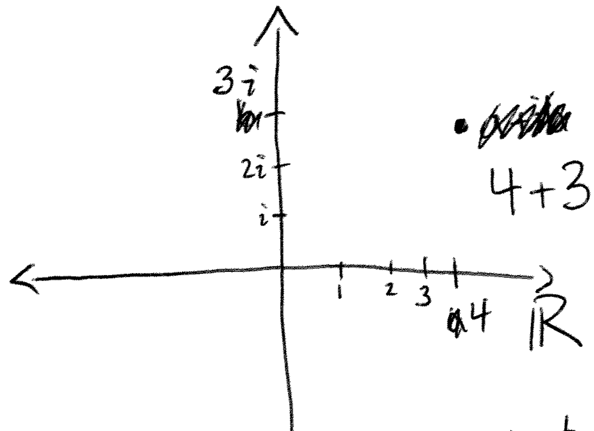
Complex Numbers

"Impossible numbers"

Working with the quadratic / cubic formula easier if you use $i = \sqrt{-1}$



such that

close bracket!

$$\mathbb{C} = \{ a + bi \mid a, b \in \mathbb{R}, \text{ and } i^2 = -1 \}$$

set of all

things like this

conditions on "things like this"

If $z = a + bi$, then $\mathrm{Re}(z) := a$ "real part of z"

$\mathrm{Im}(z) := b$ "imaginary part of z"

If $b = 0$, $z = a \in \mathbb{R}$ so $\mathbb{R} \subset \mathbb{C}$

"$\mathbb{R}$ is a subset of $\mathbb{C}$"

Arithmetic in $\mathbb{C}$

- $(a+bi) = (c+di) \iff a=c$ and $b=d$
- $(a+bi)+(c+di) = (a+c)+(b+d)i$
- $(a+bi)(c+di) = ac+(bi)c+a(di)+(bi)(di)$

$$= (ac-bd)+(bc+ad)i \quad (i^2=-1).$$

$\mathbb{C}$ satisfies all the good arithmetic of $\mathbb{R}$, but doesn't have an order – $z>0$ doesn't make sense.

Division: Can we solve $z(3+4i) = 1$ for z?

$$z = \frac{1}{3+4i} = a+bi$$

> Ex: find ~~real~~ imaginary part of $z = \frac{1}{1+i}$

trick: multiply top and bottom by $3-4i$

$$z = \frac{3-4i}{(3+4i)(3-4i)} = \frac{3-4i}{9-12i+12i+16} = \frac{3-4i}{25} = \frac{3}{25} - \frac{4}{25}i$$

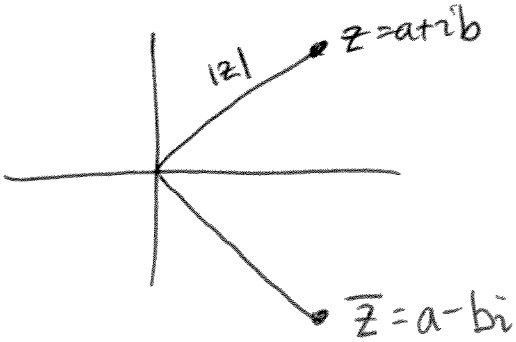
Switching the sign of the imaginary part has a special name: complex conjugate

$z = a+bi$

$\bar{z} = a-bi$

also define $|z| = \sqrt{a^2+b^2}$

$$\bar{z}z = (a-ib)(a+ib)$$
$$= a^2+b^2$$
$$= |z|^2$$



Properties:

$\overline{z+w} = \bar{z} + \bar{w}$

$\overline{zw} = \bar{z}\,\bar{w}$

$\overline{(\bar{z})} = z$

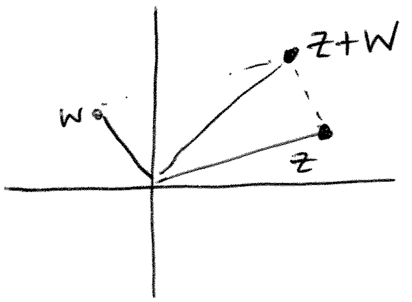
$z \in \mathbb{R} \iff \bar{z} = z$

$|\bar{z}| = |z|$

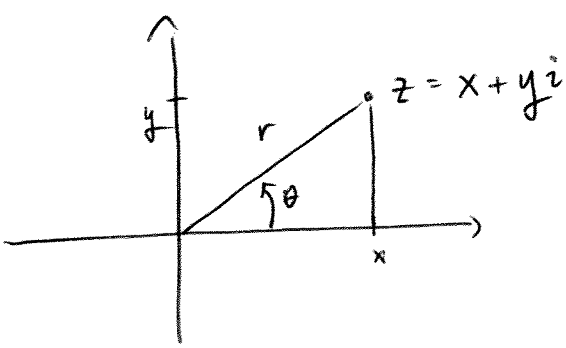
$|zw| = |z| \cdot |w|$

$|z| = 0 \iff z = 0$

$|z+w| \leq |z| + |w|$ Δ inequality.



Polar form



rename $r = |z|$

$r = \sqrt{x^2+y^2}$

$$z = x + yi$$
$$= r\left(\frac{x}{r} + \frac{y}{r} i\right) \qquad \frac{x}{r} = \cos\theta \qquad \frac{y}{r} = \sin\theta$$
$$= r(\cos\theta + \sin\theta\, i)$$

Euler: $e^z = 1 + z + \frac{z^2}{2} + \dots = \sum_{n=0}^{\infty} \frac{z^n}{n!}$ power series

$e^{i\theta} = \cos\theta + \sin\theta\, i$ (compare power series) WOW!

So $z = re^{i\theta}$ this is the polar form of $z = x + yi$

$r = \sqrt{x^2+y^2} = |z|$

θ is an angle such that $\cos\theta = \frac{x}{r}$ $\sin\theta = \frac{y}{r}$

not unique!

θ - "argument"

r - "modulus"

$$r_1 e^{i\theta_1} = r_2 e^{i\theta_2} \iff r_1 = r_2 \text{ and } \theta_1 = \theta_2 + 2\pi n$$

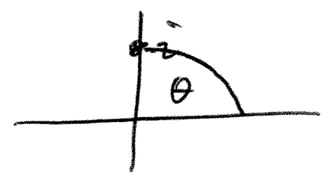
Ex: a) Find the polar form of $z = i = (0 + 1 \cdot i)$

$r = |z| = \sqrt{1^2 + 0^2} = 1$

$\sin\theta = \frac{x}{r} = \frac{1}{1} = 1$

$\cos\theta = \frac{0}{1} = 0$

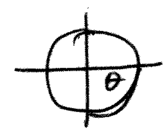
$\Rightarrow \theta = \frac{\pi}{2}$



$\Rightarrow z = i = e^{\frac{\pi}{2} i}$

b) Find the polar form of $z = -i$

$r = |z| = \sqrt{0^2 + 1^2} = 1$



$\cos\theta = \frac{0}{1} = 0$

$\sin\theta = \frac{-1}{1} = -1$

$\Rightarrow \theta = \frac{\pi}{2}$

c) $z = -1 = -1 + 0i$

$r = |z| = \sqrt{(-1)^2 + 0^2} = 1$

$\cos\theta = \frac{-1}{1} = -1$

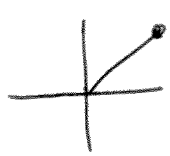
$\sin\theta = 0$

$\Rightarrow \theta = \pi$

$-1 = e^{\pi i} \Rightarrow e^{i\pi} + 1 = 0$

The 5 most important numbers in mathematics

d)



$z = 1 + i$

$|z| = \sqrt{1^2 + 1^2} = \sqrt{2}$

$\cos\theta = \frac{1}{\sqrt{2}} \quad \sin\theta = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$

$1 + i = \sqrt{2} e^{i\frac{\pi}{4}}$