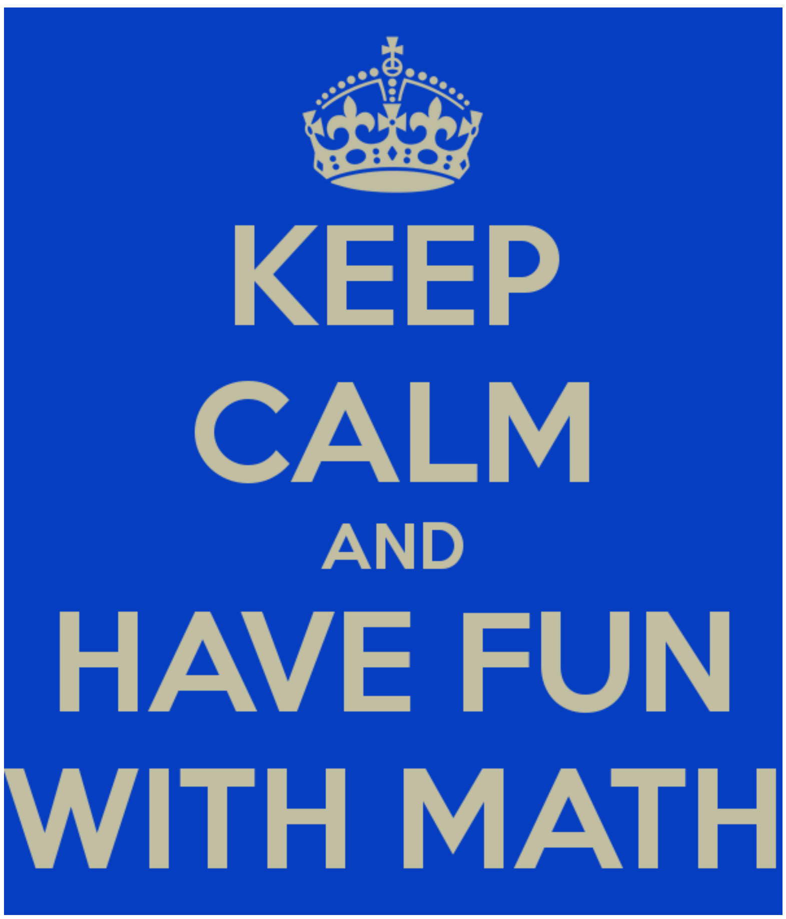
Limits, Exponentials, and Logarithms

MAT 1300 A &D

Fall 2016



1 Limits

Ex: Consider the function

$$f(x) = \frac{|x^2 - 1|}{x - 1}.$$

a) What is the domain of f?

b) As x gets really close to 1 on the left ($x < 1$) what happens to f?

c) As x gets really close to 1 on the right ($x > 1$) what happens to f?

If $f(x)$ becomes arbitrarily close to a number L as x approaches a from the left (i.e. $x < a$) then we say that L is the *left-handed limit* of $f(x)$ as x approaches a from the left. This is denoted

$$\lim_{x \to a^-} f(x) = L.$$

We similarly define the *right-handed limit* of $f(x)$ as x approaches a from the right and denote it by

$$\lim_{x \to a^+} f(x) = L.$$

In the above example the left-handed limit as x approaches 1 from the left is -2. The right-handed limit as x approaches 1 from the right is 2. The chart method we used is called the *numerical* method of finding the limit.

Ex: Find the left-handed and right-handed limits of

$$f(x) = \frac{|x^2 - 1|}{x - 1}$$

as x approaches 1 from the graph. (This is the *graphical* method of finding the limit)

Note: *When finding* $\lim_{x\to a^+} f(x)$ or $\lim_{x\to a^-} f(x)$ *it does not matter what* $f(a)$ *is or even if it is defined!*

In fact, $\lim_{x\to a^-} f(x)$ and $f(a)$ can both exist but be different!

Example:

$$f(x) = \begin{cases} 0 & \text{if } x = 1 \\ \frac{|x^2-1|}{x-1} & \text{otherwise} \end{cases}$$

If $\lim_{x\to a^+} f(x)$ and $\lim_{x\to a^-} f(x)$ both exist and have the same value (say L) then we say that the *limit* of $f(x)$ as x approaches a exists and is equal to L. This is denoted

$$\lim_{x\to a} f(x) = L.$$

2 Rules for Limits

Let c and a be real numbers and let n be any positive integer. Suppose that $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$.

Rules for Limits:

1. $\lim_{x\to a} c = c$
2. $\lim_{x\to a} x = a$
3. $\lim_{x\to a}(cf(x)) = cL$
4. $\lim_{x\to a}(f(x) + g(x)) = L + M$
5. $\lim_{x\to a}(f(x)g(x)) = LM$
6. $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{M}$ provided $M \neq 0$

7. $\lim_{x\to a}(f(x))^n = L^n$

8. $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{L}$ provided $L > 0$

Fact: If $p(x)$ is a polynomial then, for any real a,

$$\lim_{x\to a} p(x) = p(a).$$

Ex: Find the following limits if they exist.

a)

$$\lim_{x\to 2} \frac{(4x-7)^3}{x^2+5}$$

b)

$$\lim_{x\to 1} \frac{x^2 - 6x + 5}{x^2 - 1}$$

c)

$$\lim_{x\to 3} \frac{x^2 - x - 6}{|x - 3|}$$

3 Unbounded Functions

$\lim_{x\to\infty} f(x)$ is the value (if one exists) which $f(x)$ approaches as x is made arbitrarily large.

Ex: Find

$$\lim_{x\to\infty} \frac{4x^2 - 3x + 7}{2x^2 + 5x - 1}.$$

We use the notation $\lim_{x\to a} f(x) = \infty$ to represent the idea that $f(x)$ can be made arbitrarily large as x approaches a.

Ex: Find

$$\lim_{x\to -3} \frac{5}{(x+3)^2}$$

4 Continuity

Recall that if $f(x)$ is a polynomial, then $\lim_{x\to c} f(x) = f(c)$. These types of limits are easy to calculate. This leads to the following definition. Let $c \in (a, b)$ and $f(x)$ a function whose domain contains (a, b). then the function $f(x)$ is **continuous at** c if

$$\lim_{x\to c} f(x) = f(c).$$

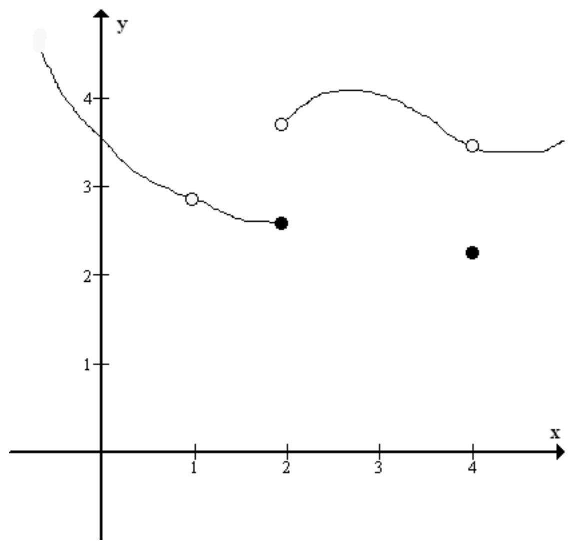
Note that this implies

1. $f(c)$ is defined,
2. the limit exists, and
3. the two are equal.

<u>Intuition:</u>

The graph of a continuous function is one that has no holes, jumps, or gaps. It can be "drawn without lifting the pencil". **This is intuition only**.

<u>Example:</u>



$f(x)$ is not continuous at

1. $x = 1$ because $f(1)$ is not defined.
2. $x = 2$ because $\lim_{x \to 2}$ does not exist.
3. $x = 4$. Here the limit exists, but is not equal to $f(4)$.

These are the three basic ways something can fail continuity.
Examples:

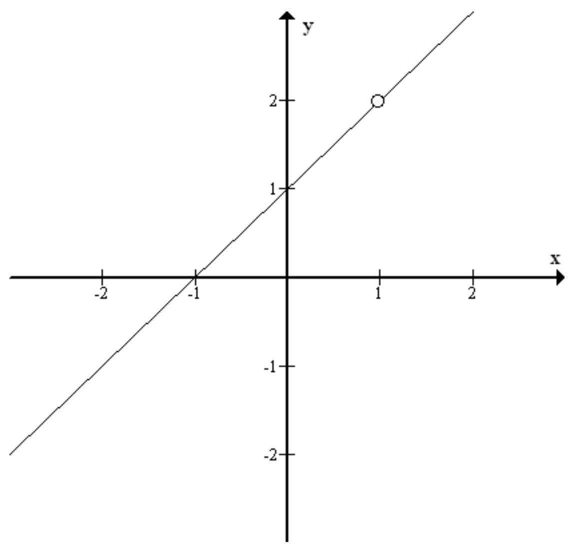
1. Any polynomial $p(x)$ is continuous everywhere.
2. A rational function is one of the form

$$f(x) = \frac{p(x)}{q(x)}$$

   where $p(x)$ and $q(x)$ are polynomials. If $f(x)$ is a rational function, it will be continuous everywhere except where $q(x) = 0$ (in these places, $f(x)$ is undefined, hence certainly not continuous).

In general, if $f(x) = \frac{p(x)}{q(x)}$, where p and q are arbitrary, then $f(x)$ is continuous everywhere that p and q are continuous and q is not 0.

3. Consider
$$f(x) = \frac{x^2 - 1}{x - 1}.$$
Let's graph it. $f(x)$ is undefined at $x = 1$. If $x \neq 1$, we get
$$\frac{x^2 - 1}{x - 1} = \frac{(x + 1)(x - 1)}{x - 1} = x + 1 \text{ – this is a line.}$$
So the graph is



This is continuous everywhere except $x = 1$. So the intervals on which it is continuous are $(-\infty, 1)$ and $(1, \infty)$.

4. The function
$$g(x) = \frac{1}{x^2 + 1}$$
is continuous everywhere because $x^2 + 1$ is never 0.

5. $f(x) = \sqrt{x}$ is continuous on the interval $[0, \infty)$

6. $f(x) = 1/x$ is continuous on the interval $(-\infty, 0)$ and $(0, \infty)$
7. The sum of continuous functions is continuous
8. The product of continuous functions is continuous

Ex: Find the intervals of the real number line on which the following functions are continuous.

a)

$$f(x) = \frac{x^2 - 5x + 2}{x + 7}$$

b)

$$f(x) = \begin{cases} 2x - 5 & \text{if } x < 3 \\ 1 & \text{if } x = 3 \\ 4 - x & \text{if } x > 3 \end{cases}$$

5 Exponential Functions and the Natural Base e

If $a > 0$ and $a \neq 1$, then *the exponential function with base* a is given by $f(x) = a^x$. An important special case is when $a = e \approx 2.71828\ldots$, an irrational number.

Properties of Exponents

Let $a, b > 0$. Then,

1. $a^0 = 1$
2. $a^x a^y = a^{x+y}$
3. $(a^x)^y = a^{xy}$
4. $(ab)^x = a^x b^x$
5. $\frac{a^x}{a^y} = a^{x-y}$
6. $(\frac{a}{b})^x = \frac{a^x}{b^x}$
7. $a^{-x} = 1/a^x$

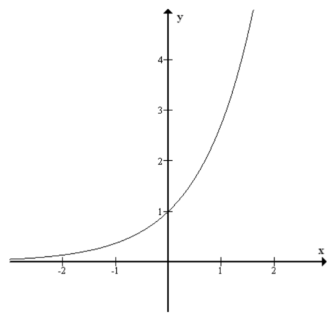
The number e is defined to be

$$\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e = \lim_{x\to 0}(1+x)^{1/x}$$

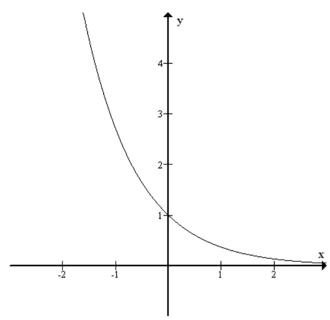
- It's possible to prove that this limit exists, but it's not obvious.
- It's an irrational number.

Graphs:

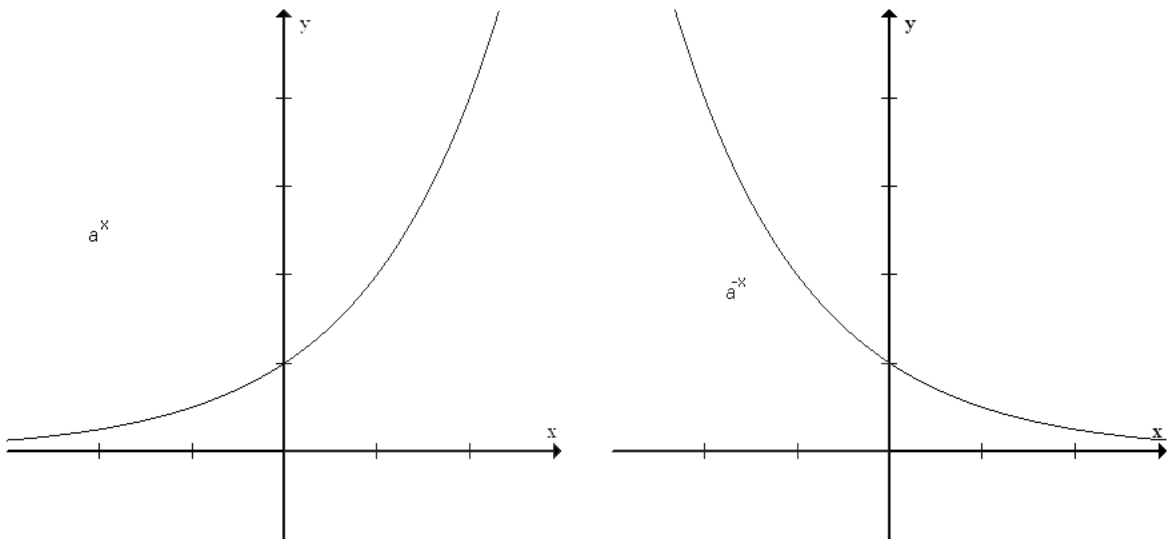
1. Exponential growth; e^x



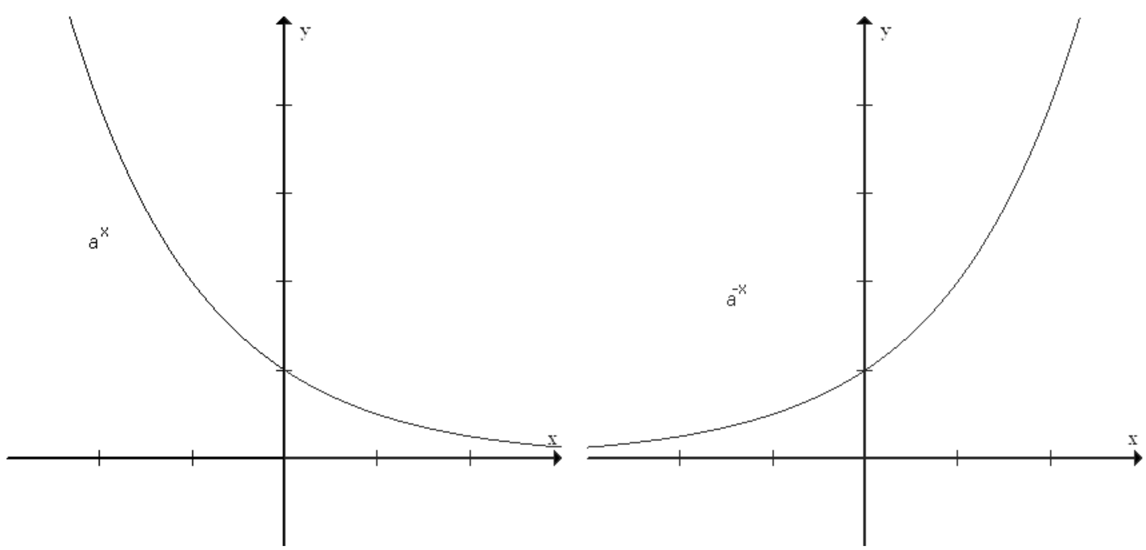
2. Exponential decay; e^{-x}



3. More generally, if $a > 1$,



If $0 < a < 1$,



Here are some sample calculations you should be able to do with exponents:

2. a)

$$\left(\frac{1}{5}\right)^3 = \frac{1}{5^3} = \frac{1}{125}$$

b)

$$\left(\frac{1}{8}\right)^{1/3} = \frac{1}{2}$$

c)

$$(64)^{2/3} = \left[(64)^{1/3}\right]^2 = 4^2 = 16$$

f)

$$4^{5/2} = \left[4^{1/2}\right]^5 = 2^5 = 32$$

You should be able to do these calculations on a test without the aid of a calculator.

Compounding Interest

Suppose I invest \$1 in a bank that pays 100% interest. Clearly, at the end of one year, I will have \$2 (it is also clear that I should be investing much more than \$1).

But suppose instead that after 6 months I withdraw my money and immediately re-invest it. How much money will I have at the end of the year?

After 6 months, we have \$1.50. If we then reinvest this at 100% interest for the rest of the year, we get

$$\$1.50\left(1+\frac{1}{2}\right) = \$2.25$$

The $\frac{1}{2}$ term above corresponds to the interest rate (100% or 1.00) divided by the number of times we compounded in the year. So by getting interest on the interest we got from the first 6 months, we ended up with more money at the end of the year. What happens if we compound it more often?

Consider this table:

# of times compounded	\$ after 1 year (approx)
1	2
2	2.25
3	2.37
4	2.44
20	2.65
100	2.70
10000	2.72

Note that:

- The more times you compound, the more money you make. However, the amount of increase gets less and less.

- The numbers on the right hand side approach a limit. Can you see what it is?

The general formula: Let

$P =$ initial deposit (or *principal*),

$r =$ interest rate, expressed as a decimal,

$n =$ number of coumpounding per year,

$t =$ number of years

Then the formula for $A(t)$, the balance after t years is

$$\boxed{A(t) = P\left[1 + \frac{r}{n}\right]^{nt}}$$

Why? You start with P dollars. When it is time for the first compounding, you multiply by $\left(1 + \frac{r}{n}\right)$(you would normally multiply by 1 plus the interest rate, but since only an nth of the year has passed we have to divide the interest rate by n). After each compounding, you multiply by another $\left(1 + \frac{r}{n}\right)$.

Example: Suppose you start with an initial deposit of \$2500 with an annual interest rate of 5%. How much do you have after 20 years if you compound yearly? Every 6 months? Every 3 months? Every month? Every day?

Solution: The ns for each other above cases are $n = 1, 2, 4, 12, 365$. So the solutions can be filled into this chart:

n	$A(t)$
1	$2500(1+0.05)^{20} \approx 6633.24$
2	$2500\left(1+\frac{.05}{2}\right)^{40}$
4	$2500\left(1+\frac{.05}{4}\right)^{80}$
12	$2500\left(1+\frac{.05}{12}\right)^{240}$
365	$2500\left(1+\frac{.05}{365}\right)^{7300}$

Note: on tests you will not be expected to simplify these numbers.

Continuous Compounding

This process of compounding repeatedly has a limit at $n \to \infty$. This is called **continuous compounding**. Let's find this formula: we will do it for $t = 1$.

$$\begin{aligned} A &= \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^n \\ &= P\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n \end{aligned}$$

If we let $x = r/n$, then $n = r/x$. Then as $n \to \infty$, $x \to 0$, so we have

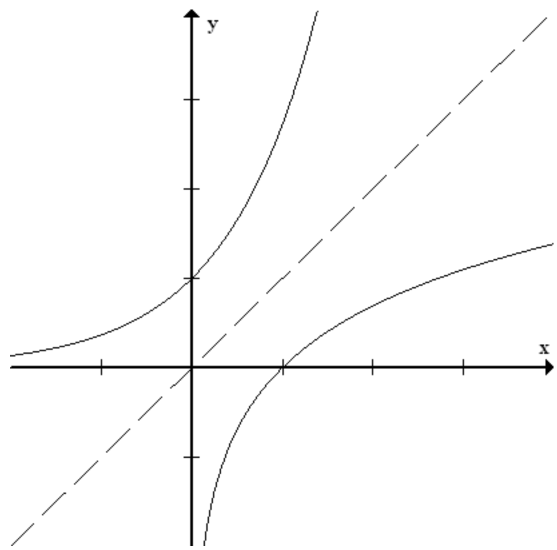
$$\begin{aligned} A &= P\lim_{x\to 0}(1+x)^{r/x} \\ &= P\left[\lim_{x\to 0}(1+x)^{1/x}\right]^r \\ &= P\cdot e^r \end{aligned}$$

In general, we have that if P is the initial amount and r is the rate, then with continuous compounding

$$\boxed{A(t) = Pe^{rt}}$$

6 Logarithms

Note the graph of e^x passes the horizontal line test, so $f(x) = e^x$ is one-to-one and therefore has an inverse function. This is also true of $f(x) = a^x$ for any $a > 0, a \neq 1$.



More generally, for any $a > 1$ the graph of a^x and its inverse look like this. If $f(x) = a^x$, then we define the inverse function f^{-1} to be the **logarithm with base** a, and write

$$f^{-1}(x) = \log_a(x)$$

Note that, since the image of a^x is only the positive numbers, the domain of $\log_a(x)$ is all positive real numbers. The key property is:

$$\boxed{\log_a x = b \Longleftrightarrow a^b = x}$$

Examples:

log equation	corresponding exponential equation
$\log_{10} 10 = 1$	$10^? = 10$
$\log_5 25 = 2$	$5^? = 25$
$\log_4 \frac{1}{2} = -\frac{1}{2}$	$4^? = \frac{1}{2}$
$\log_5 \frac{1}{125} = -3$	$5^? = \frac{1}{125}$
↑	↑
log equation	corresponding exponential equation

Log Rules

1. Most important: by the properties of inverse functions we have

$$\log_b(b^x) = x \text{ and } b^{\log_b x} = x$$

The most important case of logs is when $b = e$. Log base e has a special name, in fact we define $\log_e x = \ln(x)$. So the above becomes

$$\boxed{\ln(e^x) = x \text{ and } e^{\ln(x)} = x}$$

LEARN THIS!!

The function $\ln(x)$ is known as the **natural logarithm function**, and $\ln(x)$ should be read as "the natural logarithm of x". In class, you may also hear me read this as "lawn x", but this isn't as standard.

Other rules: (I will state for ln, but they work for every log). Suppose that $x, y > 0$

2. $\ln(xy) = \ln(x) + \ln(y)$
3. $\ln\left(\frac{x}{y}\right) = \ln x - \ln y$
4. $\ln(x^y) = y \ln(x)$

Calculations:

$$e^{3\ln(x)} = e^{\ln(x^3)} = x^3$$

$$\ln\left(\frac{1}{e}\right) = \ln(e^{-1}) = -1$$

Rewrite the following:

$$\ln\left(\frac{xy}{z}\right) = \ln(xy) - \ln(z) = \ln(x) + \ln(y) - \ln(z)$$

Compound Interest Revisited: (p324 #77)

A deposit of \$1000 is made in an account that earns interest at a rate of 5% per year. How long will it take for the balance to double if interest is compounded annually?

Solution: From our earlier formula, our balance after t years is

$$A(t) = 1000\left(1 + \frac{.05}{1}\right)^t = 1000(1.05)^t$$

We are trying to find t such that $A(t) = 2000$. So we set the formula equal to 2000:

$$2000 = 1000(1.05)^t$$

$$2 = 1.05^t$$

We have to solve this for t. The general principle we use is that if we are trying to solve for a variable in the exponent, take log of both sides. So we get

$$\ln(2) = \ln(1.05^t)$$

$$\ln(2) = t\ln(1.05)$$

In this last point we see the point of using logs - the exponent can be brought down and solved for. So,

$$t = \frac{\ln(2)}{\ln(1.05)} \approx \text{ 14.21 years.}$$

Note that this is a very typical test/exam question. The answer I would expect is $t = \frac{\ln(2)}{\ln(1.05)}$.

<u>Example 2:</u> Same question, but now interest is compounded 10 times a year.

Solution: By our formula,

$$A(t) = 1000\left(1 + \frac{.05}{10}\right)^{10t} = 1000(1.005)^{10t}$$

Again, we solve $A(t) = 2000$.

$$2000 = 1000(1.005)^{10t}$$

$$2 = 1.005^{10t}$$

$$\ln(2) = \ln(1.005^{10t})$$

$$\ln(2) = 10t\ln(1.005)$$

Therefore,

$$t = \frac{\ln(2)}{10\ln(1.005)} \approx 13.9 \text{ years.}$$

<u>Example 3:</u> Same question, but now interest is compounded continuously.

Solution: By our formula,

$$A(t) = 1000e^{.05t}$$

Again, we solve $A(t) = 2000$.

$$2000 = 1000e^{.05t}$$

$$2 = e^{.05t}$$

$$\ln(2) = \ln(e^{.05t})$$

$$\ln(2) = .05t$$

Therefore,

$$t = \frac{\ln(2)}{.05} \approx 13.86 \text{ years.}$$

Notice that continuous compounding gives a kind of "best case scenario" – no amount of compounding will get your money to double faster than approximately 13.86 years.

6.1 Solving Exponential and Logarithmic Equations

Given any equation, we can take the natural logarithm of both sides to try and find a solution!

Ex: Solve for x.

$$\ln(x) + \ln(x + 2) = \ln(15)$$