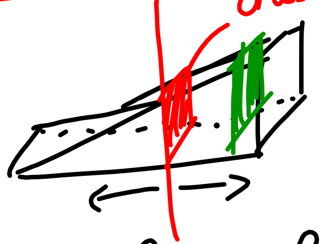
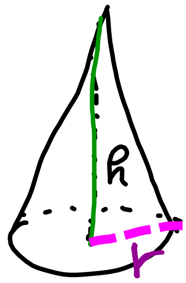
§6.2 Volumes

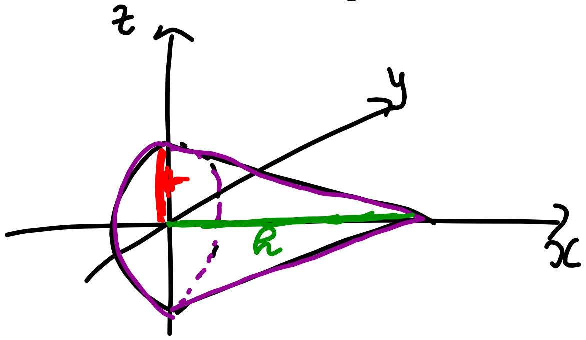
cross section area (depends on x-value)



Last time volume of wedge (half box)

Today: volume of a cone via integrals

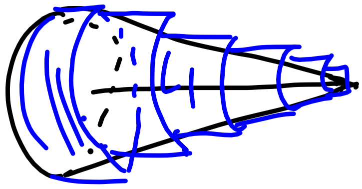




area decreases with x

disc

Want to find cross section area function.



can approx. volume by slices

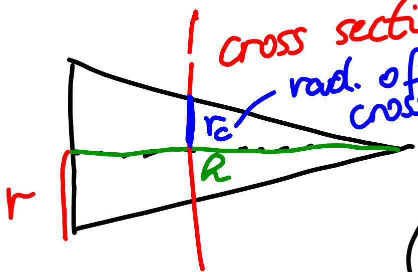
sum of volumes of slices gives an estimate for volume of cone.

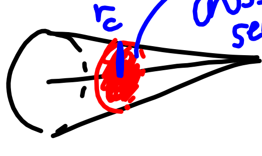
→ this is why we get the volume by integrating over the cross section area function.

Area of a circle with radius r: $A = r^2 \cdot \pi$

So we need to find the radius of the cross section.

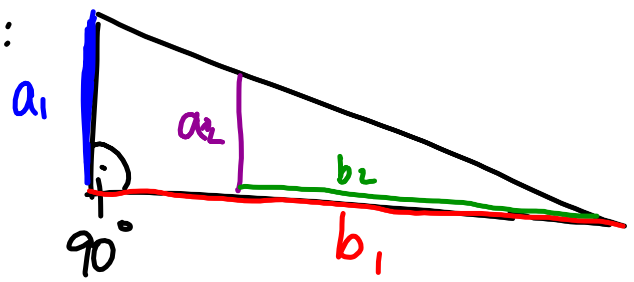
Horizontal 'slice':





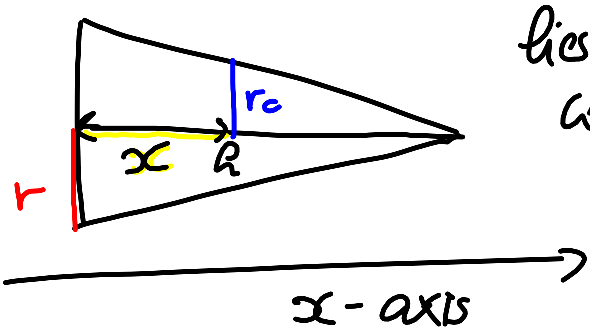
Now: try to express r_c using r and h.

ratios:



$$\frac{a_1}{b_1} = \frac{a_2}{b_2}$$

Use this!



lies in a coordinate system

b_1 corresp. to h

$a_2 \longleftrightarrow r_c$

$a_1 \longleftrightarrow r$

$b_1 - b_2 \longleftrightarrow x$

put together: $\dfrac{r}{h} = \dfrac{r_c}{h-x}$

express $\boxed{r_c(x) = \frac{r}{h}(h-x) = \frac{r\cancel{h}}{\cancel{h}} - \frac{rx}{h} = r - \frac{rx}{h}}$

Now: cross section area: area of circle

$$A(x) = (r_c(x))^2 \cdot \pi = \left(r - \frac{rx}{h}\right)^2 \cdot \pi$$

integrate for volume: want to integrate from 0 to h

$\uparrow$ top of cone!

$$V = \int_0^h A(x)\,dx$$

$$= \int_0^h \left(r - \frac{rx}{h}\right)^2 \cdot \pi\,dx = \pi \cdot \int_0^h \left(r - \frac{rx}{h}\right)^2 dx$$

$$= \pi \cdot \int_0^h \left(r^2 - 2\frac{r^2 x}{h} + \frac{r^2 x^2}{h^2}\right) dx$$

$$= \pi \cdot \left[r^2 \cdot x - \frac{\cancel{2} r^2}{h} \cdot \frac{x^2}{\cancel{2}} + \frac{r^2}{h^2} \cdot \frac{x^3}{3}\right]_0^h$$

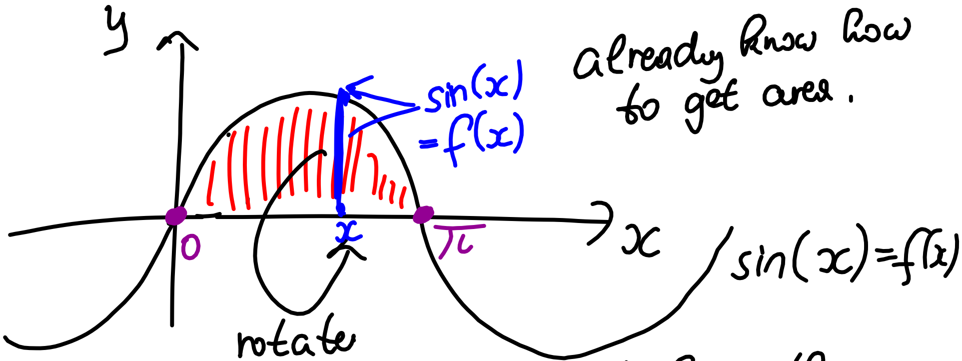
$$= \pi \left[r^2 \cdot x - \frac{r^2 x^2}{h} + \frac{r^2 x^3}{3h^2}\right]_0^h$$

$$= \pi \left(\left(\cancel{r^2 h} - \cancel{\frac{r^2 h^2}{h}} + \frac{r^2 \cdot h^3}{3h^2}\right) - \left(r^2 \cdot 0 - \frac{r^2 \cdot 0^2}{h} + 0\right)\right)$$

$$= \pi \cdot \frac{r^2 h}{3} = \frac{r^2 h \pi}{3}$$

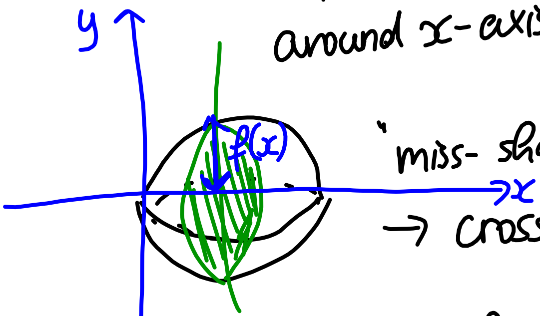
$$\boxed{\text{Difficult part: find cross section area function!}}$$

Rotating functions (volumes §6.2)



already know how to get area.

rotate around x-axis, want to know the volume!



'miss-shaped ball'

→ cross section area!!

Cross section area of a rotated function is always a circle!

Here: radius of this circle is always the value of $f(x)$ at x.

area of circle:

$A = r^2 \cdot \pi$, here:

$A = (f(x))^2 \cdot \pi$

$$V = \int_0^{\pi} (\sin(x))^2 \cdot \pi \, dx$$

trig ID:

$\sin^2(x) = \frac{1}{2}(1-\cos(2x))$

(don't expect you to know these)

$$= \pi \cdot \int_0^{\pi} (\sin(x))^2 \, dx =$$

exercise → use subst. rule for $\cos(2x)$

$$= \pi \int_0^{\pi} \frac{1}{2}(1-\cos(2x))\,dx =$$

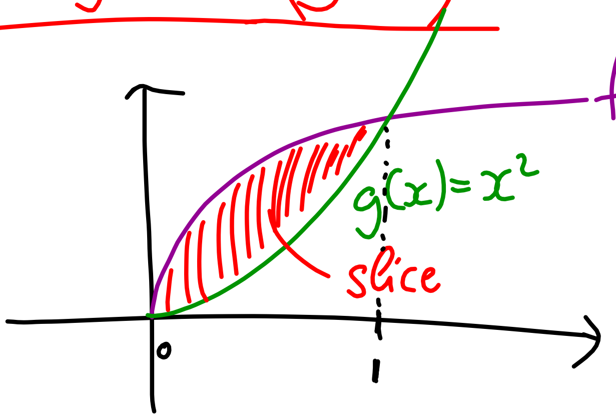
$$= \frac{\pi}{2}\left[\left(x - \frac{1}{2}\sin(2x)\right)\right]_0^{\pi} = \frac{\pi^2}{2}$$

volume of the rotated object.

Need to remember: area of a circle of radius r:

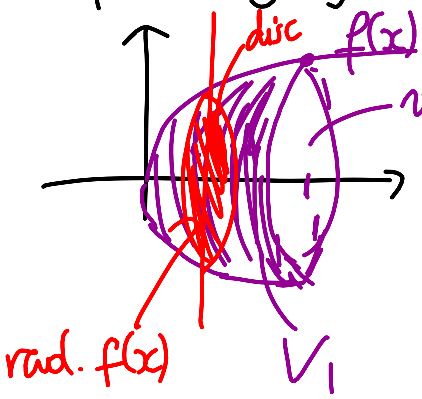
$$\boxed{A = r^2\pi}$$

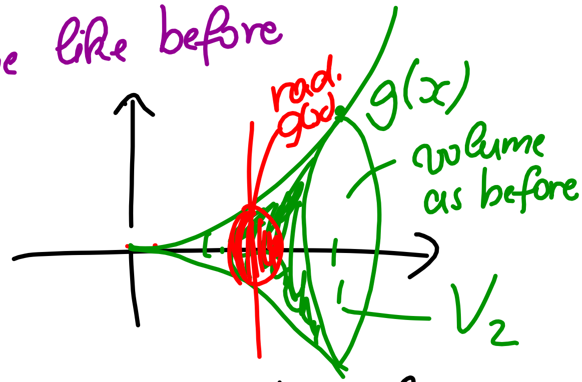
Rotating slices (§ 6.2)



Rotation of an area between two curves.

Again: volume of rotating $f(x)$, subtract volume of rotating $g(x)$. (compare to area betw. curves)





To compute the volume of the rotated slice:

Need $V = V_1 - V_2$

Ex: $f(x) = \sqrt{x}$, $g(x) = x^2$, meet at 0, 1

$$V = \pi\int_0^1 f(x)^2\,dx - \pi\int_0^1 g(x)^2\,dx = \pi\int_0^1 f(x)^2 - g(x)^2\,dx$$

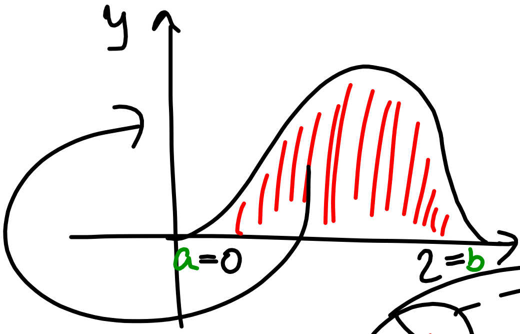
$$= \pi\int_0^1 (\sqrt{x})^2 - (x^2)^2\,dx = \pi\int_0^1 (x - x^4)\,dx$$

$$= \pi\left[\frac{x^2}{2} - \frac{x^5}{5}\right]_0^1 = \pi\left(\left(\frac{1}{2} - \frac{1}{5}\right) - \left(\frac{0}{2} - \frac{0}{5}\right)\right)$$

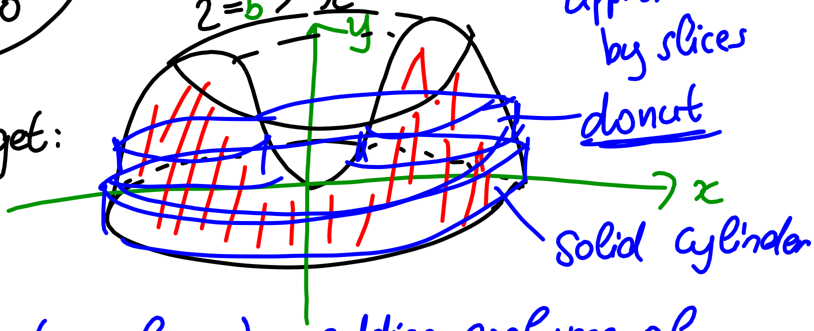
$$= \frac{3\pi}{10}$$

Rotate around y-axis (§6.3)

$y = 2x^2 - x^3$ (Ex 1 p 434)



object we get:



approximate volume by adding volume of "cylinder & donuts"

Turns out, this work:

$$\boxed{V = \int_a^b 2\pi x \cdot f(x)\,dx}$$

a, b are points where $f(x)$ crosses the x-axis (or specified - given)

For our curve:

$$V = \int_0^2 2\pi x \cdot f(x)\,dx$$

$$= 2\pi \int_0^2 x \cdot (2x^2 - x^3)\,dx = 2\pi \int_0^2 2x^3 - x^4\,dx$$

$$= 2\pi \left[\frac{2x^4}{4} - \frac{x^5}{5}\right]_0^2 \underset{\text{ex}}{=} \frac{16\pi}{5}.$$