Average value of a function §6.5

We define the average value of f on $[a,b]$ as:

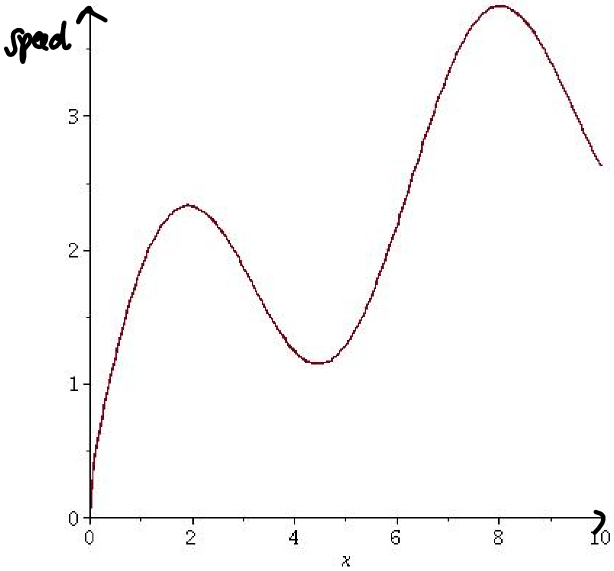
$$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx}$$

mean value thm for integrals:

If f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ s.t. $f(c) = f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$

Ex average speed of a runner: time 0 to 10 secs

Speed curve:



$f(x) = \sin(x) + \sqrt{x}$

Compute average speed via integral as above:

$$f_{ave} = \frac{1}{10-0}\int_0^{10} \sin(x)+\sqrt{x}\,dx$$

$$= \frac{1}{10}\left[-\cos(x) + \frac{x^{3/2}}{\frac{3}{2}}\right]_0^{10}$$

set calc to RAD!

$$= \frac{1}{10}\left[-\cos(10) + \frac{2}{3}\,10^{3/2} - \left(-\cos(0) + \frac{2}{3}\cdot 0^{3/2}\right)\right]$$

$$= 2.292\,\frac{m}{s}$$

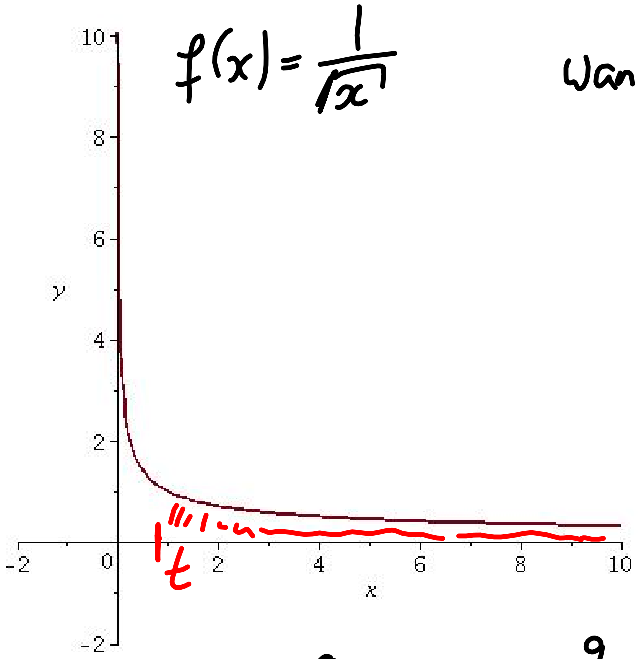
aside: $\dfrac{\frac{a}{b}}{\frac{c}{d}} = \dfrac{ad}{bc}$

Improper Integrals §7.8

know how to compute $\int_a^b f(x)\,dx$

BUT: 'small print': f is continuous!

$f(x) = \frac{1}{\sqrt{x}}$ want: $\int_0^9 f(x)\,dx$



let's work around: if $t>0$, then computing $\int_t^9 f(x)\,dx$ is no problem.

$$\int_t^9 f(x)\,dx = \int_t^9 \frac{1}{\sqrt{x}}\,dx = \int_t^9 x^{-\frac{1}{2}}\,dx = \left[2x^{\frac{1}{2}}\right]_t^9$$

$$= 2\cdot 9^{\frac{1}{2}} - 2\cdot t^{\frac{1}{2}} = \underline{2\cdot 3 - 2\cdot t^{\frac{1}{2}}}$$

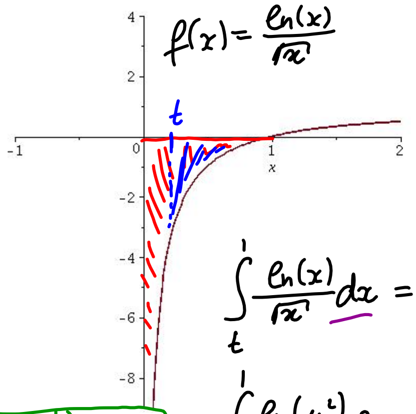
Now: want $t \to 0^+$ (approach 0 from above)

$$\lim_{t\to 0^+} \underline{6 - 2\cdot t^{\frac{1}{2}}} \underset{\text{subst. rule}}{=} \underline{\underline{6}}$$

This is an improper integral with a discontinous integrand

→ TYPE 2 (in book)

Ex2 (discon. integrand) $f(x)=\frac{\ln(x)}{\sqrt{x}}$, $\int_0^1 f(x)\,dx = ?$



compute red area.

Same as before:

Integrate from t to 1, get blue area, then let $t\to 0^+$

$$\int_t^1 \frac{\ln(x)}{\sqrt{x}}\,dx = \left| \begin{array}{l} u=\sqrt{x} \\ du=\frac{1}{2\cdot\sqrt{x}}\,dx \quad \text{or} \quad dx = 2u\,du \end{array}\right.$$

$$\boxed{\ln(a^b)=b\ln(a)} = \int_t^1 \frac{\ln(u^2)}{\cancel{u}}\, 2\cdot\cancel{u}\,du = 2\cdot\int_t^1 2\cdot\ln(u)\,du$$

$$=4\cdot\int_t^1 \ln(u)\,du = \left| \begin{array}{ll} f=\ln(u) & g'=1 \\ f'=\frac{1}{u} & g=u \end{array}\right|$$

via int. by parts

$$=4\left(u\cdot\ln(u)\Big|_t^1 - \int_t^1 \frac{1}{u}\cdot u\,du\right) = 4\left(u\cdot\ln(u)\Big|_t^1 - \int_t^1 1\,du\right)$$

$$=4\left([1\cdot\underbrace{\ln(1)}_{=0} - t\cdot\ln(t)] - u\Big|_t^1\right) = 4(-t\cdot\ln(t)-1+t)$$

rechange boundaries: get $u=\sqrt{x}$, so need $\sqrt{t}$ instead of t.

actually: $-\sqrt{t}\cdot\ln(\sqrt{t})-1+\sqrt{t}$

Now: $4\cdot\lim_{t\to 0}(-\sqrt{t}\cdot\ln(\sqrt{t})\underbrace{-1+\sqrt{t}}_{\text{subst. } t=0}) = 4\lim_{t\to 0}(-\sqrt{t}\cdot\ln(\sqrt{t}))-1$

"$0\cdot(-\infty)$" "$\frac{\infty}{\infty}$" l'Hopital

$$=4\left(-1+\lim_{t\to 0}\sqrt{t}\cdot\ln(\sqrt{t})\right) \underset{\text{rewrite}}{=} 4\left(-1+\lim_{t\to 0}\frac{\ln(\sqrt{t})}{\frac{1}{\sqrt{t}}}\right) =$$

chain rule

$$= -4+4\cdot\lim_{t\to 0}\frac{(\ln(\sqrt{t}))'}{\left(\frac{1}{\sqrt{t}}\right)'} = -4+4\cdot\lim_{t\to 0}\frac{\frac{1}{\sqrt{t}}\cdot\frac{1}{2\sqrt{t}}}{t^{-\frac{3}{2}}\cdot\left(-\frac{1}{2}\right)}$$

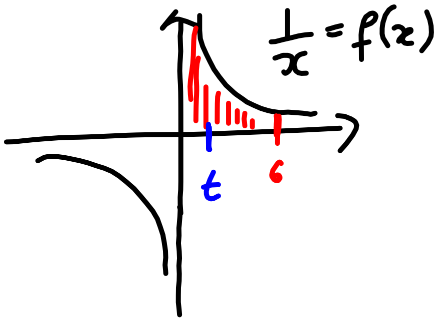
$$= -4+4\cdot\lim_{t\to 0}\frac{t^{-\frac{1}{2}}\cdot\cancel{\frac{1}{2}}\cdot t^{-\frac{1}{2}}}{-\cancel{\frac{1}{2}}\cdot t^{-\frac{3}{2}}} = -4+4\cdot\lim_{t\to 0}\frac{t^{-1}}{t^{-\frac{3}{2}}} =$$

$$= -4+4\cdot\lim_{t\to 0}\frac{t^{\frac{3}{2}}}{t} = -4+4\cdot\lim_{t\to 0} t^{\frac{1}{2}} \underset{\text{dir. subs.}}{=} -4+4\cdot 0 = \underline{\underline{-4}}$$

The entire area is below the x-axis, so a negative sign actually makes sense!

Another example

$$\int_0^6 \frac{1}{x}\,dx$$



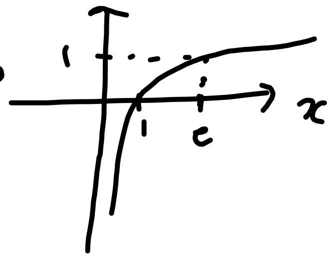
again, go from t to 6, and then $t \to 0$.

$$\int_t^6 \frac{1}{x}\,dx = \Big[\ln|x|\Big]_t^6 = \ln(6) - \ln(t)$$

(abs. value here not necessary as, $t > 0$, $6 > 0$)

NOW: $\lim\limits_{t\to 0}\left(\ln(6) - \ln(t)\right) = \ln(6) - \lim\limits_{t\to 0}\ln(t) =$

(Calc 1: $\lim\limits_{t\to 0}\ln(t) = -\infty$



)

So limit above

$= \underline{\ln(6) - (-\infty) = \infty}$

We call these integrals **divergent**!

If $\int_a^b f(x)\,dx$ has a real value despite $f(x)$ being discontinuous on $[a,b]$, then we call it **convergent**.

Otherwise, so if the area is infinite, it is called **divergent**.

This method ('Integrate to t, and then $\lim_{t\to a}$') works only if I can compute the integral.

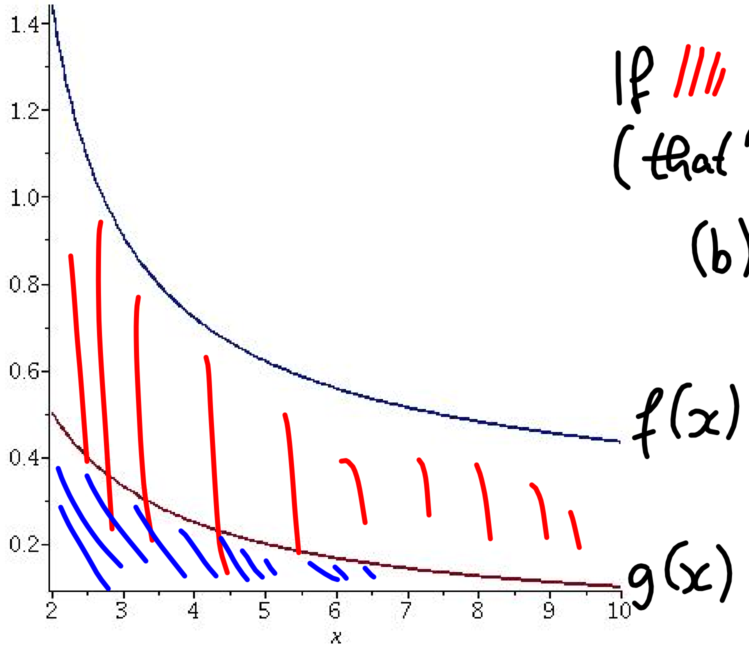
(Sometimes, we can't: e.g $\int_0^{\pi} \frac{1}{\sin(x)}\,dx$)

<u>Comparison</u>: (sim. to p. 525) f, g contin.

$f(x) \geq g(x) \geq 0$, $x \geq a$

(a) If $\int_a^b f(x)dx$ converges, so does $\int_a^b g(x)\,dx$.

(b) If $\int_a^b g(x)\,dx$ diverges, so does $\int_a^b f(x)\,dx$.



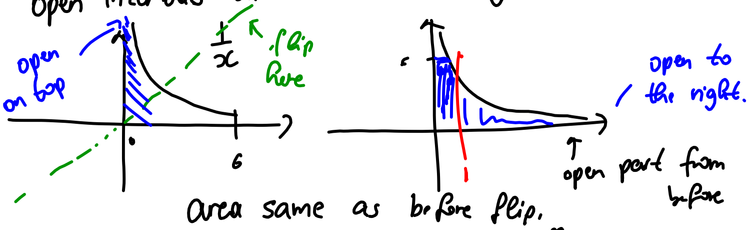
If //// is finite, so is ////.
(that's what (a) says)

(b) says, if |||| is infinite, so is \\\\.

This theorem only tells you if an area finite or infinite, not it's value.

Different type of improper integrals: (TYPE I in book)

open intervals but continuous integrand.



area same as before flip.

We computed $\int_0^6 \frac{1}{x}\,dx$, was finite. So: $\int_1^\infty \frac{1}{x}\,dx$ would be finite too. BUT: it's not! Try another one:

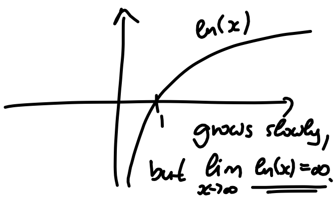
$$\int_1^\infty \frac{1}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{t} - \underbrace{\left(-\frac{1}{1}\right)}_{=1}\right)$$

(again, go up to t, then $t\to\infty$) $= 1 + \lim_{t\to\infty}\left(-\frac{1}{t}\right) = 1 + 0 = 1$

Now get back to $\frac{1}{x}$: ($t>0$ so no $|\cdot|$)

$$\lim_{t\to\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty}\left[\ln|x|\right]_1^t = \lim_{t\to\infty}\left(\ln(t) - \underbrace{\ln(1)}_{=0}\right)$$

$$= \lim_{t\to\infty}\ln(t) = \infty$$

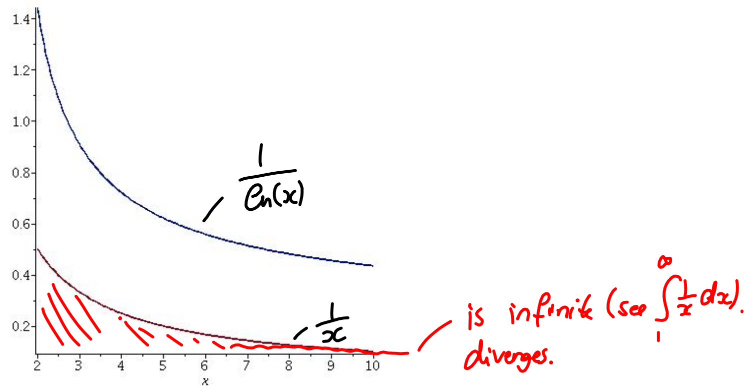


this is divergent.

This requires computing the integral.

Comparison thm as before (p. 525)

This tells that for example $\int_2^\infty \frac{1}{\ln(x)}\,dx$ diverges.



So comparison tells you, $\int_2^\infty \frac{1}{\ln(x)}\,dx$ diverges too.

Without maple: compare: $\frac{1}{x} \le \frac{1}{\ln(x)}$ is true for $x\in[2,\infty]$

$\ln(x) \le x$ for $x\in[2,\infty]$ (easy to check).

last example: (0 a gap!)

$$\int_0^1 \frac{e^{1/x}}{x^3}\,dx = \lim_{t\to 0}\underbrace{\int_t^1 \frac{e^{1/x}}{x^3}\,dx}_{I}$$

$$I = \int_t^1 \frac{e^{1/x}}{x^3}\,dx = \left|\begin{array}{l} u = \frac{1}{x} \\ du = -\frac{1}{x^2}\,dx \end{array}\right| = -\int_{1/t}^{1} \frac{e^u}{x^{\cancel{3}}}\cdot x^{\cancel{2}}\,du$$

$$= -\int_{1/t}^{1} \frac{e^u}{\frac{1}{u}}\,du = -\int_{1/t}^{1} u\cdot e^u\,du = \left|\begin{array}{ll} f = u & g' = e^u \\ f' = 1 & g = e^u \end{array}\right|$$

$$= -u\cdot e^u\Big|_{1/t}^{1} - \left(-\int_{1/t}^{1} e^u\,du\right) = -1\cdot e^1 - \left(-\frac{1}{t}\cdot e^{1/t}\right) + e^u\Big|_{1/t}^{1}$$

$$= -\cancel{e} + \frac{e^{1/t}}{t} + \cancel{e} - e^{1/t}$$

Now: $t \to 0$:

$$\lim_{t\to 0} \underbrace{e^{1/t}}_{\text{"}\infty\text{"}}\Big(\underbrace{\tfrac{1}{t}}_{\text{"}\infty\text{"}} - 1\Big) = \underline{\underline{\infty}} \qquad \underline{\text{diverges}}.$$