Differential Equations (§9.2)

$$y' = F(x, y)$$

$y' \uparrow$ $\dfrac{dy(x)}{dx}$

$x \uparrow$ independent variable

$y \uparrow$ function $y(x) = y$

Solve by: (a) $y' = 5x \rightarrow$ integrate to get $y(x)$

(b) method of separating var. e.g. $xy = \dfrac{dx}{dy}$

General solution:

Ex: $y\,dy = \dfrac{dx}{x}$ integrate

$$\int y\,dy = \int \frac{dx}{x}$$

solve for y

$$\frac{y^2}{2} = \ln|x| + C \quad \text{so} \quad y = \pm\sqrt{2\ln|x| + 2C}$$

Initial value: $y(1) = 2$ is given. This allows us to specify constant C.

$$\underset{\uparrow \text{ choose } +}{y(1) = \pm}\underbrace{\sqrt{2\ln|1| + 2C}}_{>0} = 2 \qquad (1)$$

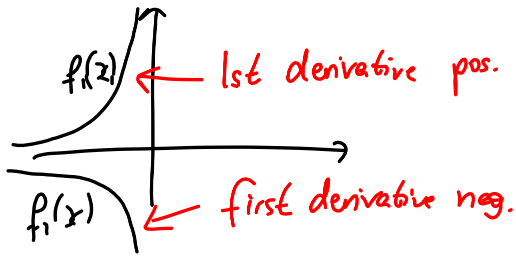
because 2 is positive

Solve (1) for C: $C = 2$ (exercise)

Direction Fields §9.2)

Depending on initial value: very different solutions.
(last time for example)

How can we tell from equation?



$\Rightarrow$ first derivative gives a hint of how the solution behaves.

Ex $\underbrace{\dfrac{dy}{dt}}_{\text{describes slope } \left(y' = \frac{dy}{dt}\right) \text{ at point } (t, y(t))} = y(t) + t^2$ ← (can't solve this yet)

table:

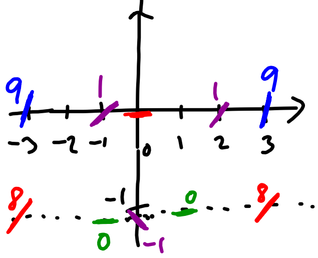
$y(t)$ \ t	-3	-1	0	1	3
0	9	1	0	1	9
-1	8	0	-1	0	8

slope at $(-3, 0)$ is 9.

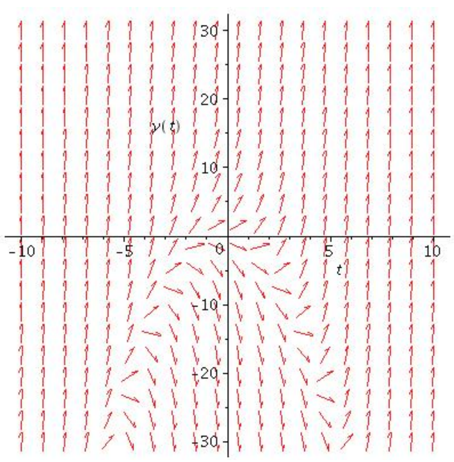
remark: no initial value, so all $(t, y(t))$ combinations are allowed.

$t = -3,\ y(t) = 0$

$y' = F(-3, 0) =$
$= 0 + (-3)^2 = 9$



If we keep doing that:



Why do we need this? Can deduce results about a solution without knowing the function of the solution!!

<u>Ex:</u> $y' = 3xy$

choose some values here (or given to you)

<u>table</u>:

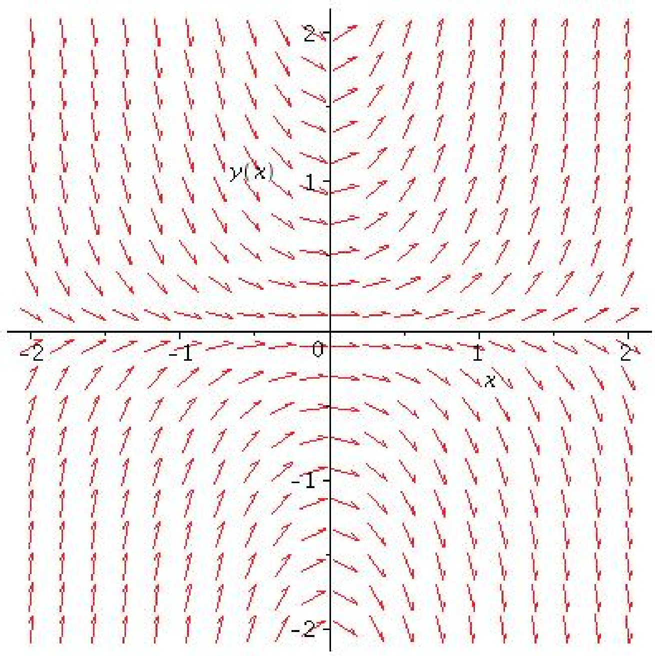
$y \backslash x$	-2	-1	0	1	2
-2	12				
-1					
0					
1					
2					

fill in (exercise)

$(-2,-2)$:

$y' = 3\cdot(-2)(-2)$

$= 12$



Can solve $y' = 3xy$ by sep. var.

$y' = \frac{dy}{dx}$ so

$$\frac{dy}{dx} = 3xy$$

$$\frac{dy}{y} = 3x\,dx$$

integrate $\int$!

$$\rightsquigarrow \ln|y| = \frac{3x^2}{2} + C$$

last week

$$\hookrightarrow \underline{y = k\cdot e^{\frac{3x^2}{2}}}$$

"Guess" a solution from the direction field if we can't compute it (default in real world)

Euler method (p 589 – 7th ed.)

Ex $y'(t) = y(t) + t^2$ (as before)

can't solve it, so guess $y(2)$ based on initial value

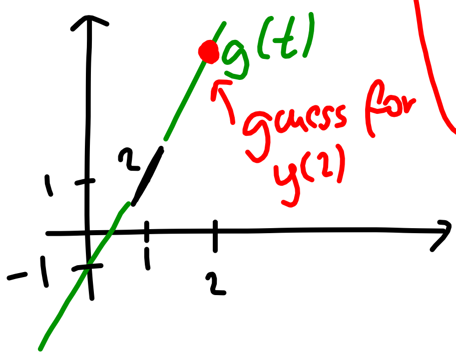
$y(0) = -2 + 6/e$

$\boxed{y(1) = 1}$ ← given

start at 1, compute slope:

$y'(1) = 1 + 1^2 = 2$

↑ Slope at (1,1) — use red eq.



one idea: take tangent at (1,1) and extrapolate to $x=2$.

tangent line equation: $g(t) = 2\cdot(t-1)+1$ at the point (1,1). x-value y-value

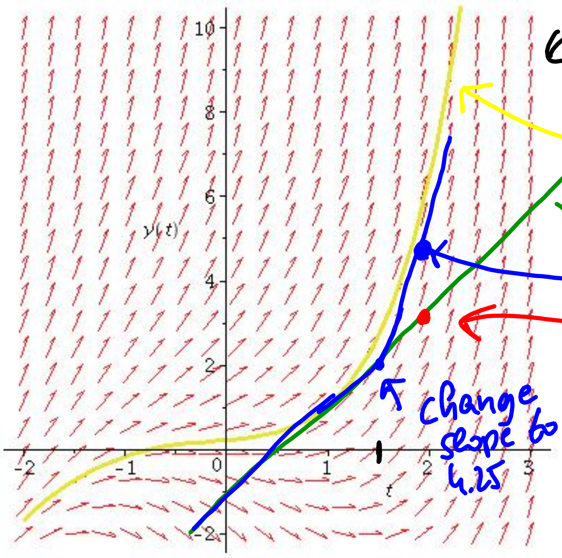
Now use $g(2)$ to "guess" $y(2)$:

$g(2) = 2\cdot(2-1)+1 = 3$ ← best guess here.

No idea here what $y(t)$ looks like. So this is our best guess.

Initial value tells you where to start, but it's a bad method to approximate values far from it.

→ make small steps and adjust direction!



Instead of going straight, ask for direction along the way.

For example: stop at $x=1.5$, get a new slope

equation: $y' = y + t^2$, so at 1.5 we readjust

We don't know $y(1.5)$, but we know $g(1.5)$, tangent line of y at (1,1)

So use $(1.5, g(1.5))$ where $g(1.5) = 2(1.5-1)+1 = 2$

Slope at $(1.5, 2)$ is $y' = 2 + (1.5)^2 = 4.25$

New tangent line:

$$g_2(t) = \underset{\text{slope}}{4.25}(t - \underset{x\text{-val.}}{1.5}) + \underset{y\text{-value}}{2}$$

Use this for our guess for $y(2)$:

$$g_2(2) = 4.25(2-1.5)+2 = 4.125$$

→ looks a lot better in picture.

General approach: **EULER'S METHOD**

approxim. values for the initial value problem

$$y' = F(x,y), \quad y(x_0) = y_0, \text{ step}$$

size h, at $x_n = x_{n-1} + h$

h

$x_0 \quad \cdots \quad x_{n-1} \; x_n$

$$y_n = y_{n-1} + \underset{\text{step size}}{h} \cdot \underbrace{F(x_{n-1}, y_{n-1})}_{\text{slope at } (x_{n-1}, y_{n-1})}$$

Ex (Exercise 22, §9.2 - 7th ed.)

Use Euler's method, step size 0.2 to estimate $y(1)$

of $\quad y' = xy - x^2, \quad y(0) = 1$

$y_0 = 1$

First: $x_0 = 0$

target $x_n = 1$

Euler's meth:

$x_n = x_{n-1} + h = x_{n-1} + 0.2$

$y_n = y_{n-1} + \underbrace{0.2}_{=h} \cdot F(x_{n-1}, y_{n-1})$

$x_0 = 0 \quad 0.2 \quad 0.4 \quad 0.6 \quad 0.8 \quad x_n = 1$

5 steps, so $n = 5$

We have $x_0 = 0$, $y_0 = 1$ from $y(0) = 1$

Now: $\underline{x_1} = x_0 + h = 0 + 0.2 = \underline{0.2}$

$\underline{y_1} = y_0 + 0.2 \cdot F(x_0, y_0)$ $\quad$ (had $F(x,y) = \underbrace{xy - x^2}_{=y'}$)

$= 1 + 0.2 \cdot F(0,1) = 1 + 0.2(0 \cdot 1 - 0^2) = \underline{1}$

have x_1, y_1, then reuse formula for x_2, y_2!!

$x_2 = 0.4$ and $y_2 = 1.032$ ... and so on exercise

$y_3 = 1.08256$

$y_4 = 0.8524672$ $\quad$ not rounded!!

$y_5 = 0.348861952$ ← guess for $\underline{\underline{y(1)}}$!