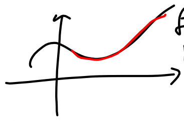
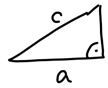
Arc length § 8.1



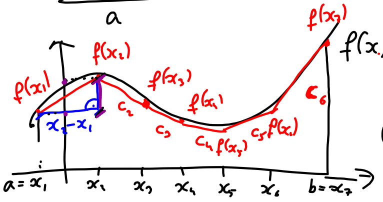
How long is the red part?

recall: right triangle



Pythagores: $c^2 = a^2 + b^2$

or $c = \sqrt{a^2+b^2}$

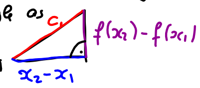


estimate length btw. a and b.

(1) choose $x_1, \dots, x_6$

(2) compute length of red path to estimate curve length

Can express the sides of the triangle as



$$c_1 = \sqrt{(x_2-x_1)^2 + (f(x_2)-f(x_1))^2}$$

Can do the same for all c_i.

So the length of red path is

$$L = \underbrace{\sqrt{(x_2-x_1)^2+(f(x_2)-f(x_1))^2}}_{c_1} + \underbrace{\sqrt{(x_3-x_2)^2+(f(x_3)-f(x_2))^2}}_{c_2} \cdots$$

$$= \sum_{i=1}^{7} \underbrace{\sqrt{(x_{i+1}-x_i)^2 + (f(x_{i+1})-f(x_i))^2}}_{= c_i}$$

Want a better estimate: make gaps smaller, more x_i

$$L = \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{(x_{i+1}-x_i)^2 + (f(x_{i+1})-f(x_i))^2}$$

Mean value theorem (Calc I): (f differ.)

$$\exists c \text{ s.t. } \frac{f(x_{i+1})-f(x_i)}{x_{i+1}-x_i} = f'(c).$$

Can replace: $f(x_{i+1})-f(x_i) = f'(c)\cdot(x_{i+1}-x_i)$

fill into root:

$$\sqrt{(x_{i+1}-x_i)^2 + (f'(c)(x_{i+1}-x_i))^2}$$

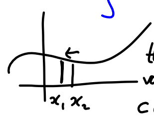
$$= (x_{i+1}-x_i)\cdot\sqrt{1+(f'(c_i))^2} \quad \text{(verify, ask TAs)}$$

So: $$L = \lim_{n\to\infty}\sum_{i=1}^{n} \underbrace{(x_{i+1}-x_i)}_{\text{difference of } x\text{'s}}\cdot\sqrt{1+f'(c_i)}$$

difference of x's + lim ⇝ integral

$$\boxed{L = \int_{x_1=a}^{x_7=b} \sqrt{1+f'(x)^2}\,dx}$$

arc length of $f(x)$ between a and b

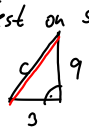


as $x_2 \to$ the mean value. $c \in [x_1, x_2]$ ⇝ $c = x_2 = x_1$ in the limit

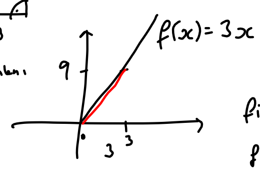
(Need to know formula, but not how we got it)

Test on something we can check:

Ex: right triangle, find c with an integral.



function:



arc length between 0, 3 is the length of c

first, need $f'(x)$:

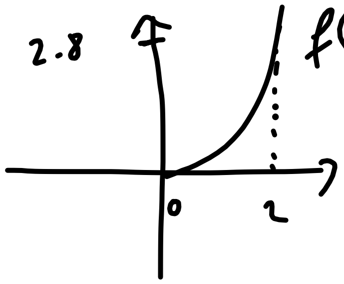
$f'(x) = 3$

put into formula:

$$L = \int_0^3 \sqrt{\underbrace{1+(3)^2}_{=10}}\,dx = \int_0^3 \sqrt{10}\,dx = \left[\sqrt{10}\cdot x\right]_0^3$$

Now more elaborate examples:

Ex $f(x) = x^{3/2}$ on $0, 2$



again: $f'(x) = \frac{3}{2} \cdot x^{3/2 - 1} = \frac{3}{2} \cdot x^{1/2}$

$$L = \int_0^2 \sqrt{1 + (f'(x))^2}\, dx = \int_0^2 \sqrt{1 + \left(\tfrac{3}{2} x^{\frac{1}{2}}\right)^2}\, dx$$

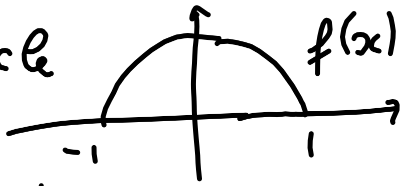
subst. rule

$$= \int_0^2 \sqrt{1 + \tfrac{9}{4} \cdot x}\, dx \overset{\checkmark}{=} \left| \begin{array}{l} u = 1 + \frac{9}{4} x \\ du = \frac{9}{4} dx \end{array} \right| = \int_{1+\frac{9}{4}\cdot 0}^{1+\frac{9}{4}\cdot 2} \sqrt{u} \cdot \tfrac{4}{9}\, du$$

$$= \frac{4}{9} \int_1^{1+\frac{9}{2}} u^{1/2}\, du = \frac{4}{9} \left[\frac{2}{3} \cdot u^{3/2} \right]_1^{11/2} = \frac{4}{9} \left(\frac{2}{3} \cdot \left(\frac{11}{2}\right)^{3/2} - \frac{2}{3} \cdot 1^{3/2} \right) = \underline{\underline{3{,}526}}$$

→ makes sense.

Ex: Half circle



$f(x) = \sqrt{1 - x^2}$

$f'(x) = \frac{1}{2 \cdot \sqrt{1 - x^2}} \cdot (-2x)$ ← inner derivative (chain rule)

$$L = \int_{-1}^{1} \sqrt{1 + \left(\frac{-\cancel{2}x}{\cancel{2}\sqrt{1 - x^2}} \right)^2}\, dx$$

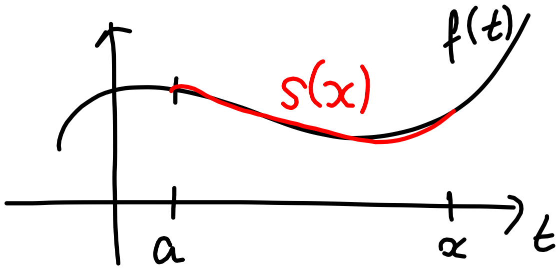
Hard to compute!!

can't express the integral as function! can't compute it.

(later: via parametrisation - CALC III)

formula: very sensitive to details.

Arc length function (p541 – 7th edition)



no fixed interval. Start at a go to x, keep x variable.

$s(x)$ ... arc length from a to x.

$$s(x) = \int_a^x \sqrt{1+(f'(t))^2}\,dt$$

One example: (application)

check as exercise

Ex: $f(x) = \left(\frac{3}{2}x\right)^{\frac{2}{3}} + 1$, $f'(x) = \left(\frac{3}{2}x\right)^{-\frac{1}{3}}$

$$s(t) = \int_a^t \sqrt{1+\left(\left(\tfrac{3}{2}x\right)^{-\frac{1}{3}}\right)^2}\,dx$$

$\frac{2}{3}\cdot\left(\frac{3}{2}\cdot x\right)^{-\frac{1}{3}}\cdot\frac{3}{2}$

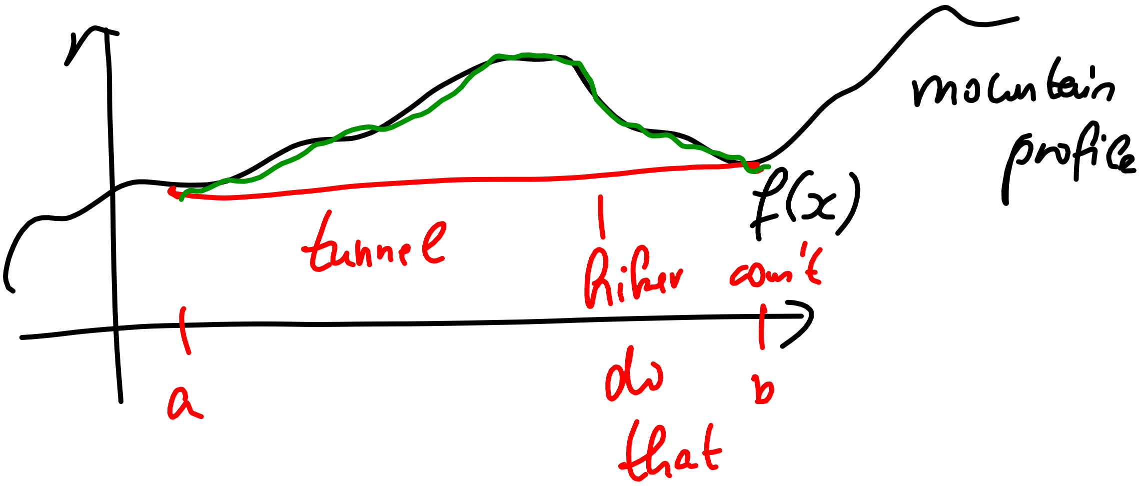
exponent, function, inner der.

$$= \int_a^t \sqrt{1+\left(\frac{1}{\left(\frac{3}{2}x\right)^{\frac{1}{3}}}\right)^2}\,dx = \int_a^t \sqrt{\frac{\left(\left(\frac{3}{2}x\right)^{\frac{1}{3}}\right)^2 + 1}{\underbrace{\left(\left(\frac{3}{2}x\right)^{\frac{1}{3}}\right)^2}_{=\left(\frac{3}{2}x\right)^{\frac{2}{3}}}}}\,dx$$

$=$ (subst. rule) $\left| \begin{array}{l} u = \left(\frac{3}{2}x\right)^{\frac{2}{3}} + 1 \\ du = \left(\frac{3}{2}x\right)^{-\frac{1}{3}} dx \end{array} \right|$ $\left(\frac{3}{2}x\right)^{-\frac{1}{3}} = f'(x)$

$$= \int_{\left(\frac{3}{2}a\right)^{\frac{2}{3}}+1}^{\left(\frac{3}{2}t\right)^{\frac{2}{3}}+1} \frac{\sqrt{u}}{\cancel{\left(\frac{3}{2}x\right)^{\frac{1}{3}}}}\cdot\cancel{\left(\tfrac{3}{2}x\right)^{\frac{1}{3}}}\,du$$

$$= \int_{\left(\frac{3}{2}a\right)^{\frac{2}{3}}+1}^{\left(\frac{3}{2}t\right)^{\frac{2}{3}}+1} \sqrt{u}\,du = \left[\frac{2}{3}u^{\frac{3}{2}}\right]_{\left(\frac{3}{2}a\right)^{\frac{2}{3}}+1}^{\left(\frac{3}{2}t\right)^{\frac{2}{3}}+1} = \frac{2}{3}\left[\left(\left(\tfrac{3}{2}t\right)^{\frac{2}{3}}+1\right)^{\frac{3}{2}} - \left(\left(\tfrac{3}{2}a\right)^{\frac{2}{3}}+1\right)^{\frac{3}{2}}\right]$$


mountain profile
f(x)
tunnel
Hiker can't do that
a
b

application to physics §8.3

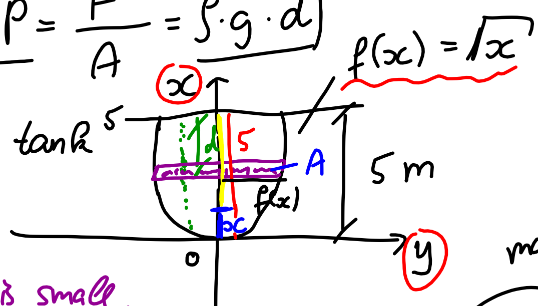
Hydrostatic pressure

$$F = m \cdot g = \rho \cdot g \cdot \underbrace{A \cdot d}_{\text{volume}}$$

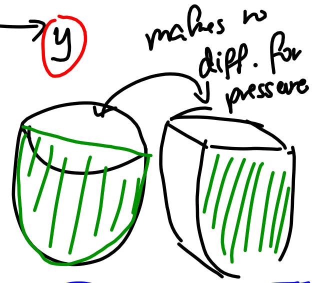
force, mass, gravity constant, density, area, depth

pressure: $\boxed{P = \frac{F}{A} = \rho \cdot g \cdot d}$

Ex water tank



if width is small, pressure almost constant.



depth at point x is given by:

$$d(x) = (5 - x)$$

$$F = 1000 \frac{kg}{m^3} \cdot 9.81 \frac{m}{s^2} \cdot (5 - x) \cdot \overbrace{(\underbrace{2f(x)}_{\text{width}} \cdot \Delta x)}^{\text{area } A}$$

↑approx. ↑density ↑g ↑width of slice

(Δx: difference of x's: eg $x_2 - x_1 = \Delta x$)

$$F = \int 9.81 \cdot 1000 \cdot (5-x) \cdot 2f(x)\, dx$$

limit of diff.

bounds: 0 m is bottom, 5 m ... top

→ from 0 to 5 over x.

$$= \int_0^5 9810 \cdot (5-x) \cdot 2 \cdot \sqrt{x}\, dx = 19620 \int_0^5 (5-x)\sqrt{x}\, dx$$

$$= 19620 \int_0^5 (5\sqrt{x} - x \cdot x^{\frac{1}{2}})\, dx =$$

exercise

$$= 19620 \left(\int_0^5 5\sqrt{x}\, dx - \int_0^5 x^{\frac{3}{2}}\, dx \right) = \underline{292\,477.69\,N}$$