^{\infty} \frac{\ln(k)}{k}$

$$\geq \frac{\ln(2)}{2} + \underbrace{\sum_{k=3}^{\infty} \frac{1}{k}}_{\text{divergent!}} \text{ " }= \infty\text{ "}.$$

So $\sum_{k=1}^{\infty} \frac{\ln(k)}{k}$ diverges.

Limit comparison

Idea: $\sum a_i$, $\sum b_i$ if $\lim \frac{a_i}{b_i} = c$, $\boxed{c > 0}$

then either both series converge or diverge.

Ex $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges, show by limit comp.

$\sum_{n=1}^{\infty} \frac{1}{2^n}$ geom. series!

$a_n = \frac{1}{2^n - 1}$

$b_n = \frac{1}{2^n}$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{1}{2^n-1}}{\frac{1}{2^n}} = \lim_{n\to\infty} \frac{2^n}{2^n - 1} = \lim_{n\to\infty} \frac{\frac{2^n}{2^n}}{\frac{2^n - 1}{2^n}}$$

$$= \lim_{n\to\infty} \frac{1}{1 - \underbrace{\frac{1}{2^n}}_{\to 0}} = \frac{1}{1-0} = \underline{\underline{1}} > 0$$

So $\sum_{n=1}^{\infty} \frac{1}{2^n - 1}$ converges because $\sum_{n=1}^{\infty} \frac{1}{2^n}$ does.